Luminosity distance for Born-Infeld electromagnetic waves propagating in a cosmological magnetic background

Matías Aiello,¹,² Gabriel R. Bengochea,¹ and Rafael Ferraro¹,²

¹Instituto de Astronomía y Física del Espacio, Casilla de Correo 67, Sucursal 28, 1428 Buenos Aires, Argentina
²Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina

We solve the Born-Infeld equations for electromagnetic plane waves propagating in a static background magnetic field. We extend the solutions to the case of a spatially flat FRW universe by resorting to an adiabatic approximation. The non-linear character of Born-Infeld equations causes an influence of the background field on the amplitude and the phase velocity of the wave. These effects modify the luminosity distance of a source, which gains a new dependence on the redshift that is governed by the background field.

I. INTRODUCTION

In 1934 Born and Infeld ¹ ² proposed a non-linear electrodynamics with the aim of obtaining a finite value for the self-energy of a point-like charge. Born-Infeld Lagrangian leads to field equations whose spherically symmetric static solution gives a finite value for the electrostatic field at the origin. The constant $b$ appears in Born-Infeld Lagrangian as a new universal constant. Following Einstein, Born and Infeld considered the metric tensor $g_{\mu\nu}$ and the electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as the symmetric and anti-symmetric parts of a unique field $b g_{\mu\nu} + F_{\mu\nu}$. Then they postulated the Lagrangian density

$$\mathcal{L} = -\frac{1}{4\pi} \sqrt{|\det(b g_{\mu\nu} + F_{\mu\nu})|} - \sqrt{-\det(b g_{\mu\nu})}$$

where the second term is chosen so that Born-Infeld Lagrangian tends to Maxwell Lagrangian when $b \to \infty$. In four dimensions, this Lagrangian results to be

$$\mathcal{L} = -\frac{b^2}{4\pi} \left(1 - \sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^4}}\right)$$

where $S$ and $P$ are the scalar and pseudoscalar field invariants

$$S = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad P = \frac{1}{4} * F_{\mu\nu} F^{\mu\nu}$$

Born-Infeld Lagrangian is usually mentioned as an exceptional Lagrangian because of the properties of being the unique structural function which: 1- Assures that the theory has a single characteristic surface equation; 2- Fulfills the positive energy density and the non-space like energy current character conditions; 3- Fulfills the strong correspondence principle. As a consequence of these conditions, the Lagrangian has time-like or null characteristic surfaces.

Some attention has been devoted to the dynamics of a universe governed by matter described by Born-Infeld-like Lagrangians ⁴. In particular, it has been shown that a matter Born-Infeld field, under certain assumptions, can produce accelerated expansion ⁵ as the current observations suggest ⁶.

It is a well established fact in non-linear electrodynamics that the presence of a background field modifies the speed of an electromagnetic wave. This issue is studied in Ref. ⁷ ⁸ ⁹ ¹⁰ ¹¹ by considering the propagation of discontinuities. The result is that the surfaces of discontinuity do not propagate on the light cone when a background field is present. However the rays can be described as null geodesics of an effective geometry which depends on the background field.

In this work we study the propagation of Born-Infeld waves in Minkowski geometry, and obtain exact solutions when a static background magnetic field is present. We use an adiabatic approximation to extend these solutions to the case of Born-Infeld waves propagating in a spatially flat FRW expanding universe. We obtain that both the amplitude and the phase are modified by the expansion. In fact, the expanding scale factor $a(t)$ appears in the amplitude and the phase under the combination $|\mathbf{B}|^2/(a^4b^2)$, where $\mathbf{B}$ is the background field. This feature affects the redshift $z$ and the luminosity distance $d_L$ of a source. We compute $d_L(z)$ at the lowest order in $|\mathbf{B}|^2/(a^4b^2)$. We obtain that a background magnetic field in non-linear electrodynamics modifies the shape of the curve $d_L$ vs $z$.

