Emission spectra and quantum efficiency of single-photon sources in the cavity-QED strong-coupling regime

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We derive analytical formulas for the forward emission and side emission spectra of cavity-modified single-photon sources, as well as the corresponding normal-mode oscillations in the cavity quantum electrodynamics strong-coupling regime. We investigate the effects of pure dephasing, treated in the phase-diffusion model based on a Wiener-Levy process, on the emission spectra and normal-mode oscillations. We also extend our previous calculation of quantum efficiency to include the pure dephasing process. All results are obtained in the Weisskopf-Wigner approximation for an impulse-excited emitter. We find that the spectra are broadened, the depths of the normal-mode oscillations are reduced and the quantum efficiency is decreased in the presence of pure dephasing.

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I. INTRODUCTION

Single-photon sources (SPS) have important uses in quantum communication [1, 2], quantum computing [3], and metrology [4]. Since the turn of this century, significant progress has been made in generating single photons from a microscopic quantum emitter such as an atom or ion [5, 6], an organic molecule [7, 8, 9], a semiconductor quantum dot (QD) or nanocrystal [10, 11, 12, 13, 14, 15, 16, 17], or a color center in diamond [18, 19, 20]. Most of these SPS are based on spontaneous emission, whose lifetime eventually limits the emission rate and linewidth of the SPS, and whose isotropic nature prevents high collection efficiency. A more promising scheme for producing well-controlled single photons is cavity quantum electrodynamics (QED). An atom or QD inside a high-finesse microcavity is prepared in an excited state and is allowed to spontaneously emit. The emission rate and linewidth can be considerably altered in the cavity-QED weak-coupling regime. If the cavity-QED cavity-diagram limit, the altered emission rate is obtained from Fermi’s golden rule using a modified density of states to account for the cavity boundary conditions. In the weak-coupling regime, the atomic excitation is irreversibly lost to the continuum of all available photon states, including the side modes (leak modes) and cavity modes. The side emission and forward emission spectra are single-peaked, either enhanced or inhibited.

If the cavity volume is sufficiently small, and the finesse is high enough such that the coherent interaction rate between the quantum emitter and cavity exceeds the decay rates of the composite system to be in the cavity-QED strong-coupling regime, the emission process is reversible and a photon emitted into the cavity can be coherently reabsorbed before it is emitted out of the cavity. The initially excited quantum emitter undergoes single-quantum Rabi oscillations. The emissions, still allowed to both the side and forward direction of the cavity, will show double-peaked spectra. In this regime, the coupling of the emission to the single cavity mode, however, is far stronger than its coupling to the side modes. If, in addition, there is almost no dephasing of the quantum emitter during the emission process, the emission process can be nearly deterministic [5, 6], emitting a photon into a well-defined “forward” beam outside the cavity.

Most if not all single-quantum systems, however, inevitably interact with certain heat baths, leading to dephasing or loss of coherence, which results from a randomization of the phases of the emitter’s wave functions by thermal fluctuations in the environmental fields. Population relaxation processes contribute to dephasing with a dephasing rate given by half the population decay rate. It is often necessary to account for other dephasing interactions, such as elastic collisions in an atomic vapor, or elastic phonon scattering in a solid, the so-called pure dephasing process. Pure dephasing causes the coherent overlap of the upper and lower state wave functions to decay in time, while not affecting the state populations. For example, the pure dephasing rate can be small and ignored for resonant excitation of a single QD at low temperature (6 K) and power density [21]. While at elevated temperature, however, experiments [22, 23] reveal a pure dephasing contribution that dominates excitonic dephasing. Our results are directly applicable to experimental data presented in Refs. [24, 25, 20], for which no theoretical predictions were previously available.

In this paper, we derive the normal-mode oscillations and spectra of the photon emitted both in the side direction and in the forward beam in the cavity-QED strong-coupling regime. We first review the derivation of the normal mode oscillations and emission spectra obtained in the Weisskopf-Wigner approximation (WWA) Secs. 10 and 11. Then we focus on the influence of the pure dephasing process, treated in the phase-
where $\Delta = \omega_0 - \omega_c$ and $\sigma_+ = \sigma_0$. Later we allow it to fluctuate, to model pure dephasing.

Given that there is only one excitation in the system, the state vector can be written as

\[
|\psi(t)\rangle = E(t)|e, 0\rangle|0\rangle_R_1 |0\rangle_R_2 + C(t)|g, 1\rangle|0\rangle_R_1 |0\rangle_R_2 \\
+ \sum_p S_p(t)|g, 0\rangle|1_p\rangle_R_1 |0\rangle_R_2 \\
+ \sum_k O_k(t)|g, 0\rangle|0\rangle_R_1 |1_k\rangle_R_2
\]

where $|m, n\rangle$ ($m = e, g; n = 0, 1$) denotes the emitter state (excited state, ground state) with $n$ photons in the cavity, $|\rho_p\rangle R_1 |1_k\rangle R_2$ ($j, l = 0, 1$) corresponds to $j$ photons in the $p$ mode (other than the privileged cavity mode) of the emitter reservoir $R_1$ and $l$ photons in a single-mode ($k$) traveling wave of the one-dimensional photon reservoir $R_2$ (output beam). $E(t), C(t), S_p(t)$, and $O_k(t)$ are the slowly varying probability amplitudes.

The equations of motion for the probability amplitudes are obtained by substituting $|\psi(t)\rangle$ and $\dot{H}_I(t)$ into the Schrödinger equation and then projecting the resulting equations onto different states respectively. In the WWA [27, 31], we obtain

\[
\dot{E}(t) = -i\gamma g e^{i\Delta t} C(t) - \gamma E(t), \\
\dot{C}(t) = -i\gamma g e^{-i\Delta t} E(t) - \kappa C(t) \\
S_p(t) = -iA^*_p \int_0^t dt'' e^{i\delta_p t''} E(t''), \\
O_k(t) = -iB^*_k \int_0^t dt'' e^{i\delta_k t''} C(t')
\]

where $\gamma$ and $\kappa$ are one-half the radiative decay rates of the atomic population (other than the privileged cavity mode) and the intracavity field, respectively. Dots indicate time derivatives. The general solutions to the coupled differential Eqs. 3 and 4 are

\[
\left( \begin{array}{c} E(t) \\ C(t) \end{array} \right) = e^{-(\kappa/2)t} \left( \begin{array}{cc} e^{i\Delta t/2} & 0 \\ 0 & e^{-i\Delta t/2} \end{array} \right) \left( \begin{array}{cc} 1/2 - \frac{\delta_p}{2\lambda} + \frac{i\gamma + \Delta}{4\lambda} & -\frac{\delta_k}{2\lambda} - \frac{\gamma}{2} \\ -\frac{\delta_k}{2\lambda} - \frac{\gamma}{2} & 1/2 + \frac{i\gamma + \Delta}{4\lambda} \end{array} \right) \\
\left( \begin{array}{c} E(0) \\ C(0) \end{array} \right)
\]
where \( K \equiv \kappa + \gamma \), \( \Gamma \equiv \kappa - \gamma \), and \( \lambda = \sqrt{g_0^2 - (\Gamma - i \Delta)/2}^2 \).

