Quantum field theory with an interaction on the boundary

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Abstract

We consider quantum theory of fields \( \phi \) defined on a \( D \) dimensional manifold (bulk) with an interaction \( V(\phi) \) concentrated on a \( d < D \) dimensional surface (brane). Such a quantum field theory can be less singular than the one in \( d \) dimensions with an interaction \( V(\phi) \). It is shown that scaling properties of fields on the brane are different from the ones in the bulk. We discuss as an example fields on de Sitter space.

1 Introduction

Models with an interaction concentrated on a \( d < D \) dimensional submanifold (brane) of a \( D \) dimensional manifold (bulk) are interesting for high energy physics as well as for statistical physics. In the first case we consider the visible universe as a submanifold (a brane \([1][2]\) of a higher dimensional space. Field theoretic models with an interaction on the boundary come also from string theory \([3]\). In ref.\([3]\) the eleven dimensional gravity is interacting with ten dimensional gauge fields living on the boundary. In statistical physics we may consider materials with a boundary and an interaction of some constituents placed on the boundary \([4][5][6][7]\). It is an experimental fact \([8]\) that correlation functions of field variables depending on the boundary points have critical exponents different from the bulk correlation functions.

In this paper we discuss field theoretic models with an interaction on the brane. We concentrate on the scalar field but some methods and results can be generalized to gravitational and gauge field interactions. We begin with the free propagator. We admit any boundary condition preserving the symmetries
of the free Lagrangian. We show that a differential operator which is singular close to the brane has the Green function which is more regular on the brane than the one for operators with constant coefficients. Subsequently, we discuss models with an interaction concentrated on the brane. The functional measure is defined \[9\] by its covariance (the Green function) and its mean. The mean breaks symmetries of the classical action. We average over mean values in order to preserve the symmetries. As a consequence of the more regular behaviour of the Green function the model with an interaction concentrated on the brane has milder ultraviolet divergencies. We give examples of nonrenormalizable theories in the bulk which become superrenormalizable when restricted to the boundary. We work mainly with the imaginary time version of quantum field theory. In the last section we discuss the scattering theory. The free particle is treated as a packet of waves on a curved manifold. We calculate the scattering matrix of such particles resulting from the \( V(\phi) \) interaction concentrated either on the boundary or at the time \( z = 0 \) (a kick at a fixed moment).

2 Green functions on a boundary

We consider a \( D = d + m \) dimensional Riemannian manifold of the warped form \( \mathcal{M}_g = \mathcal{M}_m \times g \mathcal{R}^d \) \[10\] with the boundary \( \mathcal{R}^d \) whose metric close to the boundary takes the form

\[ ds^2 = G_{AB}(X)dX^A dX^B = g_{\mu\nu}(y)dx^\mu dx^\nu + g_{jk}(y)dy^j dy^k \]  (1)

where \( X = (y, x) \) are local coordinates on \( \mathcal{M}_g \), \( g_{jk} \) is the Riemannian metric induced on \( \mathcal{M}_m \) and \( g_{\mu\nu} : \mathcal{M}_m \to \mathcal{R}^d \) is a positive definite \( d \times d \) matrix function defined on \( \mathcal{M}_m \). The action for the free field \( \phi \) reads

\[ W_0 = \int dX \sqrt{G} G^{AB} \partial_A \phi \partial_B \phi \]  (2)

The free (Euclidean) quantum field can be defined as the one whose propagator is determined by the Green function

\[ -\mathcal{A}G = \partial_A G^{AB} \sqrt{G} \partial_B G = \delta \]  (3)

where \( G = \det G_{AB} \). In the metric (1) eq.(3) can be expressed as

\[ \left( g^{\mu\nu}(y) \sqrt{G}(y) \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} + \frac{\partial}{\partial y^j} g_{jk}(y) \sqrt{G}(y) \frac{\partial}{\partial y^k} \right) G = \delta \]  (4)

The solution of eq.(3) is not unique. If \( G' \) is another solution of eq.(3) then \( G' = G + R \) where \( R \) is a solution of the equation

\[ \mathcal{A}R = 0 \]  (5)
We can determine \(G\) unambiguously imposing some additional requirements, e.g., requiring that \(G = 0\) on the boundary or that \(G\) be scale invariant on the boundary.