The rest of this paper is organized as follows. In the next section we obtain solutions to Born-Infeld equations for plane waves in a background magnetic field. In section III we study the energy flux for plane and spherical waves and we work out the dependence of the luminosity distance on the redshift in a cosmological context when non-linear electrodynamics effects are present. In section

*Electronic address: aiello@iafe.uba.ar; ANPCyT Fellow
¹Electronic address: gabriel@iafe.uba.ar; CONICET Fellow
²Electronic address: ferraro@iafe.uba.ar; Member of Carrera del Investigador Científico (CONICET, Argentina)
IV we display the conclusions.

II. BORN-INFELD PLANE WAVE EQUATIONS

We assume a homogeneous, isotropic and flat universe with the standard Friedmann-Robertson-Walker (FRW) metric,
\[ ds^2 = a(\eta)^2 [d\eta^2 - dx^2 - dy^2 - dz^2] \quad (4) \]
where \( \eta \) is the conformal time defined by \( d\eta = a(t)^{-1} dt \) and \( a \) is the scale factor of the universe. Born-Infeld field satisfies,
\[ dF = 0 \quad (5) \]
\[ d^*F = 0 \quad (6) \]
where \( F \) is a 2-form defined as
\[ F = \frac{F - \frac{P}{b^2} * F}{\sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^2}}} \quad (7) \]
Then \( F \) can be written as
\[ d \left[ \ln \left( \sqrt{1 + \frac{2S}{b^2} - \frac{P^2}{b^2}} \right) \right] \wedge (*F + \frac{P}{b^2} F) = d \left[ *F + \frac{P}{b^2} F \right] \quad (8) \]

In order to study waves propagating in a uniform background magnetic field \( B \), we will separately consider the cases of transversal, longitudinal and a random background magnetic fields. To begin with, let us consider Minkowski space. So the scale factor \( a \) is constant.

A. B is transversal and \( P = 0 \)

We propose the solution
\[ F = E(\xi) \, d\xi \wedge dx + B_B \, dx \wedge dz \quad (9) \]
where
\[ \xi = z - \beta \eta \quad (10) \]
is the sole variable in the solution and \( B_B \) is the transversal background magnetic field. The first term in \( (9) \) is the wave and \( \xi \) is its phase. \( \beta \leq 1 \) in \( (10) \) takes account of the fact that waves might propagate inside the light cone. The solution \( (9) \) fulfills \( (8) \) for any function \( E(\xi) \). Since \( a \) is constant and \( P = 0 \), \( (8) \) implies
\[ \left[ \ln \left( \sqrt{1 + \frac{2S}{b^2}} \right) \right] ' d\xi \wedge *F = d *F \quad (11) \]
The 2-form \( *F \) corresponding to the field \( (9) \) is
\[ *F = E(\xi) (d\eta - \beta dz) \wedge dy - B_B \, d\eta \wedge dy \quad (12) \]
Then, \( (9) \) turns out to be
\[ \ln \left( \sqrt{1 + \frac{2S}{b^2}} \right) ' = \frac{E'(\xi)}{E(\xi) - (1 - \beta^2)^{-1}B_B} \quad (13) \]
which means that
\[ \sqrt{1 + \frac{2S}{b^2}} = C \left( E(\xi) - (1 - \beta^2)^{-1}B_B \right) \quad (14) \]
for some integration constant \( C \). Since
\[ 2S = a^{-4}[(1 - \beta^2)E(\xi)^2 - 2B_B E(\xi) + B_B^2] \quad (15) \]
one can easily check that \( (14) \) is satisfied for any function \( E(\xi) \) whenever \( \beta \) has the value
\[ \beta = \left( 1 + \frac{B_B^2}{a^4 b^2} \right)^{-1/2} \quad (16) \]
The integration constant in \( (14) \) is \( C = -a^{-4}b^{-2} \beta B_B \), which means that
\[ \sqrt{1 + \frac{2S}{b^2}} = \beta \left[ 1 + \frac{B_B}{a^2 b^2} (B_B - E(\xi)) \right] \quad (17) \]
The field \( *F \) turns out to be
\[ *F = \frac{(1 + \frac{P^2}{b^2}) (B_B - E(\xi))}{1 + \frac{B_B}{a^2 b^2} (B_B - E(\xi))} \, d\xi \wedge dy + B_B \, dy \wedge dz \quad (18) \]