In the strong-coupling regime, defined by \( g_0 \gg \kappa, \gamma \), the real part of \( \lambda \) is much larger than its imaginary part. Then \( \lambda \) can be approximated as \( \lambda \approx g = \sqrt{g_0^2 + (\Delta/2)^2 - (\Gamma/2)^2} \), which is the generalized vacuum Rabi frequency. Note that for the case when the emitter and cavity are exactly at resonance, \( \Delta = \omega_0 - \omega_c = 0 \), the complex frequency \( \lambda \) is purely real and equals \( \sqrt{g_0^2 - (\Gamma/2)^2} \). The solutions to the probability amplitudes are then

\[
\begin{pmatrix}
E(t) \\
C(t)
\end{pmatrix} = e^{-(K/2)t} \begin{pmatrix}
e^{i\Delta t/2} & 0 \\
0 & e^{-i\Delta t/2}
\end{pmatrix} \begin{pmatrix}
\cos(gt) + \Gamma - i\Delta \\
-\frac{\Gamma - i\Delta}{2g} \sin(gt)
\end{pmatrix} \begin{pmatrix}
E(0) \\
C(0)
\end{pmatrix}. \tag{6}
\]

In this context, we assume the quantum emitter is prepared in an excited state \( E(0) = 1 \), \( C(0) = 0 \) at time \( t_0 = 0 \) (more generally, it can be prepared in an arbitrary single-quantum state). The solutions subject to this initial condition are

\[
E(t) = e^{-(K - i\Delta)/2t} \left[ \cos(gt) + \frac{\Gamma - i\Delta}{2g} \sin(gt) \right] \tag{7}
\]

\[
C(t) = e^{-(K + i\Delta)/2t} \left[ -\frac{i\gamma_0}{g} \sin(gt) \right]. \tag{8}
\]

\( S_E(t) \) and \( O_C(t) \) can be obtained by carrying out the integrations in Eq. (4).

### III. NORMAL-MODE OSCILLATIONS AND EMISSION SPECTRA OF SPS IN THE CAVITY-QED STRONG-COUPLING REGIME

The strong interaction between an excited quantum emitter and a single cavity mode leads to single-quantum Rabi oscillation (normal-mode oscillation) in the time domain or a frequency splitting in the frequency domain, the so-called normal-mode splitting, which arises from the coherent interaction of two degenerate systems—the single quantum emitter and the single cavity mode. In this section, we discuss the normal-mode oscillations and emission spectra of SPS in the cavity-QED strong-coupling regime. We first investigate the normal-mode oscillations by calculating the probabilities of finding the composite system in different states. Then we define and calculate the spectra of SPS appropriate for this case in the long-time limit.

#### A. Normal-mode oscillations

The normal-mode oscillations can be viewed either in the dressed-state picture or in the bare-state picture. Here we look at them in the bare-state picture where the oscillations can be relatively easier to find. The probability of finding the system in the excited atomic state is

\[
P_c(t) = |E(t)|^2 = \frac{e^{-Kt}}{2} \left[ 1 + \frac{\Gamma^2 + \Delta^2}{4g^2} \right] \cos(2gt) + \frac{\Gamma}{g} \sin(2gt). \tag{9}
\]

The probability of finding the system in the single cavity mode is

\[
P_c(t) = |C(t)|^2 = \frac{q_0^2}{g^2} e^{-Kt} \sin^2(gt). \tag{10}
\]

Consider the case when the emitter and cavity are exactly at resonance, \( \Delta = \omega_0 - \omega_c = 0 \). Figure 2 are plots of the two probabilities with both linear and logarithmic scales. The probabilities oscillate sinusoidally with an exponential decay envelope. However, they have opposite phases, which indicate the coherent oscillatory energy exchange between the excited emitter and the cavity field.

Define the emission probability \( P_o(t) \) to be the probability of finding a single photon in the output mode of the cavity between the initial time \( t_0 = 0 \) and a later time \( t \). This equals

\[
P_o(t) = 2\kappa \int_0^t dt' |C(t')|^2 = \eta_q \left\{ 1 - e^{-Kt} \left[ 1 + \frac{K^2}{2g^2} \sin^2(gt) + \frac{K}{2g} \sin(2gt) \right] \right\}. \tag{11}
\]

where \( \eta_q \equiv \frac{[g_0^2/(g_0^2 + \kappa \gamma)]}{[\kappa/(\kappa + \gamma)]} \) is the single-photon QE, given by the single-photon emission prob-
ability \( P_o(t) \) in the sufficiently long-time limit \( t \gg K^{-1} \), which has been discussed in our previous publication [29].

**B. Emission spectra**

It may seem strange to talk about the spectrum of a single-mode field since we normally associate a single mode with a single frequency. Here we are dealing, however, with what should more correctly be called a quasi-mode, a mode defined in a leaky optical cavity, which therefore has a finite linewidth. For a stationary and ergodic process, the Wiener-Khintchine theorem [31] states that the spectrum is given by the Fourier transform of the two-time correlation function of the radiated field. In the strong-coupling regime and for an impulsive excitation of the system, however, this relation between the correlation function and spectrum fails because the coherent interaction overwhelms the relaxations here. There is no time \( t \) after which the correlation functions depend only on the time difference. Thus the dipole correlation and the emitted field correlation cannot be stationary. We use a generalized definition of the Wiener-Khintchine spectrum applicable in this case (Appendix A).

The side emission and forward emission spectra are defined as, in the long-time limit \( (t \gg K^{-1}) \)

\[
S_{s.e.}(\Omega) = 2\gamma \frac{1}{\pi} \text{Re} \left\{ \int_0^\infty d\tau e^{i\Omega\tau} \left[ \int_0^\infty dt E(t+\tau) E^*(t) \right] \right\}, \quad (12)
\]

\[
S_{f.e.}(\Omega) = 2\kappa \frac{1}{\pi} \text{Re} \left\{ \int_0^\infty d\tau e^{i\Omega\tau} \left[ \int_0^\infty dt C(t+\tau) C^*(t) \right] \right\}, \quad (13)
\]

where \( \Omega \equiv \omega - \omega_c \) is the emission frequency centered at the cavity resonance \( \omega_c \). Substituting the Eqs. (7) and (8) into Eqs. (12) and (13), we obtain the unnormalized spectra

\[
S_{s.e.}(\Omega) = \frac{\gamma}{\pi} \frac{\kappa - i(\Omega + \Delta)}{(K/2 - i\Delta/2 - i\Omega)^2 + g^2}^2, \quad (14)
\]

\[
S_{f.e.}(\Omega) = \frac{\kappa}{\pi} \frac{-i\omega_0}{(K/2 + i\Delta/2 - i\Omega)^2 + g^2}^2. \quad (15)
\]

The side emission spectrum here is the same as the spontaneous emission spectrum calculated elsewhere [32].

The forward emission spectrum is what we expect to measure by an ideal detection system at the output of the cavity in the forward direction, and has not been presented previously, to our knowledge. For zero atom-cavity detuning \( \Delta = 0 \), where the atom and cavity resonances are degenerate, both the side emission and forward emission spectra show the normal-mode splittings, which however, are different. The splittings are \( \Delta \omega_s = 2\sqrt{[g_0^2 + 2g_0\kappa(\kappa + \gamma)]^{1/2} - \kappa^2} \) for the side emission, and \( \Delta \omega_f = 2\sqrt{g_0^2 - (\kappa^2 + \gamma^2)/2} \) for the forward emission, as shown by the thicker red curves in Figs. 3(a) and 3(b) respectively. Both are different from the generalized Rabi splitting \( 2g \).
Beyond the energy-splitting difference at zero atom-cavity detuning, it is also illuminating to investigate the dependence of the energy eigenvalue structure on the atom-cavity detuning. Shown in Fig. 3 are plots of the spectra, in the strong-coupling regime, for seven different values of atom-cavity detuning $\Delta$. As $|\Delta|$ increases, the vacuum Rabi splitting also increases for both the side emission and forward emission spectra. At the same time, for the side emission spectra, the cavitylike peak features stronger emission and the atomlike peak grows smaller. While for the forward emission spectra, however, both peaks show the same emission intensity.

IV. INFLUENCE OF PURE DEPHASING ON THE NORMAL-MODE OSCILLATIONS AND EMISSION SPECTRA

Generally speaking, pure dephasing means the decay of the dipole coherence without change in the populations of the system. Any real transition to other states leads to population decay. Thus the pure dephasing is caused by virtual processes which start from a relevant state and, through some excursion in the intermediate states, return to the same initial state. These virtual processes give rise to the temporal fluctuations of phases of the wave functions, which consequently lead to pure dephasing.

