Black Holes Are Almost Optimal Quantum Cloners

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We show that black holes clone incoming quantum states with a fidelity that depends on the black hole’s absorption coefficient. Perfectly reflecting black holes are optimal universal quantum cloners of the type described by Simon, Weihs, and Zeilinger \cite{1}, and operate on the principle of stimulated emission. In the limit of perfect absorption, the fidelity of clones is equal to what can be obtained via quantum state estimation methods, which is suboptimal. But for any absorption probability less than one, the cloning fidelity is nearly optimal as long as \(\omega/T \geq 10\), a common parameter for modest-sized black holes.

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Black holes are quantum objects with intriguing characteristics. They are formed in the stellar collapse of stars with sufficient mass because such objects become relativistically unstable. While classically no signal can emerge from within the event horizon, Hawking showed that in curved-space quantum field theory, black holes must emit thermal radiation with a temperature \(T\) that is inversely proportional to the black hole’s mass \(M\): \(T = 1/(8\pi M)\), in convenient units \cite{2}.

But Hawking’s calculation also created an apparent paradox. Because the eponymous radiation takes its energy from the mass of the black hole, the latter might ultimately evaporate. And if the emitted radiation is strictly thermal as the calculation suggests, the initial data about the formation of the black hole would have to be erased with the concomitant evaporation, something the laws of physics simply should not permit. The same appears to hold true for information directed at the event horizon after the formation of the black hole (this means “at late times” in the parlance of gravity, because according to stationary observers the black hole only forms for \(t \rightarrow \infty\)). If the black hole final state does not depend on whether Shakespeare’s or Goethe’s works are absorbed by the black hole—given that both \textit{œuvres} have the same mass—then space time dynamics would appear to be irreversible even if black holes never evaporate at all.

Recently, we found that from the point of view of quantum communication theory, black holes are not so special after all \cite{3}. Rather, they appear to be relatively ordinary noisy quantum communication channels, with a capacity to transmit classical information given by the expression proposed by Holevo \cite{4}, because any matter or radiation absorbed by the black hole must stimulate the emission of \textit{exact copies} outside of the event horizon \cite{5}. Because the absorbed particles are quantum states, we may wonder whether stimulated emission at the black hole horizon violates the quantum no-cloning theorem \cite{6,7}. It does not: the spontaneously emitted Hawking radiation prevents precisely that. But we \textit{may} ask how well quantum states are copied by black holes, because this will shed light on how black holes treat \textit{quantum}—rather than classical—signal states. We find that a black hole’s cloning fidelity depends on the probability with which quantum states are absorbed by the black hole, \(\Gamma_0\), with a cloning fidelity ranging from the classical measurement limit to the universal optimal cloner \cite{8,9}.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{penrose_diagram.png}
\caption{Penrose diagram of the spacetime of a black hole. Modes \(a\), \(b\), \(c\) and \(A\) are concentrated in a region of null infinity indicated by the letter (note that \(a\) and \(b\) actually overlap on \(I^-\)).}
\end{figure}

Quantum black holes can be described succinctly by an operator \(H\) that transforms the so-called \textit{Boulware} vacuum of curved-space quantum field theory, (which becomes the ordinary Minkowski vacuum far from the black hole) into a non-trivial vacuum separated by an event horizon into an “inside” region (denoted by “II” in Fig. 1) and an “outside” (region I): the Unruh vacuum
\[|\psi_{\text{out}}\rangle = e^{-iH}|\psi_{\text{in}}\rangle.\]
The Boulware vacuum is annihilated by the operators $a_k$ and $b_k$ describing modes that in the future travel just outside and just inside the event horizon, and are related to the operators that annihilate the Urruh vacuum $A_k$ (annihilating outside states) and $B_k$ ("inside") by a unitary transformation. Even though there is no "inside" and "outside" for the early-time vacuum, such a separation is useful because it guarantees that time evolution does not mix positive and negative frequency modes (see, e.g., [10]). In a quantum theory of complex scalar fields, the Bogoliubov transformation relating $A_k$ to $a_k$ and $b_k$ is, for example,

$$A_k = e^{-iH}a_ke^{iH} = \alpha_k a_k - \beta_k b_{-k}^\dagger,$$

which can be implemented by the Hermitian operator

$$H = ig_k(a_k^\dagger b_{-k} - a_k b_{-k} + k \rightarrow -k)$$

describing the entanglement of particles and antiparticles outside and inside the horizon (for mode $k$). In terms of the interaction strength $g_k$, we find

$$\alpha_k^2 = \cosh^2 g_k = \frac{1}{1 - e^{-\omega/T}},$$

$$\beta_k^2 = \sinh^2 g_k = \frac{1}{e^{\omega/T} - 1},$$

where $\omega = |k|$ is the frequency associated with mode $k$ and $T$ is the Hawking temperature. The relationship between the coefficients $\alpha_k$, $\beta_k$, the frequency of the modes and the Hawking temperature $T$ is enforced by requiring that the solution to the free-field equations in a black hole background is analytic across the horizon, as usual [2]. Note that while the full Hamiltonian is a sum over modes with all $k$, we focus here on a single mode because they do not mix in this 1+1 dimensional theory.