B. B is transversal and \( B \parallel E \)

We propose the solution
\[ F = E(\xi) \, d\xi \wedge dx + B_E \, dy \wedge dz \quad (19) \]
where now \( B_E \) is the transversal background magnetic field parallel to \( E \).

Then
\[ *F = E(\xi) (d\eta - \beta dz) \wedge dy + B_E \, d\eta \wedge dx \quad (20) \]
and
\[ P = a^{-4} \beta E(\xi) \, B_E \quad (21) \]
In this case the l.h.s. in \( (18) \) will get a component \( d\eta \wedge dx \wedge dz \) that cannot be equaled to any similar component in the r.h.s. So, in order that \( (18) \) to be a solution of \( (18) \), the argument in the logarithm should be a constant:
\[ 2S - b^{-2}P^2 = a^{-4}[(1 - \beta^2)E(\xi)^2 + B_E^2 - a^{-4} \beta^2 b^{-2}E(\xi)^2 B_E^2] = D \quad (22) \]
Thus \( \mathbf{F} \) results
\[
\left( 1 - \beta^2 \right) - \beta^2 \frac{B_E^2}{a^4 b^2^2} E'(\xi) = 0 \quad (23)
\]
This equation is fulfilled for any function \( E(\xi) \) when \( \beta \) has the value
\[
\beta = \left( 1 + \frac{B_E^2}{a^4 b^2} \right)^{-1/2} \quad (24)
\]
Therefore, according to (22), it is
\[
\sqrt{1 + \frac{2 S}{b^2} - \frac{P^2}{b^4}} = \beta^{-1} \quad (25)
\]
The field \( \ast \mathbf{F} \) results
\[
\ast \mathbf{F} = -E(\xi) \, d\xi \wedge dy + \frac{B_E E(\xi)^2}{a^4 b^2 + B_E^2} \, d\xi \wedge dx + \left( 1 + \frac{B_E^2}{a^4 b^2} \right)^{-1/2} B_E \, d\eta \wedge dy \quad (26)
\]
C. \( B \) is longitudinal

The solution including a longitudinal background magnetic field \( B_L \) is
\[
F = E(\xi) \, d\xi \wedge dx + B_L \, dx \wedge dy \quad (27)
\]
Then
\[
\ast F = E(\xi) \, (d\eta - \beta dz) \wedge dy + B_L \, d\eta \wedge dz \quad (28)
\]
In this case it is
\[
P = 0, \quad 2 S = B_L^2 + (1 - \beta^2) E(\xi)^2 \quad (29)
\]
and \( \mathbf{F} \) becomes
\[
\left[ \ln \left( \sqrt{1 + \frac{2 S}{b^2}} \right) \right]' \, \left( 1 - \beta^2 \right) E(\xi) = (1 - \beta^2) E'(\xi) \quad (30)
\]
Then \( \beta = 1 \), so the wave travels on the light cone.