A. Phase-diffusion model of pure dephasing

The effects of pure dephasing can be calculated numerically, based on the Green function formalism by considering the microscopic details of various virtual processes [33]. Instead, for simplicity we treat this problem analytically in the phase-diffusion model where the incoherence due to elastic collisions or elastic phonon scattering is described by a stochastic model of random frequency modulation, as shown in Fig. 4 replacing the atomic transition frequency or the phase of the wave function by an instantaneous one

$$\omega_0 \rightarrow \omega_0(t) = \omega_0 + f(t) \quad \text{or} \quad \omega_0 t \rightarrow \int_0^t dt' \omega_0(t') = \omega_0 t + \varphi(t),$$

where $f(t)$ is the instantaneous deviation of the transition frequency due to the elastic collisions or scattering process and $\varphi(t) \equiv \int_0^t f(t')dt'$ is the instantaneous stochastic phase of the wave function.

![FIG. 4: (Color online) Schematic diagram of pure dephasing process in the phase-diffusion model.](image-url)
$\hat{\varphi}(t)$ is a random, stationary, Gaussian variable with the mean value and the mean square correlation given by

$$\langle f(t) \rangle = 0, \quad \langle f(t)f(t') \rangle = 2\gamma_p \delta(t-t').$$  \hspace{1cm} (17)

The angular brackets indicate a statistical average over the random variables of the stochastic process. The Markovian nature of the process is reflected by the presence of the delta function $\delta(t-t')$. The Gaussian property is introduced such that all higher correlation functions can be obtained from the second-order correlation function by permutations and multiplications $[31]$. $2\gamma_p$ is the pure dephasing rate.

Taking into account the pure dephasing modeled by the stochastic process, the net change for Eq. (3) is that the phase term $e^{i\Delta t}$ should be replaced, such that

$$\dot{E}(t) = -i\gamma_0e^{i(\Delta t+\varphi(t))}C(t) - \gamma E(t)$$  \hspace{1cm} (18)

$$\dot{C}(t) = -i\gamma_0e^{-i(\Delta t+\varphi(t))}E(t) - \kappa C(t).$$  \hspace{1cm} (19)

We note that the above equations with stochastic random variables are examples of a multiplicative stochastic process, studied intensively by Refs. [34, 35, 36]. We solve these equations exactly, using the method developed by Wodkiewicz [36] for a multiplicative stochastic process described by the following general vector equation

$$\frac{d}{dt}\vec{v}(t) = [M_0 + if(t)M_1]\vec{v}(t),$$  \hspace{1cm} (20)

where $\vec{v}(t)$ is an $n$-dimensional vector, $M_0$ and $M_1$ are arbitrary $n \times n$ matrices, in general complex and time independent, and $f(t)$ is the random variable of the stochastic process described by Eq. (17). The equations of the type (20) can be solved for the quantum expectation value of $\vec{v}(t)$ exactly [37]. For a Wiener-Levy process, the stochastic average of the equation satisfies the following differential equation

$$\frac{d}{dt}\langle \vec{v}(t) \rangle = [M_0 - \gamma_p M_1^2] \langle \vec{v}(t) \rangle.$$  \hspace{1cm} (21)

The solution to Eq. (21) can be written in the Laplace-transform form

$$\langle \vec{v}(t) \rangle = \int_C \frac{dz}{2\pi i} \exp(zt)N^{-1}(z) \langle \vec{v}(0) \rangle,$$  \hspace{1cm} (22)

where the matrix $N^{-1}(z)$ is the inverse to $N(z)$, which itself is given by the formula $N(z) = z\mathbb{I} - (M_0 - \gamma_p M_1^2)$.

In Eq. (22), the contour of integration $C$ lies parallel to the imaginary axis in the complex $z$ plane, to the right of all singularities of the integrands. In order to find the time behavior of $\langle \vec{v}(t) \rangle$, we have to invert the matrix $N(z)$, whose determinant plays an essential role because the roots of its secular equation are the poles of the integration in Eq. (22).

**B. Normal-mode oscillations and quantum efficiency in the presence of pure dephasing**

We proceed to solve the stochastic Eqs. (18) and (19). For simplicity, we consider the case when the emitter and the cavity are in resonance, $\Delta = 0$. The detuning can always be put back without difficulty. First, by making the substitutions $E(t) = \tilde{E}(t)e^{-\gamma t}$ and $C(t) = \tilde{C}(t)e^{-\kappa t}$ for convenience, we obtain the following simpler equations

$$\dot{\tilde{E}}(t) = -i\gamma_0 e^{-\Gamma t}e^{i\varphi(t)}\tilde{C}(t)$$  \hspace{1cm} (23)

$$\dot{\tilde{C}}(t) = -i\gamma_0 e^{\Gamma t}e^{-i\varphi(t)}\tilde{E}(t).$$  \hspace{1cm} (24)

Then by defining variable $Y(t) = e^{-\Gamma t+i\varphi(t)}\tilde{C}(t)$ so that the differential equation for $\tilde{E}(t)$ does not explicitly depend on the random variable $\varphi(t)$, we obtain a matrix equation of the type (20) for a multiplicative stochastic process, with

$$\tilde{v}(t) = \begin{pmatrix} \tilde{E}(t) \\ Y(t) \end{pmatrix}, \quad M_0 = \begin{pmatrix} 0 & -i\gamma_0 \\ -i\gamma_0 & -\Gamma \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$  \hspace{1cm} (25)

and a statistically independent initial condition $\langle \tilde{v}(0) \rangle = \langle \tilde{E}(0), Y(0) \rangle$.

The inverse matrix to the matrix $N(z)$ is

$$N^{-1}(z) = \frac{1}{\det[N(z)]} \begin{pmatrix} z + \Gamma + \gamma_p & -i\gamma_0 \\ -i\gamma_0 & -i\gamma_0 \end{pmatrix}.$$  \hspace{1cm} (26)

Plugging Eq. (26) back into Eq. (22), using the Laplace-transform technique and choosing properly the contour of integration $C$, we obtain the quantum expectation value of the probability amplitude $\langle \tilde{E}(t) \rangle$,

$$\langle \tilde{E}(t) \rangle = \int_C \frac{dz}{2\pi i} e^{zt} \frac{\tilde{E}(0) - i\gamma_0 Y(0)}{(z-z_1)(z-z_2)}$$

$$e^{-(\Gamma+\gamma_p)t/2} \left\{ \cos(g_1t) + \frac{\Gamma + \gamma_p}{2g_1} \sin(g_1t) \right\} \tilde{E}(0) - \left[ \frac{i\gamma_0}{g_1} \sin(g_1t) \right] Y(0),$$  \hspace{1cm} (27)

where $g_1 \equiv \sqrt{\gamma_0^2 - (\Gamma + \gamma_p)^2/4}$. Similarly, if we define $X(t) = e^{\gamma t-i\varphi(t)}\tilde{E}(t)$, while keep-
\[
\langle \hat{C}(t) \rangle = e^{(\Gamma - \gamma_p)t/2} \left\{ \cos(g_2 t) - \frac{\Gamma - \gamma_p}{2g_2} \sin(g_2 t) \right\} \hat{C}(0) - \left[ \frac{ig_0}{g_2} \sin(g_2 t) \right] X(0), \tag{28}
\]

where \( g_2 \equiv \sqrt{g_0^2 - (\Gamma - \gamma_p)^2/4} \). Taking into account the definitions of \( \hat{E}(t) \) and \( \hat{C}(t) \), as well as the fact that \( X(0) = \hat{E}(0) = E(0) \) and \( Y(0) = \hat{C}(0) = C(0) \), we transform back to \( E(t) \) and \( C(t) \),

\[
\left( \langle E(t) \rangle \langle C(t) \rangle \right) = e^{-(\kappa + \gamma_p)t/2} \begin{pmatrix} \cos(g_1 t) + \frac{\Gamma + \gamma_p}{2g_1} \sin(g_1 t) & -\frac{ig_0}{g_1} \sin(g_1 t) \\ -\frac{2g_0}{g_2} \sin(g_2 t) & \cos(g_2 t) - \frac{\Gamma - \gamma_p}{2g_2} \sin(g_2 t) \end{pmatrix} \begin{pmatrix} E(0) \\ C(0) \end{pmatrix}. \tag{29}
\]

The generalized Rabi frequencies for \( \langle E(t) \rangle \) and \( \langle C(t) \rangle \) now are different from each other, as compared to Eq. (6), where they were the same for both \( \langle E(t) \rangle \) and \( \langle C(t) \rangle \). This implies the destroying of coherence between the two eigenstates of the system, due to the pure dephasing process. We will show this phase-destroying effect on the normal-mode oscillations explicitly later in this section.