Quantum cloning refers to the duplication or copying of quantum information, a process that is known to be impossible to achieve perfectly [3, 4] but that can be performed approximately. A quantum cloning machine attempts to maximize the probability that a quantum state $|\psi\rangle$ (or by extension $N$ identically prepared states $|\psi\rangle^\otimes N$) is cloned into a state $|\psi_{\text{out}}\rangle = U |\psi_{\text{in}}\rangle |0\rangle_X$ with a unitary transformation $U$ acting on the input state $|\psi_{\text{in}}\rangle$, a blank state $|0\rangle$ that is to hold multiples of the cloned state, and an ancillary state $|X\rangle$. The formalism of quantum cloning machines was introduced by Bužek and Hillery [5], and has received considerable attention (see, e.g., the reviews [11, 12]). The fidelity of one of the $M$ cloned states $\rho_{\text{out}}^j$ is given by

$$F_j = \text{in}(\langle \psi | \rho_{\text{out}}^j | \psi \rangle),$$

and the optimal $N \rightarrow M$ cloning fidelity of universal quantum cloning machines (devices that copy any input state $|\psi\rangle$ with the same fidelity) is [3, 13]

$$F_{\text{opt}} = \frac{M(N+1)+N}{M(N+2)}.$$
The fidelity of these anticlones is

$$F_{\text{anti}} = \frac{N+1}{N+2}$$  \hspace{1cm} (10)$$

for each anticlone, which is the fidelity of the best possible state preparation via state estimation from a finite quantum ensemble [16]. Note that the optimal cloning fidelity [7] tends to the state estimation fidelity in the limit $M \to \infty$, which we recognize as the classical measurement limit of quantum cloning.

Consider the fidelity of cloning when $N$ antiparticles in mode $b$ are sent in, traveling just inside the horizon: $|\psi\rangle_{\text{in}} = |0,0\rangle_a|0,N\rangle_b$. We are now interested in the fidelity of the anticlones generated on the other side. The wavefunction is

$$|\psi\rangle_M \sim \sum_{j=0}^{M} \sqrt{\left(\frac{j+N}{N}\right)} |j,M-j\rangle_a |M-j,j+N\rangle_b$$  \hspace{1cm} (11)$$
giving the probability to observe $j$ particles and $M-j$ antiparticles outside the horizon

$$p(j,M-j|N) = \frac{\left(j+N\right)}{\sum_{j=0}^{N} \left(j+N\right)}.$$  \hspace{1cm} (12)$$

The fidelity of these anticlones clones is also $N+1/N+2$, equal to the fidelity of the anticlones when sending in $|N,0\rangle_a$, but the anticlones of the antiparticle states are of course clones of the particle states. In summary, sending $N$ particles in mode $a$ creates optimal particle clones in region I, while sending in $N$ antiparticles in mode $b$ gives rise to classical particle clones in region I instead.

We now discuss the more realistic cloning scenario where we send quantum states into the already-formed black hole at late times, by introducing an additional mode $c$ (see Fig. 1) that is strongly blue-shifted with respect to the early-time modes $a$ and $b$, and therefore commutes with them. The $c$-modes are turned into $a$ and $b$ modes via absorption, reflection, and emission at the horizon, and were used earlier as signal states to study the fate of information sent into black holes [3]. The Bogoliubov transformation that connects these operators (for each mode $k$) is (we omit the index $k$ from the Bogoliubov coefficients from now on)

$$A_k = e^{-iH}a_k e^{iH} = \alpha a_k - \beta b^\dagger_k + \gamma c_k,$$  \hspace{1cm} (13)$$
which can be implemented with the Hamiltonian

$$H = ig(a^\dagger_k b^\dagger_{-k} - a_k b_{-k}) + ig'(a^\dagger_k c_k - a_k c^\dagger_k) + k \to -k.$$  \hspace{1cm} (14)$$

Given the analogy between a black hole and a parametric down-converter, this Hamiltonian has an interesting quantum optical interpretation as the sum of an active optical element—a parametric amplifier described by a squeezing Hamiltonian with gain $g$ and a passive one: a beam-splitter with a phase $g'$ (see, e.g., [17]).

Obtaining the coefficients $\alpha, \beta,$ and $\gamma$ from (13) is straightforward, and yields

$$\alpha^2 = \cos^2(\sqrt{g^2 - g''^2}) = \Gamma_0 = \frac{1}{1 - e^{-w/T}}$$  \hspace{1cm} (15)$$
$$\beta^2 = \frac{g' g}{g''} \sin^2(\sqrt{g^2 - g''^2}) = \frac{\Gamma}{e^{w/T} - 1}$$  \hspace{1cm} (16)$$
$$\gamma^2 = \frac{\sin^2(\sqrt{g^2 - g''^2})}{w^2} = 1 - \Gamma.$$  \hspace{1cm} (17)$$

where $w^2 = 1 - (g/g'')^2$. Note that unitarity implies $\alpha^2 - \beta^2 + \gamma^2 = 1$. The coefficients $\alpha$ and $\beta$ together again set the black hole temperature via $\beta^2/\alpha^2 = e^{-w/T}$. The quantum absorption probability of the black hole, $\Gamma_0$, is set by $\alpha^2$ alone, while the classical absorption probability $\Gamma$ is set by $\alpha^2 - \beta^2$. Thus, $\Gamma$ is always smaller than $\Gamma_0$ by $\beta^2$, which is the strength of the quantum effects. In the above formulas, $g' \geq g$ so that $w$ is real and $0 < \Gamma_0 < 1$ [18] is a probability.$^3$