D. Random background \( B \)

When the background is an arbitrary uniform magnetic field, i.e.
\[
F_B = B_B \, dx \wedge dz + B_E \, dy \wedge dz + B_L \, dx \wedge dy \quad (31)
\]
then the left and right sides of \( \mathbf{F} \) are built from
\[
d\xi \wedge (\ast F + b^{-2} PF) = \left[ (1 - \beta^2) E(\xi) - B_B - a^{-4} b^{-2} \beta^2 E(\xi) B_E^2 \right] d\eta \wedge dy \wedge dz + \left[ B_E - a^{-4} b^{-2} \beta^2 E(\xi) B_B B_E \right] d\eta \wedge dz \wedge dx + dz - a^{-4} b^{-2} \beta^2 E(\xi) B_E B_L \, d\eta \wedge dy \wedge dz - \beta^{-1} B_E B_L \, d\eta \wedge dz \wedge dx \quad (32)
\]
\[
d(\ast F + b^{-2} PF) = \left[ (1 - \beta^2) - a^{-4} b^{-2} \beta^2 B_E^2 \right] E'(\xi) \wedge d\eta \wedge dy \wedge dz - a^{-4} b^{-2} \beta^2 E'(\xi) B_B B_E \, d\eta \wedge dz \wedge dx + B_E B_L \, d\eta \wedge dx \wedge dy \wedge dz \quad (33)
\]
Now \( \mathbf{F} \) decomposes in four components that are incompatible. This means that, in this case, the solution is not the mere sum of the wave plus the background magnetic fields but some additional terms (presumably of order \( b^{-2} \)) are necessary. Nevertheless we can consider a background magnetic field having non-correlated random components,
\[
\langle B_B \rangle = \langle B_E \rangle = \langle B_L \rangle = 0 \quad (34)
\]
Thus, only the component \( d\eta \wedge dy \wedge dz \) remains in \( \mathbf{F} \).

On the other hand, since the argument in the logarithm of \( \mathbf{F} \) is
\[
2 S - b^{-2} P^2 = a^{-4} \beta^2 \left( (\beta^{-2} - 1) E(\xi)^2 - 2 \beta^{-2} B_B E(\xi) + \beta^{-2} B^2 - a^{-4} b^{-2} E(\xi)^2 B_E^2 \right) \quad (35)
\]
where \( B^2 = B_B^2 + B_E^2 + B_L^2 \), it is also true that only the component \( d\eta \wedge dy \wedge dz \) is left in \( \mathbf{F} \). So we are going to solve the equation
\[
\frac{1}{2} \left[ b^{-2} S - b^{-4} P^2 \right]' \quad (36)
\]
subjected to the conditions \( \mathbf{F} \). Remarkably, the term linear in \( B_B \) in \( \mathbf{F} \) will survive combined with a similar term coming from \( \mathbf{F} \). After the factor \( E'(\xi) \) has been suppressed in \( \mathbf{F} \), one realizes that the terms that are proportional to \( E^2 \) cancel out. The remaining equation is
\[
\beta^{-2} \frac{B_E^2}{a^4 b^2} = \left( \frac{\beta^{-2} - 1}{a^4 b^2} \right) \left( 1 + \frac{\langle B_E^2 \rangle}{a^4 b^2} \right) \quad (37)
\]
so the value of \( \beta \) is
\[
\beta = \left[ 1 + 2 \frac{\langle B_E^2 \rangle}{a^4 b^2} \right]^{-1/2} \quad (38)
\]
We will compute the field \( \ast \mathbf{F} \) at the lowest order in \( b^{-2} \) (the rest of the paper will be worked at this order). In that case \( \ast \mathbf{F} \) can be approximated as (see \( \mathbf{F} \))
\[
\ast \mathbf{F} = \ast F \left( 1 - \frac{S}{b^2} \right) + \frac{P}{b^2} F + \mathcal{O}(b^{-4}) \quad (39)
\]
According to \( \mathbf{F} \) and \( \mathbf{F} \), \( S \) and \( P \) are
\[
- \frac{S}{b^2} = \frac{1}{2} \left( 2 B_B E(\xi) + B_E^2 + \mathcal{O}(b^{-4}) \right) \quad (40)
\]
\[
\frac{P}{b^2} = \frac{B_E E(\xi)}{a^3 b^2} + \mathcal{O}(b^{-4}) \quad (41)
\]