We are more interested in finding the influence of the pure dephasing on the probabilities \( |C(t)|^2 \) and \( |E(t)|^2 \), or \( I(t) \equiv |\hat{C}(t)|^2 \) and \( J(t) \equiv |\hat{E}(t)|^2 \), because they give the normal-mode oscillations and are what one measures in experiment. In order to find the equations of motion for them, we have to introduce two other one-time functions \( H(t) \equiv \hat{E}(t)\hat{C}^*(t) \) and \( H^*(t) \equiv \hat{E}^*(t)\hat{C}(t) \). The equations of motion for these functions are

\[
\frac{d}{dt} \begin{pmatrix} H(t) \\ H^*(t) \end{pmatrix} = \begin{pmatrix} \Gamma & 0 \\ 0 & \Gamma \end{pmatrix} \begin{pmatrix} H(t) \\ H^*(t) \end{pmatrix} - \begin{pmatrix} ig_0 e^{-\Gamma t + i\varphi(t)} & ig_0 e^{\Gamma t - i\varphi(t)} \\ ig_0 e^{\Gamma t + i\varphi(t)} & -ig_0 e^{-\Gamma t - i\varphi(t)} \end{pmatrix} \begin{pmatrix} I(t) \\ J(t) \end{pmatrix}. \tag{30}
\]

We still solve these equations assuming the quantum emitter is prepared in an excited state \( E(0) = 1 \), \( C(0) = 0 \) at time \( t_0 = 0 \). We solve these one-time functions one by one as we did above for solving \( \langle E(t) \rangle \) and \( \langle C(t) \rangle \). For example, to find the solution to \( \langle I(t) \rangle \), defining \( U_I(t) = e^{\Gamma t - i\varphi(t)}H(t), U_{I*} = e^{\Gamma t + i\varphi(t)}H^*(t), Z_I(t) = e^{2\Gamma t}J(t) \) and keeping \( I(t) \) unchanged, we obtain a matrix equation as the standard vector form of Eq. (20), with

\[
\langle I(t) \rangle = \frac{g_0^2}{2g^2} e^{[\Gamma - \gamma_p (1 + \varepsilon)]/2} t \left[ e^{\gamma_p (1 + 3\varepsilon)t/2} - \frac{\gamma_p}{4g} \sin(2gt) - \cos(2gt) \right]. \tag{32}
\]
where \( \varepsilon \equiv (\Gamma/2g)^2 \), and \( g \equiv \sqrt{g_0^2 - (\Gamma/2)^2} \) is the generalized Rabi frequency, as defined before. Treating \( \gamma_p/g \) as a perturbation parameter, we kept the order to \( O(\gamma_p/g) \) in the coefficients and the order to \( O(\gamma_p\varepsilon/g) \) in the exponential arguments.

Similarly, after some tedious algebra, we find the time evolutions of \( \langle J(t) \rangle \) and \( \langle H(t) \rangle \) are (see Appendixes B2 and B3):

\[
\langle J(t) \rangle = \frac{g_0^2}{2g^2} e^{-[\Gamma+\gamma_p(1+\varepsilon)]t/2} \left\{ e^{\gamma_p(1+3\varepsilon)t/2} - \left[ \frac{\gamma_p}{4g} - \frac{g(\Gamma - \gamma_p/2)}{g_0^2} \right] \sin(2gt) - \left( 1 - \frac{2g^2}{g_0^2} \right) \cos(2gt) \right\}.
\]

\[
\langle H(t) \rangle = \frac{g_0^2}{2g} e^{-\gamma_p(3+\varepsilon)t/2} \left[ \frac{3\Gamma}{2g} \gamma_p(1-\varepsilon)t/2 - \frac{\Gamma - \gamma_p}{g} e^{-\gamma_p(1-\varepsilon)t/2} - \frac{\Gamma + 2\gamma_p}{g} \cos(2gt) + \sin(2gt) \right].
\]

Finally, the quantum expectation value of the complex conjugate of \( H(t) \) is just the complex conjugate of its quantum expectation value \( \langle H^*(t) \rangle = \langle H(t) \rangle^* \). Using the definitions of \( \bar{E}(t) \) and \( \bar{C}(t) \), we can easily find the solutions for \( \langle |E(t)|^2 \rangle = e^{-2\gamma t} \langle J(t) \rangle \) and \( \langle |C(t)|^2 \rangle = e^{-2c(t)\langle I(t) \rangle} \).

Therefore, the probability of finding the system in the excited atomic state, including the pure dephasing, is

\[
\langle P_e(t) \rangle = \langle |E(t)|^2 \rangle = \frac{g_0^2}{2g^2} e^{-[\Gamma+\gamma_p(1+\varepsilon)]t/2} \left\{ e^{\gamma_p(1+3\varepsilon)t/2} - \left[ \frac{\gamma_p}{4g} - \frac{g(\Gamma - \gamma_p/2)}{g_0^2} \right] \sin(2gt) - \left( 1 - \frac{2g^2}{g_0^2} \right) \cos(2gt) \right\}.
\]

And the probability of finding the system in the single cavity mode with pure dephasing process is

\[
\langle P_c(t) \rangle = \langle |C(t)|^2 \rangle = \frac{g_0^2}{2g^2} e^{-[\Gamma+\gamma_p(1+\varepsilon)]t/2} \left[ e^{\gamma_p(1+3\varepsilon)t/2} - \frac{\gamma_p}{4g} \sin(2gt) - \cos(2gt) \right].
\]

Shown in Figs 5(a) and 5(b) are three plots of each probability in the presence of pure dephasing,

The modulation depths of the red-dot and blue-triangle curves, with pure dephasing rates \( \gamma_p/2\pi = (1.0, 2.5) \text{ GHz} \), are reduced, as compared with the black-square curves where there is no pure dephasing. The normal-mode oscillation frequency seems unaffected because we solved for the probabilities only up to first order in \( \gamma_p/g \). In fact, it will change slightly from \( 2g \) to \( 2g(1 - \gamma_p^2/32g^2) \) if we approximate to second order in \( \gamma_p/g \). The normal-mode oscillations are smeared in the presence of the pure dephasing process.