Consider first $1 \to M$ cloning. Because the extended Hamiltonian [13] is still rotationally invariant, we can again restrict ourselves to study the effect of the black hole on one particular state. To clone the state $|1\rangle_L = |0,0\rangle_a|0,0\rangle_b|1,0\rangle_c$, for example, we obtain:

$$\rho_a = \text{Tr}_{bc} \langle U |1\rangle_L |1\rangle_L^\dagger = \rho_{k|1} \otimes \rho_{-k|0},$$  \hspace{1cm} (18)$$

where (with $\xi = \frac{\gamma^2}{\alpha^2 \beta^2}$)

$$\rho_{k|1} = \sum_m p(m|1) |m\rangle\langle m|$$

$$= \frac{\alpha^2}{(1 + \beta^2)} \sum_{m=0}^{\infty} \left( \frac{\beta^2}{1 + \beta^2} \right)^m (1 + m\xi) |m\rangle\langle m|,$$  \hspace{1cm} (19)$$

$$\rho_{k|0} = \frac{1}{1 + \beta^2} \sum_{m=0}^{\infty} \left( \frac{\beta^2}{1 + \beta^2} \right)^m |m\rangle\langle m|.$$  \hspace{1cm} (20)$$

From (19,20) the $1 \to M$ cloning fidelity can be calculated as before

$$F_{1 \to M} = \sum_{M=0}^{\infty} \frac{M+\xi}{M} \sum_{m=0}^{M} p(M-j|1)p(j|0) = \frac{3 + \xi + 2\xi M}{3(2 + \xi M)}.$$  \hspace{1cm} (21)$$

Let us investigate this result in a number of physical limits. As the black hole becomes more and more reflective, $\Gamma_0 = \alpha^2 \to 0$, which implies $\xi \to \infty$. In this case, the fidelity (21) approaches the optimal value

$$\lim_{\alpha_a \to 0} F_{1 \to M} = \frac{2}{3} + \frac{1}{3M},$$  \hspace{1cm} (22)$$
as is seen in Fig. 2. For arbitrary $N$, the limit is the Gisin-Massar optimal fidelity [1]. We can recognize this result as the special case we treated earlier: If the black hole perfectly reflects incoming states, the black hole behaves just as if early-time modes ($a$-modes) were traveling just outside the horizon (except for the redshift).
Another limit of note is that of full absorption: $\Gamma_0 \to 1$. In that case $\xi \to 1$ and $F_{1 \to M} \to 2/3$, the fidelity of a classical cloning machine. It can be shown in general that for full absorption, the $N \to M$ cloning fidelity is equal to $N + 1/N + 2$ independently of $\omega/T$, which is the result we obtained earlier when sending $N$ antiparticles in mode $b$ directly behind the horizon. This is again not surprising, as the absorption of $c$-particles stimulates the emission of $b$ antiparticles behind the horizon, who in turn give rise to anticolones of the antiparticles: clones.

While in this case the best we can do to reconstruct the state is to make classical measurements that allow us to optimally estimate the quantum state, note that in the limit $N \to \infty$ the probability to do this correctly tends to one, implying that the quantum state information can be reconstructed with arbitrary accuracy. This result mirrors the one we obtained earlier [2] showing that in the limit $N \to \infty$ the capacity of the quantum black hole channel to transmit classical information becomes equal to the noiseless channel capacity, even for full absorption. Another interesting limit is that of large black holes, where the Hawking temperature approaches zero. In that case, as $\omega/T \to \infty$ implies $\xi \to \infty$ (as long as $\Gamma_0 < 1$), we again recover the optimal universal quantum cloning fidelity $[22]$. This behavior can be seen in Fig. 3. Note that as even modest-sized black holes have $\omega/T \geq 10$, most black holes are nearly-optimal universal quantum cloners unless the absorption probability is exactly equal to 1. These results extend to $N \to M$ cloning machines. Just as for the $N = 1$ case, the black hole cloner approaches the optimal cloner in the limit $T \to 0$ or $\Gamma_0 \to 0$, and turns into a classical cloning machine in the limit $\Gamma_0 \to 1$ and for $M \to \infty$.

In conclusion, the conservation of information in black hole dynamics can be understood from a quantum information-theoretic point of view by observing that black holes are nearly-optimal universal quantum cloners, implying that any information, classical or quantum, is cloned as well as can be expected given the laws of physics, and left outside the event horizon before the carriers of information are absorbed by the black hole.

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