Taking into account that \( \langle F_b \rangle = \langle *F_b \rangle = 0 = \langle P \rangle \), the field \( \langle \mathcal{F} \rangle \) turns out to be

\[
\langle \mathcal{F} \rangle \simeq \left( 1 + \frac{\langle B^2 \rangle}{2 a^4 b^2} \right) F^{\text{wave}} - b^2 \langle S * F_b - P F_b \rangle
\]

\[
= \left( 1 + \frac{\langle B^2 \rangle}{2 a^4 b^2} \right) E(\xi) (d\eta - \beta dz) \wedge dy
\]

\[
- \frac{E(\xi)}{3 a^4 b^2} (d\eta + dz) \wedge dy
\]

\[
\simeq - \left( 1 + \frac{\langle B^2 \rangle}{2 a^4 b^2} \right) E(\xi) d\xi \wedge dy
\]

where \( F^{\text{wave}} \) alludes to \( E(\xi) d\xi \wedge dx \).

### E. Expanding universe

When the scale factor is not a constant but a function \( a(\eta) \), the variable \( \xi \) should be understood as

\[
\xi = z - \int_0^\eta \beta(\eta') \, d\eta'
\]

i.e., \( d\xi = dz - \beta(\eta)d\eta \). In this way, if the same functional dependence for \( \beta \) is kept (but the constant \( a \) is replaced by \( a(\eta) \)), then a substantial part of the previous computation will remain valid. However it is no more true that all magnitudes depend only on \( \xi \) because now there are factors \( a(\eta) \) in \( S, P \) and \( \mathcal{F} \). Even so, the former solutions can be useful as adiabatic approximations. Let us consider a wave \( E(\xi) = E \cos \omega \xi \) whose period \( T = 2\pi/(\omega \beta) \) is negligible compared with the characteristic time of the universe expansion \((a(\eta)/a(\eta))^{-1}\). By computing the adiabatic invariant

\[
J = \int \rho \, dq = 2 \int_0^{T/2} \rho' \, d\eta
\]

one can get the adiabatic correction to the amplitude. When \( a \) is a constant the electromagnetic potential is

\[
A = -\frac{E}{\omega \beta} \sin[\omega(z - \beta \eta)] \, dx + A_b
\]

The momentum conjugated to \( q = A_z \) is \( p = \sqrt{-g} \mathcal{F}^{\eta x} = -\mathcal{F}_{\eta x} \). For instance, the adiabatic invariant for the field \( B_z \) is

\[
J = \frac{E^2}{\omega} \left( 1 + \frac{5}{6} \frac{\langle B^2 \rangle}{a^4 b^2} \right) + \mathcal{O}(b^{-4})
\]

(hereafter we will approximate the results by expressions that are linear in \( b^{-3} \)). In order that \( J \) remains approximately constant when the expansion is slow, then the amplitude \( E \) in \( \text{(12)} \) should get a time dependent correction:

\[
E \rightarrow E \left( 1 - \frac{5}{12} \frac{\langle B^2 \rangle}{a(\eta)^4 b^2} \right)
\]

Therefore

\[
\langle \mathcal{F} \rangle \simeq - \left( 1 + \frac{\langle B^2 \rangle}{12 a(\eta)^4 b^2} \right) E \cos \omega \xi \, d\xi \wedge dy
\]

#### III. ENERGY FLUX, REDSHIFT AND LUMINOSITY DISTANCE IN A FRW METRIC

Let us now consider the energy flux for a Born-Infeld wave. When a plane wave propagates along the \( z \) direction, the \( \eta - z \) component of the energy-momentum tensor is given by

\[
T^\eta_z = -\frac{1}{4\pi} F^{\mu \nu} \mathcal{F}_{\mu z}^{\nu z}
\]