Consequently, the emission probability of a single photon into the forward beam and the QE with the pure dephasing process, as defined before, are

\[
\langle P_o(t) \rangle = 2 \kappa \int_0^t dt' \langle |C(t')|^2 \rangle = \frac{g_0^2}{2g^2} \left\{ \frac{1 - e^{-[\Gamma+\gamma_p(1+\varepsilon)]t/2}}{K - \gamma_p^2} - \frac{K + \gamma_p(1+\varepsilon/2)}{[K + \gamma_p(1+\varepsilon/2)]^2 + (2g)^2} \right\} + \frac{\kappa g_0^2}{g^2} e^{-[\Gamma+\gamma_p(1+\varepsilon)]t/2} \left\{ [K + \gamma_p(1+\varepsilon/2)] \left[ \frac{\gamma_p}{4g} \sin(2gt) + \cos(2gt) \right] + 2g \left[ \frac{\gamma_p}{4g} \cos(2gt) - \sin(2gt) \right] \right\}
\]

\[
\eta_{\gamma_p} = \langle P_o(t \to \infty) \rangle = \frac{g_0^2}{g^2} \frac{\kappa}{K - \gamma_p^2} \left\{ 1 - \frac{[K - \gamma_p^2][K + \gamma_p(1+\varepsilon/2)]}{[K + \gamma_p(1+\varepsilon/2)]^2 + (2g)^2} \right\}.
\]
emission probabilities with and without pure dephasing, and Fig. 5(b) is the QE \( \eta_q \) as a function of the pure dephasing rate. The emission probability is also smeared for a pure dephasing rate \( \gamma_p/2\pi = 4 \text{ GHz} \) compared with no pure dephasing. The QE decreases only about 1% as the dephasing rate increases from 0 to 4 GHz.

C. Two-time correlation functions and the emission spectra in the presence of pure dephasing

In order to calculate the emission spectra, we need to find the two-time correlation functions because the emission spectra in the long-time limit are proportional to the Fourier transform of their convolutions. The two-

time correlation functions are defined as follows:

\[
Q(t, t') \equiv \tilde{E}(t)\tilde{E}^*(t'), \quad R(t, t') \equiv \tilde{C}(t)\tilde{E}^*(t') \\
F(t, t') \equiv \tilde{E}(t)\tilde{C}^*(t'), \quad G(t, t') \equiv \tilde{C}(t)\tilde{C}^*(t').
\]

Of these \( Q(t, t') \) and \( G(t, t') \) are required for calculating the emission spectra.

The quantum regression theorem [37, 38], which provides a framework to calculate two-time correlation functions, states that the equations of motion for two-time correlation functions \( Q(t, t') \) and \( R(t, t') \) and \( F(t, t') \) and \( G(t, t') \) with respect to variable \( t \) obey the same equations of motion as those for \( \tilde{E}(t) \) and \( \tilde{C}(t) \), respectively,

\[
\frac{\partial}{\partial t} Q(t, t') = -ig_0 e^{-\Gamma t} e^{i\omega(t)} R(t, t'), \\
\frac{\partial}{\partial t} R(t, t') = -ig_0 e^{\Gamma t} e^{-i\omega(t)} Q(t, t'), \\
\frac{\partial}{\partial t} F(t, t') = -ig_0 e^{-\Gamma t} e^{i\omega(t)} G(t, t'), \\
\frac{\partial}{\partial t} G(t, t') = -ig_0 e^{\Gamma t} e^{-i\omega(t)} F(t, t').
\]
but now with initial conditions
\[
Q(t = t', t') = |\tilde{E}(t')|^2 \equiv J(t'), \\
R(t = t', t') = \tilde{C}(t')\tilde{E}^*(t') \equiv H^*(t'), \\
F(t = t', t') = \tilde{E}(t')\tilde{C}^*(t') \equiv H(t'), \\
G(t = t', t') = |\tilde{C}(t')|^2 \equiv I(t'),
\]
which are already solved and given explicitly by Eqs. 32 to 34.

We now specialize to the case \( t \geq t' \) and define \( t = t' + \tau \). We are particularly interested in the expectation values of \( \langle Q(t' + \tau, t') \rangle \) and \( \langle G(t' + \tau, t') \rangle \) as pointed out before. The solutions for their expectation values, according to the quantum regression theorem, have the same forms as the solutions for the one-time averages of \( \langle \tilde{E}(t) \rangle \) and \( \langle \tilde{C}(t) \rangle \) in Eqs. 27 and 28, with the initial conditions given above:

\[
\langle Q(t' + \tau, t') \rangle = e^{-(\Gamma + \gamma_p)\tau/2} \left\{ \cos(g_1\tau) + \frac{\Gamma + \gamma_p}{2g_1} \sin(g_1\tau) \right\} \langle J(t') \rangle - \left[ \frac{i\gamma_0}{g_1} \sin(g_1\tau) \right] \langle H^*(t') \rangle, \\
\langle G(t' + \tau, t') \rangle = e^{(\Gamma + \gamma_p)\tau/2} \left\{ \cos(g_2\tau) - \frac{\Gamma - \gamma_p}{2g_2} \sin(g_2\tau) \right\} \langle I(t') \rangle - \left[ \frac{i\gamma_0}{g_2} \sin(g_2\tau) \right] \langle H(t') \rangle.
\]

By substituting \( \tilde{E}(t') = E(t')e^{\gamma_0t'} \), \( \tilde{C}(t') = C(t')e^{\gamma_0t'} \) and the initial conditions into Eqs. 17 and 18, we obtain the explicit solutions for \( \langle E(t' + \tau)E^*(t') \rangle \) and \( \langle C(t' + \tau)C^*(t') \rangle \). The side emission and forward emission spectra are then

\[
S_{s,e}(\Omega) = \frac{2\gamma}{\pi} \text{Re} \left\{ \int_0^\infty dt \int_0^\infty dr |E(t')|^2 \langle E(t' + \tau)E^*(t') \rangle \right\} = \frac{g_0^2}{g^2} \text{Re} \left\{ \frac{\gamma/\pi}{(K + \gamma_p/2 - i\Omega)^2 + g_1^2} \right\}
\]
\[
\left\{ \frac{-3\gamma}{2(K + \gamma_p + 4\gamma_p\varepsilon)} - \frac{2g^2 - [K + \gamma_p (3 + \varepsilon)/2](\Gamma/2 + \gamma_p)}{[K + \gamma_p (3 + \varepsilon)/2]^2 + (2g)^2} + \frac{\Gamma - \gamma_p}{K + 2\gamma_p - 4\gamma_p\varepsilon} \right\}
\]
\[
\frac{g_0^2}{g^2} \text{Re} \left\{ \frac{(\gamma/\pi)(\kappa + \gamma_p - i\Omega)}{[(K + \gamma_p)/2 - i\Omega]^2 + g_1^2} \right\} \left\{ \frac{1}{K - \gamma_p\varepsilon} - \frac{1}{K + \gamma_p (1 + \varepsilon)/2^2 + (2g)^2} \right\}, \quad (49)
\]

\[
S_{f,e}(\Omega) = \frac{2\gamma}{\pi} \text{Re} \left\{ \int_0^\infty dt \int_0^\infty dr |C(t')|^2 \langle C(t' + \tau)C^*(t') \rangle \right\} = \frac{g_0^2}{g^2} \text{Re} \left\{ \frac{\kappa/\pi}{(K + \gamma_p/2 - i\Omega)^2 + g_2^2} \right\}
\]
\[
\left\{ \frac{3\gamma}{2(K + \gamma_p + 4\gamma_p\varepsilon)} + \frac{2g^2 - [K + \gamma_p (3 + \varepsilon)/2](\Gamma/2 + \gamma_p)}{[K + \gamma_p (3 + \varepsilon)/2]^2 + (2g)^2} - \frac{\Gamma - \gamma_p}{K + 2\gamma_p - 4\gamma_p\varepsilon} \right\}
\]
\[
\frac{g_0^2}{g^2} \text{Re} \left\{ \frac{(\kappa/\pi)(\gamma + \gamma_p - i\Omega)}{[(K + \gamma_p)/2 - i\Omega]^2 + g_2^2} \right\} \left\{ \frac{1}{K - \gamma_p\varepsilon} - \frac{K + \gamma_p (1 + \varepsilon)/2}{[K + \gamma_p (1 + \varepsilon)/2]^2 + (2g)^2} \right\}. \quad (50)
\]

Shown in Figs. a and b are the side emission and forward emission spectra in the long-time limit, for different dephasing rates \( \gamma_p \), while other parameters are the same as those in Fig. 2. The effect of pure dephasing is twofold. The phase fluctuations decrease the peak intensities of the spectra, and broaden the linewidths of the two peaks and hence smear out the splittings, which correspond to damping rates of the Rabi oscillations in the time domain as shown in Fig. 5. This effect is further seen to increase with increasing values of pure dephasing rate \( \gamma_p \).

V. CONCLUSION

We derived analytical formulas for the side and forward (useful cavity output) emission spectra of single-photon sources in the cavity-QED strong-coupling regime. We
also studied the influence of the pure dephasing process on the emission spectra and the QE, in the case that the pure dephasing rate is significantly less than the coherent coupling rate, that is, up to first order in $\gamma_p/g$. These results should be useful in analyzing photoluminescence spectra from strongly coupled semiconductor-QD microcavities, where pure dephasing cannot always be assumed negligible because often temperature tuning of the QD has to be used to tune through cavity resonance [24, 25, 26]. One can use this method, for example, to model the time jitter of solid-state SPS, where the excited state of the QD or color center in diamond is often populated by spontaneous phonon emission, by averaging over nonradiative relaxation time. One may also calculate the two-photon interference visibility assuming having two independent but identical SPS and investigate how the pure dephasing processes affect the indistinguishability of the emitted single photons.

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**APPENDIX A: FORWARD EMISSION AND SIDE EMISSION SPECTRA**

For a stationary and ergodic process, the Wiener-Khintchine theorem [31] states that the spectrum is given by the Fourier transform of the two-time correlation function of the radiated field. One can easily generalize the definition of the Wiener-Khintchine spectrum to that of a nonstationary spectrum appropriate in this case. We define both the side emission, the spontaneous emission of the excited emitter into the free-space other than the cavity (side modes or leak modes), and the forward emission, the emission of single photons through the cavity mirror into a single wave-packet, outward-traveling wave. We recognize from Eq. (4) in the text that $O_k(t)$ is proportional to the Fourier transform of the probability amplitude $C(t')$,

$$O_k(t) = -iB_k^2 \int_0^t dt' e^{i(\omega_k - \omega_c)t'} C(t').$$  \hspace{1cm} (A1)

We define the spectrum as the absolute value squared of the Fourier transform of the probability amplitude in the long-time limit, which is proportional to the Fourier transform of the convolution of the probability amplitude, as will be shown later. For simplicity, we consider the case that the atom/QD-cavity is at resonance. Therefore, the forward emission spectrum is given by

$$S_{f.e.}(\omega - \omega_c) = \lim_{t \to \infty} D(\omega_c)|O_\omega(t)|^2,$$  \hspace{1cm} (A2)

where we have changed the probability amplitude from $O_k(t)$ to $O_\omega(t)$ by using the density of states for the one-dimensional photon reservoir $D(\omega_c) = L/2\pi \epsilon c$ [38]. Using the solution to the probability amplitude $C(t)$ and the expression of $O_k(t)$ in Eq. (A1), we can calculate the spectrum

$$S_{f.e.}(\omega - \omega_c) = \lim_{t \to \infty} D(\omega_c)|B(\omega_c)|^2 \int_0^t dt' e^{i(\omega - \omega_c)t'} C(t') \int_0^t dt'' e^{-i(\omega - \omega_c)t''} C^*(t'').$$  \hspace{1cm} (A3)

Then using the definition of the decay rate of the intra-cavity field $\kappa = \pi D(\omega_c)|B(\omega_c)|^2$ [38] and defining a new...
variable $\Omega \equiv \omega - \omega_c$, the emission frequency centered at the cavity resonance $\omega_c$, and $\tau \equiv t' - t''$, we obtain the

\[ S_{f.e.}(\Omega) = 2\kappa \frac{1}{\pi} \text{Re} \left\{ \int_0^\infty d\tau e^{i\Omega \tau} \left[ \int_0^\infty dt' C(t' + \tau) C^*(t') \right] \right\}. \quad (A4) \]

The normalized forward emission spectrum is

\[ s_{f.e.} = \left( 2\kappa \int_0^\infty dt |C(t)|^2 \right)^{-1} S_{f.e.}. \quad (A5) \]

\[ S_{s.e.}(\Omega) = 2\gamma \frac{1}{\pi} \text{Re} \left\{ \int_0^\infty d\tau e^{i\Omega \tau} \left[ \int_0^\infty dt' E(t' + \tau) E^*(t') \right] \right\} \quad (A6) \]

\[ s_{s.e.} = \left( 2\gamma \int_0^\infty dt |E(t)|^2 \right)^{-1} S_{s.e.}. \quad (A7) \]

\[ \boxed{ \text{APPENDIX B: THE APPROXIMATE SOLUTIONS FOR THE EXPECTATION VALUES OF } I(t) \text{, } J(t) \text{ AND } H(t) } \]

1. Approximate roots of the secular equation of the matrix $N_I(z)$ and the solution for $\langle I(t) \rangle$ in the limit $(4g_0^2 - \Gamma^2) \gg \Gamma^2, \gamma_p^2$

From Eq. 33 in the text, define matrix $M = M_0 - \gamma_p M_1^2$, given explicitly by

\[ M = \begin{pmatrix} \Gamma - \gamma_p & 0 & -i\gamma_0 & i\gamma_0 \\ 0 & \Gamma - \gamma_p & i\gamma_0 & -i\gamma_0 \\ -i\gamma_0 & i\gamma_0 & 0 & 0 \\ i\gamma_0 & -i\gamma_0 & 0 & 2\Gamma \end{pmatrix}. \quad (B1) \]

Then the matrix $N_I(z) \equiv zI - M$ and its determinant are

\[ N_I(z) = \begin{pmatrix} z + \gamma_p - \Gamma & 0 & -i\gamma_0 & i\gamma_0 \\ 0 & z + \gamma_p - \Gamma & i\gamma_0 & -i\gamma_0 \\ i\gamma_0 & -i\gamma_0 & 0 & 0 \\ -i\gamma_0 & i\gamma_0 & 0 & z - 2\Gamma \end{pmatrix}. \quad (B2) \]

\[ \text{det} [N_I(z)] = (z + \gamma_p - \Gamma) [z (z - 2\Gamma) (z + \gamma_p - \Gamma) + 4g_0^2 (z - \Gamma)]. \quad (B3) \]

The secular equation is given by the vanishing of the determinant Eq. 33, which reduces to a cubic equation, for $z_1 = \Gamma - \gamma_p$,

\[ (z - \Gamma) [(z - \Gamma)^2 + \gamma_p (z - \Gamma) + 4g_0^2 - \Gamma^2] = \gamma_p \Gamma^2, \quad (B4) \]

forward emission spectrum

\[ \text{solution for } \langle I(t) \rangle \text{ is } \pm \sqrt{\frac{2}{\pi \kappa}}. \]

Similarly the side emission spectrum and the normalized side emission spectrum, in the long time limit, are given by

\[ s_{s.e.} = \left( 2\gamma \int_0^\infty dt |E(t)|^2 \right)^{-1} S_{s.e.}. \]

which is the standard Torrey equation \[49, 50\]. This cubic equation can be solved exactly \[49\], although only in an implicit form. As Torrey has pointed out, in the special case of interest, this equation has a relatively simple explicit solution. We solve it in the strong-coupling regime, $(4g_0^2 - \Gamma^2) \gg \Gamma^2, \gamma_p^2$, in which case there are two kinds of roots. The first of these follows the assumption that $(z - \Gamma)^2$ is small compared with $(4g_0^2 - \Gamma^2)$, allowing one to rearrange the cubic equation (B4) and solve by iteration

\[ z - \Gamma = \frac{\gamma_p \Gamma^2}{4g_0^2 - \Gamma^2} \left[ 1 + \frac{(z - \Gamma)(z - \Gamma + \gamma_p)}{4g_0^2 - \Gamma^2} \right]^{-1}, \]

\[ z_2 \approx \Gamma + \gamma_p \varepsilon + O \left( \frac{\gamma_p}{g} \right)^3, \quad (B5) \]

where $\varepsilon \equiv (\Gamma/2g)^2$, and note that $g \equiv \sqrt{g_0^2 - (\Gamma/2)^2}$. The second kind of root occurs when $(z - \Gamma)^2$ is as large as $(4g_0^2 - \Gamma^2)$, but with opposite sign. The cubic equation (B4) can be written as

\[ (z - \Gamma)^2 + 4g_0^2 - \Gamma^2 = -\gamma_p (z - \Gamma) \left[ 1 - \frac{\Gamma^2}{(z - \Gamma)^2} \right]. \quad (B6) \]