The plane wave energy flux can be easily converted into a spherical flux by equaling \( T^{\eta \eta} \) to \( r^{-2} T^{\eta z} \) and replacing \( z \) by \( r \) in the phase. In fact, the energy-momentum conservation law \( T^\mu_\nu = 0 \) can be written as

\[
[\sqrt{-g} T^{\mu \nu}]_{,\nu} + \Gamma^\mu_{\nu \rho} \sqrt{-g} T^{\rho \nu} = 0
\]

We can consider solutions of this equation in cartesian coordinates \( (\eta, x, y, z) \) or in spherical coordinates \( (\eta, r, \theta, \varphi) \). In the last case, the spatially flat FRW metric is

\[
ds^2 = a(\eta)^2 \left[ d\eta^2 - dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right]
\]

Remarkably, both of cases have equal Christoffel symbols: the only non-null ones are equal to \( a(\eta)'/a(\eta) \). This means that both of cases have equal solutions \( \sqrt{-g} T^{\mu \nu} \). Since the metric determinant changes from \( \sqrt{-g} = a(\eta)^4 \) to \( \sqrt{-g} = a(\eta)^4 r^2 \sin \theta \), but the factor \( \sin \theta \) is cancelled out for spherical symmetric solutions, then one concludes that \( r^2 T^{\nu \nu} \) takes the place of \( T^{\eta z} \) when the plane energy flux is replaced by a spherically symmetric one.

We will evaluate the energy flux for the wave \( \text{(48)} \). So, the sole components taking part in \( \text{(49)} \) are

\[
F_{\eta z}^{\text{wave}} \simeq - \left( 1 + \frac{\langle B^2 \rangle}{2 a^4 b^2} \right) \left( 1 - \frac{3}{4} \frac{\langle B^2 \rangle}{a(\eta)^4 b^2} \right) E \cos \omega \xi
\]

and

\[
F_{z z}^{\text{wave}} \simeq - \left( 1 + \frac{\langle B^2 \rangle}{2 a^4 b^2} \right) \left( 1 - \frac{\langle B^2 \rangle}{4 a(\eta)^4 b^2} \right) E \cos \omega \xi
\]

where \( 1 + \frac{\langle B^2 \rangle}{2 a^4 b^2} \) is a constant which normalizes the amplitude \( E \) in such a way that the flux of energy at the time of emission \( \eta_e \) is just proportional to \( E^2 \) (see
So, if a ray is emitted at time \( \eta \) at \( r = \text{constant} \) during a time \( dt = a(\eta) \, d\eta \) is

\[
\int T^{\eta} \, d\Sigma = \int_{\Omega} T^{\eta} \, d\Omega = 4\pi a(\eta)^4 r^2 \, T^{\eta \sigma}(\eta) \, dt
\]

i.e. the energy flux per unit time and area is

\[
\mathcal{F} = a(\eta)^2 \, T^{\eta \sigma}(\eta)
\]

The luminosity of an object results from integrating the flux at the time of emission \( \eta_e \). By using (53) one obtains

\[
L = \frac{E^2}{2 \, a_e^2}
\]

Combining (53) and (56) the flux of energy can be written as

\[
\mathcal{F} \approx \frac{L}{4\pi a(\eta)^2} \left[ 1 + \frac{(B^2)}{a(\eta)^4} \left( a_e^4 - 1 \right) \right]
\]

The luminosity distance \( d_L \) is defined as (see for instance [11])

\[
d_L^2 = \frac{L}{4\pi^{\frac{2}{3}} a_o^{\frac{1}{3}}}
\]

where \( a_o \) is the flux measured at time \( \eta_o \) at the position of the observer \( r_o \) (the source is at \( r = 0 \)). Therefore

\[
d_L = \frac{a_o^2 r_o}{a_e} \left[ 1 - \frac{(B^2)}{2 \, a_e^2} \left( a_e^4 - 1 \right) \right] + \mathcal{O}(b^{-4})
\]