To first order in $\gamma_p$, the factor $(z - \Gamma)^2$ on the right-hand side, Eq. (B6) can be replaced by $-4g_0^2 - \Gamma^2$. This gives a quadratic equation for $(z - \Gamma)$, $(z - \Gamma)^2 + \gamma_p (1 + \varepsilon)(z - \Gamma) + 4g^2 = 0$, whose solutions are the third and fourth
roots
\[ z_{3,4} \approx \Gamma - \frac{\gamma_p}{2} (1 + \varepsilon) \pm i2g\sqrt{1 - (\gamma_p/4g)^2} + O \left( \frac{\gamma_p}{g} \right)^2. \]

The inverse matrix to the matrix \( N_I(z) \) in Eq. \( \text{(B2)} \) is

\[
N_I^{-1}(z) = \frac{1}{\det [N_I(z)]} \begin{pmatrix}
1 & 2 & 3 & 4 \\
3 & n_{31} & n_{32} & n_{33} \\
2 & n_{33} & n_{34} \\
1 & n_{34} & n_{31} & n_{32}
\end{pmatrix}
\]

where we have used the initial condition that \( \langle \bar{v}_I(0) \rangle ^T = (0, 0, 0, 1) \) Treating \( \gamma_p/g \) as a perturbation parameter, we set the order to \( O(\gamma_p/g) \) in the coefficients and the order to \( O(\gamma_p \varepsilon/g) \) in the exponential arguments.

2. Approximate roots of the secular equation of the matrix \( N_J(z) \) and the solution for \( \langle I(t) \rangle \) in the limit \( (4g_0^2 - \Gamma^2) \gg \Gamma^2, \gamma_p^2 \).

As it is clear from the definition of \( \bar{v}_J(t) \), we can only obtain the solution for \( \langle I(t) \rangle \) in the above calculation. In order to obtain the solution for \( \langle J(t) \rangle \), we have to derive another equation of the type Eq. \( \text{(20)} \) with the fold definitions of the vector and matrices:

\[
\bar{v}_J(t) = \begin{pmatrix}
U_J(t) \\
U'_J(t) \\
W_J(t) \\
J(t)
\end{pmatrix}, \quad M_1 = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad M_0 = \begin{pmatrix}
-\Gamma & 0 & -i g_0 & i g_0 \\
0 & -\Gamma & i g_0 & -i g_0 \\
i g_0 & i g_0 & -2\Gamma & 0 \\
i g_0 & -i g_0 & 0 & 0
\end{pmatrix}
\]

and with the initial condition \( \langle \bar{v}_J(0) \rangle ^T = (0, 0, 0, 1) \), where \( U_J(t) = e^{-\Gamma t - i \phi(t)} H(t), U'_J(t) = e^{-\Gamma t + i \phi(t)} H^*(t), W_J(t) = e^{-2\Gamma t} H(t) \).

The calculation of \( N_J(z) \) is almost the same as the calculation in Appendix \( \text{B3} \). The matrix \( M \equiv M_0 - \gamma_p M_1^2 \) is

\[
M = \begin{pmatrix}
-\Gamma - \gamma_p & 0 & -i g_0 & i g_0 \\
0 & -\Gamma - \gamma_p & i g_0 & -i g_0 \\
i g_0 & i g_0 & -2\Gamma & 0 \\
i g_0 & -i g_0 & 0 & 0
\end{pmatrix}
\]

with

\[
\begin{align*}
n_{31} &= -i g_0 (z - 2\Gamma) (z + \gamma_p - \Gamma) \\
n_{32} &= i g_0 (z - 2\Gamma) (z + \gamma_p - \Gamma) \\
n_{33} &= (z + \gamma_p - \Gamma) (z - 2\Gamma) (z + \gamma_p - \Gamma + 2g_0^2) \\
n_{34} &= 2g_0^2 (z + \gamma_p - \Gamma)
\end{align*}
\]

where we only calculate the elements of the third row of \( N_J^{-1}(z) \) because they are required to calculate \( \langle I(t) \rangle \), which is then

\[
\langle I(t) \rangle = \int_C \frac{dz}{2\pi i} e^{zt} (z - z_1)(z - z_2)(z - z_3)(z - z_4) = \int_C \frac{dz}{2\pi i} e^{zt} \left( z_4^2 \sin(2gt) - \cos(2gt) \right),
\]

Then the matrix \( N_J(z) \equiv z I - M \) and its determinant are, respectively,

\[
N_J(z) = \begin{pmatrix}
z + \gamma_p + \Gamma & 0 & i g_0 & -i g_0 \\
0 & z + \gamma_p + \Gamma & -i g_0 & i g_0 \\
i g_0 & -i g_0 & z + 2\Gamma & 0 \\
i g_0 & i g_0 & 0 & z
\end{pmatrix}
\]

and

\[
\det [N_J(z)] = (z + \gamma_p + \Gamma) \left[ z (z + 2\Gamma) (z + \gamma_p + \Gamma + 4g_0^2 (z + \Gamma)) \right],
\]

which is the same as Eq. \( \text{(B3)} \) provided that we change \( \Gamma \) to \( -\Gamma \). So the roots of the secular equation of the matrix \( N_J(z) \) are

\[
\begin{align*}
z_1 &\approx -\Gamma - \gamma_p, \quad z_2 \approx -\Gamma + \gamma_p \varepsilon + O \left( \frac{\gamma_p}{g} \right)^3, \\
z_{3,4} &\approx -\Gamma - \frac{\gamma_p}{2} (1 + \varepsilon) \pm i2g + O \left( \frac{\gamma_p}{g} \right)^2.
\end{align*}
\]

The inverse matrix to the matrix \( N_J(z) \) is therefore

\[
N_J^{-1}(z) = \frac{1}{\det [N_J(z)]} \begin{pmatrix}
1 & 2 & 3 & 4 \\
3 & n_{41} & n_{42} & n_{43} \\
2 & n_{43} & n_{44} \\
1 & n_{44} & n_{41} & n_{42}
\end{pmatrix}
\]

with

\[
\begin{align*}
n_{41} &= i g_0 (z + 2\Gamma) (z + \gamma_p + \Gamma) \\
n_{42} &= -i g_0 (z + 2\Gamma) (z + \gamma_p + \Gamma) \\
n_{43} &= 2g_0^2 (z + \gamma_p + \Gamma) \\
n_{44} &= (z + \gamma_p + \Gamma) [(z + 2\Gamma) (z + \gamma_p + \Gamma) + 2g_0^2]
\end{align*}
\]
where we only calculate the elements of the fourth row of $N^{-1}_J(z)$ because they are required to calculate $\langle J(t) \rangle$, which is then

\[
\langle J(t) \rangle = \int \frac{dz}{C 2\pi i} e^{zt} \frac{n_{44}}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)} = \int \frac{dz}{C 2\pi i} e^{zt} \frac{(z + 2\Gamma)(z + \Gamma + \gamma_p) + 2g_0^2}{(z - z_2)(z - z_3)(z - z_4)}
\]

where we have used the initial condition that $\langle \vec{v}_j(0) \rangle^T = (0, 0, 0, 1)$ and kept the order to $O(\gamma_p/g)$ and $O(\Gamma/g)$ in the coefficients and the order to $O(\gamma_p^2/g)$ in the exponential arguments.

3. Approximate roots of the secular equation of the matrix $N_H(z)$ and the solution for $\langle H(t) \rangle$ in the limit $(4g_0^2 - \Gamma^2) \gg \Gamma^2, \gamma_p^2$

In order to obtain the solution for $\langle H(t) \rangle$, we derive another equation of the type Eq. (20) with the following definitions of the vector and matrices:

\[
\vec{v}_H(t) = \begin{pmatrix} H(t) \\ U_H(t) \\ W_H(t) \\ Z_H(t) \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

\[
M_0 = \begin{pmatrix} 0 & 0 & -i\gamma_0 & i\gamma_0 \\ 0 & 0 & i\gamma_0 & -i\gamma_0 \\ -i\gamma_0 & i\gamma_0 & -\Gamma & -\gamma_p \\ i\gamma_0 & -i\gamma_0 & 0 & \Gamma \end{pmatrix}
\]

and with the initial condition $\langle \vec{v}_H(0) \rangle^T = (0, 0, 0, 1)$, where $U_H(t) = e^{i2\vec{v}(t)H^*(t)}$, $W_H(t) = e^{-i\Gamma t + i\vec{v}(t)I(t)}$, $Z_H(t) = e^{i\Gamma t + i\vec{v}(t)J(t)}$.

The matrix $M$ for the vector $\vec{v}_H(t)$ is

\[
M = M_0 - \gamma_p M_1^2 = \begin{pmatrix} 0 & 0 & -i\gamma_0 & i\gamma_0 \\ 0 & -4\gamma_p & i\gamma_0 & -i\gamma_0 \\ -i\gamma_0 & i\gamma_0 & -\Gamma & -\gamma_p \\ i\gamma_0 & -i\gamma_0 & 0 & \Gamma - \gamma_p \end{pmatrix}
\]

Therefore

\[
N_H(z) = \begin{pmatrix} z & 0 & ig_0 & -ig_0 \\ 0 & z + 4\gamma_p & -ig_0 & ig_0 \\ -ig_0 & ig_0 & z + (\Gamma + \gamma_p) & 0 \\ 0 & 0 & z - (\Gamma - \gamma_p) & 0 \end{pmatrix}
\]

and the determinant of the matrix $N_H(z)$ is

\[
det[N_H(z)] = (z + 4\gamma_p)^2 - \Gamma^2 + 4g_0^2 (z + \gamma_p)(z + 2\gamma_p).
\]

The secular equation is given by setting $\det[N_H(z)] = 0$, which gives

\[
(z + \gamma_p)(z + 2\gamma_p)^2 + 4g_0^2 (z + \gamma_p)(z + 2\gamma_p) - \Gamma^2 (z + 2\gamma_p)^2 - 4\gamma_p^2(z + \gamma_p)^2 = -4\gamma_p^2\Gamma^2.
\]

In the most general case, no simple factorizations occur, and a quartic equation must be solved. Again the roots are implicit in the general case, but explicit in the strong-coupling regime. Similarly, there are two kinds of roots in the strong-coupling regime. The first of these follows from the assumption that both $(z + \gamma_p)^2$ and $(z + 2\gamma_p)^2$ are small compared with $(4g_0^2 - \Gamma^2)$, in which case it is natural to rearrange Eq. (B15) into the form

\[
(z + \gamma_p)(z + 2\gamma_p) \approx \Gamma^2, \gamma_p^2
\]

where we used the assumptions $(4g_0^2 - \Gamma^2) \gg \Gamma^2, \gamma_p^2$. Then

\[
\frac{z + \gamma_p}{z + 2\gamma_p} \approx 1 \quad \text{and} \quad \frac{z + 2\gamma_p}{z + \gamma_p} \approx 1.
\]

\[
(z + \gamma_p)(z + 2\gamma_p) \approx \frac{-4\gamma_p^2\Gamma^2}{4g_0^2 - \Gamma^2 - 4\gamma_p^2} \left[1 + \frac{(z + \gamma_p)(z + 2\gamma_p)}{4g_0^2 - \Gamma^2 - 4\gamma_p^2}\right]^{-1},
\]

(B17)
which is solved by iteration. The roots are

\[
\begin{align*}
    z_1 & \approx -\gamma_p (1 + 4\varepsilon) + O \left( \frac{2\gamma_p}{g} \right)^3 \\
    z_2 & \approx -2\gamma_p (1 - 2\varepsilon) + O \left( \frac{2\gamma_p}{g} \right)^3.
\end{align*}
\]

(B18)

The second kind of root occurs if \((z + \gamma_p) (z + 2\gamma_p)\) is as large as \((4\gamma_p^2 - \Gamma^2)\), but has the opposite sign. Then the alternative rearrangement of Eq. (B15) is

\[
(z + \gamma_p) (z + 2\gamma_p) + (4\gamma_p^2 - \Gamma^2 - 4\gamma_p^2)
\approx \gamma_p \Gamma^2 \left( \frac{z - 2\gamma_p}{z + \gamma_p} \right) (z + 2\gamma_p).
\]

(B19)

To first order in \(\gamma_p\), the factor \((z + \gamma_p) (z + 2\gamma_p)\) on the right hand side of Eq. (B20) can be replaced by \(- (4\gamma_p^2 - \Gamma^2)\). This gives a simple quadratic equation for \(z\), \(z^2 + \gamma_p (3 + \varepsilon) z + 4\gamma_p^2 - 4\gamma_p^2 + 2\gamma_p^2 (1 - \varepsilon) = 0\), whose solutions are the third and fourth roots

\[
\begin{align*}
    z_{3,4} & \approx -\frac{\gamma_p}{2} (3 + \varepsilon) \pm i2g \sqrt{1 - (\gamma_p/g)^2} + O \left( \frac{\gamma_p}{g} \right)^2 \\
    & \approx -\frac{\gamma_p}{2} (3 + \varepsilon) \pm i2g + O \left( \frac{\gamma_p}{g} \right)^2.
\end{align*}
\]

(B20)

The inverse matrix to \(N_H(z)\) is

\[
N_H^{-1}(z) = \frac{1}{\det[N(z)]} \begin{pmatrix} n_{11} & n_{12} & n_{13} & n_{14} \\
\vdots & \vdots & \vdots & \vdots \\
\end{pmatrix},
\]

with

\[
\begin{align*}
    n_{11} & = (z + 4\gamma_p) \left( (z + \gamma_p)^2 - \Gamma^2 \right) + 2g^2 (z + \gamma_p), \\
    n_{12} & = 0, \\
    n_{13} & = -ig_0 (z + 4\gamma_p) (z + \gamma_p - \Gamma), \\
    n_{14} & = ig_0 (z + 4\gamma_p) (z + \gamma_p + \Gamma).
\end{align*}
\]

Therefore \(\langle H(t) \rangle\) is given by

\[
\langle H(t) \rangle = \int_C \frac{dz}{2\pi i} e^{2\gamma_p \gamma_p t/2 (z - z_1) (z - z_2) (z - z_3) (z - z_4)}
\approx \frac{i g_0}{2g} e^{-\gamma_p (3 + \varepsilon) t/2} \left[ \frac{3\Gamma}{2g} e^{-\gamma_p (1 - \varepsilon) t/2} - \frac{\Gamma - \gamma_p}{g} e^{-\gamma_p (3 + \varepsilon) t/2} - \frac{\Gamma + 2\gamma_p}{2g} \cos(2gt) + \sin(2gt) \right],
\]

(B21)

where we have used the initial condition that \(\langle \hat{v}(0) \rangle^T = (0, 0, 0, 1)\) and kept the order to \(O(\gamma_p/g)\) and \(O(\Gamma/g)\) in the coefficients and the order to \(O(\gamma_p \varepsilon/g)\) in the exponential arguments.

[17] J. Vuchovic, D. Fattal, C. Santori, G. S. Solomon,