In order to relate the luminosity distance with the redshift let us consider the motion of a ray: \( dr = \beta(\eta) \, d\eta \). So, if a ray is emitted at time \( \eta \) from a source located at \( r = 0 \), and arrives at time \( \eta_o \) to the position \( r_o \) of the observer, then the wavefront emitted at time \( \eta_e + \delta\eta_e \) will arrive at the observer at time \( \eta_o + \delta\eta_o \) in such a way that

\[
\int_{\eta_e}^{\eta_o} \beta \, d\eta = \int_{\eta_e + \delta\eta_e}^{\eta_o + \delta\eta_o} \beta \, d\eta
\]

or, equivalently

\[
\int_{\eta_e}^{\eta_e + \delta\eta_e} \beta \, d\eta = \int_{\eta_e}^{\eta_o + \delta\eta_o} \beta \, d\eta
\]

Thus \( \beta_c \, \delta\eta_c = \beta_o \, \delta\eta_o \), i.e. \( \beta_c \, a_e^{-1} \, \delta t_e = \beta_o \, a_e^{-1} \, \delta t_o \). Therefore the redshift is

\[
1 + z = \frac{\nu_e}{\nu_o} = \frac{\delta t_c}{\delta t_o} = \frac{a_o}{a_e} \left[ 1 + \frac{(B^2)}{3 \, a_e^2 b^2} \left( 1 - a_e^4 \right) \right] + \mathcal{O}(b^{-4})
\]

By inverting this relation one obtains

\[
\frac{a_o}{a_e} \simeq (1 + z) \left[ 1 + [(1 + z)^4 - 1] \frac{(B^2)}{3 \, a_e^2 b^2} \right]
\]

This quotient is one of the components in the luminosity distance \( \mathcal{L} \). The other one is the proper distance \( a_o r_o \). This distance depends on how the universe evolves. In fact, following the motion of the ray, it results

\[
a_o r_o = \int_{\eta_o}^{\eta_e} a_o \beta(\eta) \, d\eta = \int_{z}^{0} a_o \beta(z') \, \frac{dz'}{dz'}
\]

where \( z' \) is the redshift of a wave emitted at time \( t \leq t_o \). One can replace \( dz'/dt \) in terms of the Hubble parameter \( H \equiv \frac{\dot{a}}{a} = \frac{d}{dt} \log \left( \frac{a(t)}{a_o} \right) \) by performing the derivative of the logarithm of (53),

\[
H(z) \simeq -\frac{1}{(1 + z)} \left[ 1 + (1 + z) \frac{4 \, (B^2)}{3 \, a_e^2 b^2} \right] \frac{dz}{dt}
\]

Thus

\[
a_o r_o \simeq \int_{0}^{z} a_o \beta(z') \left[ 1 + (1 + z') \frac{4 \, (B^2)}{3 \, a^2 b^2} \right] \frac{dz'}{(1 + z')H(z')}
\]

\[
\simeq \int_{0}^{z} \left[ 1 + (1 + z')^{4} - 1 \right] \frac{(B^2)}{3 \, a^2 b^2} \frac{dz'}{H(z')}
\]

According to Einstein equations the Hubble parameter for a spatially flat universe dominated by matter and cosmological constant is [11]

\[
H(z)^2 = H_0^2 \left[ \Omega_m \left( \frac{a_o}{a} \right)^3 + \Omega_\Lambda \right]
\]

where \( \Omega_m \) and \( \Omega_\Lambda \) are the contributions from matter and a cosmological constant to the total density of the universe (so we are considering here \( \Omega_m + \Omega_\Lambda = 1 \)). From (53) one knows that

\[
H(z)^2 = \frac{\Omega_m + \Omega_\Lambda Z^3}{Z^3}
\]

Therefore the proper distance (66) times the present Hubble parameter results

\[
H_o a_o r_o \simeq \int_{1}^{1+z} \frac{1}{\sqrt{\Omega_\Lambda + \Omega_m Z^3}}
\]

\[
+ \frac{(B^2)}{a^2 b^2} \left( \frac{4 \, Z^4 - 1}{3 \, \sqrt{\Omega_\Lambda + \Omega_m Z^3}} - \frac{\Omega_m Z^3 \, (Z^4 - 1)}{2 \, (\Omega_\Lambda + \Omega_m Z^3)^{\frac{3}{2}}} \right)
\]

By replacing this integral in (68) and combining it with (53), the luminosity distance turns out to be

\[
H_o d_L \simeq (1 + z) \int_{1}^{1+z} \frac{1}{\sqrt{\Omega_\Lambda + \Omega_m Z^3}} \, dz + \frac{(B^2)}{a^2 b^2} f_L(z)
\]
The approximation is valid when \((1 + z)^4 \frac{(B^2)}{a_o^{-4} b^{-2}} \ll 1\); see for instance [13].

[1] The approximation is valid when \((1 + z)^4 \frac{(B^2)}{a_o^{-4} b^{-2}} \ll 1\); see for instance [13].

[2] The modulus of the magnetic field results from reading the background field \(F_b\) in the normalized basis \((a dy, a dx, a dy, a dz)\).
linear electrodynamics should be considered as a source of degeneration in the curve \(d_L\) vs \(z\). Figure 4 compares the curve of the standard cosmology (\(\Omega_m = 0.3, \Omega_\Lambda = 0.7\), \(b \to \infty\)) with the one resulting from Born-Infeld electrodynamics for the same values of \(\Omega_m\) and \(\Omega_\Lambda\). The curves intersect at \(z_r = 2.55\). If \(z < z_r\) the curves get their maximum separation at \(z = 1.91\). If \(z > z_r\) the luminosity distance predicted by Born-Infeld electrodynamics becomes smaller than the one from standard cosmology.

In this last case the curves seem to go dramatically apart (however this feature should be confirmed by extending the calculus up to a higher order of approximation). The degree of separation of both curves is governed by the value of \(\langle B^2 \rangle a^{-4}b^{-2}\). Future observations could allow a test of these features to obtain a constraint for this value.

### IV. CONCLUSIONS

In this work we have solved the Born-Infeld equations for electromagnetic plane waves propagating in a background magnetic field. In the absence of background field, Born-Infeld plane waves are equal to Maxwell ones. On the contrary, in the presence of a background magnetic field \(\mathbf{B}\) the non-linear effects modify both the phase and the amplitude of the wave with corrections that depend on the combination \(|\mathbf{B}|^2 a^{-4} b^{-2}\), where \(a\) is the scale factor of the universe. It is remarkable that Born-Infeld electrodynamics depends on \(a\) and \(b\) only through the combination \(a^4 b^2\). This means that the Maxwellian approximation \((b \to \infty)\) also corresponds to the limit \(a \to \infty\). So, although the electromagnetic field is presently well described by Maxwell equations for a wide range of phenomena, the non-linear Born-Infeld electrodynamics could have an influence in the past when the scale factor was smaller. Therefore the expanding universe is a good laboratory to test Born-Infeld electrodynamics; many non-linear aspects of its equations could be relevant when highly redshifted objects are observed.

In this work we have begun the search for this kind of effects. The influence of Born-Infeld electrodynamics on the luminosity distance exhibits interesting features that could be experimentally established by means of more precise supernova observations and a better knowledge of the cosmological background fields. Firstly, the experimental data for \(d_L\) vs \(z\) could be fitted without invoking dark energy, although there is no observational evidence for the background field that would be needed. Secondly, the shape of the curve \(d_L\) vs \(z\) predicted by the standard cosmology \((\Omega_m = 0.3, \Omega_\Lambda = 0.7, b \to \infty)\) for high redshifts differs appreciably from the one predicted by Born-Infeld electrodynamics, which opens the possibility of detecting non-linear electrodynamics effects in the future.

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