We assume that the metric is scale invariant

\[
g_{\mu\nu}(\lambda y) = \lambda^{2\alpha} g_{\mu\nu}(y) \quad (6)
\]

and

\[
g_{jk}(\lambda y) = \lambda^{2\beta} g_{jk}(y) \quad (7)
\]

It follows that

\[
G(\lambda y) = \lambda^{2\alpha+2m\beta} G(y)
\]

Let us consider \(G_{\lambda,\rho}(x, y; x', y') \equiv G(\rho x, \lambda y; \rho x', \lambda y').\) It satisfies the equation

\[
\lambda^m \rho^d \left( \lambda^{-2\alpha+m\beta+da} g^{\mu\nu}(y) \sqrt{G(y)} \frac{\partial}{\partial \rho x^\mu} \frac{\partial}{\partial \rho x^\nu} + \lambda^{m\beta+da-2\beta} \frac{\partial}{\partial \lambda y^j} g^{jk}(y) \sqrt{G(y)} \frac{\partial}{\partial \lambda y^k} \right) G = \delta(x - x') \delta(y - y') \quad (8)
\]

Eq.(8) is identical with eq.(4) but expressed in rescaled coordinates (the scale invariance (6)-(7) has been applied). Let us choose \(\rho\) such that the scale factors in the two terms in eq.(8) are equal

\[
\lambda^{-2\alpha+m\beta+da} \rho^{-2} = \lambda^{m\beta+da-2\beta-2} \quad (9)
\]

Hence,

\[
\rho = \lambda^{1-\alpha+\beta} \quad (10)
\]

Then, it follows that

\[
\left( g^{\mu\nu}(y) \sqrt{G(y)} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} + \frac{\partial}{\partial y^j} g^{jk}(y) \sqrt{G(y)} \frac{\partial}{\partial y^k} \right) \hat{G} = \delta \quad (11)
\]

where

\[
\hat{G}(y, x; y', x') = \lambda^{(D-2)(1+\beta)} \hat{G}(\lambda y, \lambda^{1-\alpha+\beta} x; \lambda y', \lambda^{1-\alpha+\beta} x') \quad (12)
\]

\(\hat{G}\) and \(G\) satisfy the same equation (4). Hence, for the scale invariant solution \(\hat{G} = G\) (there are many solutions of eq.(4) but the scale invariant solution is unique).

We are interested in the boundary Green function \(G_E(0, x; 0, x') \equiv G_E(|x - x'|).\) \(G_E\) depends solely on the Euclidean distance \(|x - x'|\) as a consequence of the translational invariance of eq.(4) and its rotational invariance when \(y = y' = 0.\)

Choosing in eq.(12)

\[
\lambda = |x - x'|^{-\frac{1}{1-\alpha+\beta}} \quad (13)
\]

we obtain

\[
G_E(|x - x'|) = K |x - x'|^{-(D-2)\sigma} \quad (14)
\]

where

\[
\sigma = \frac{1+\beta}{1-\alpha+\beta} \quad (15)
\]
and $K(\alpha, \beta, D) = \mathcal{G}(0, e; 0, 0)$ (here $e \in \mathbb{R}^d$ is an arbitrary vector such that $|e| = 1$). Some authors \cite{?} in de Sitter case require that $\mathcal{G} = 0$ at $y = 0$. In such a case $K = 0$ in eq.(14) and our result is trivial. However, in other models of QFT on de Sitter space \cite{12}\cite{13}\cite{14}\cite{15}\cite{16} $\mathcal{G}$ does not vanish at $y = 0$.

It will be useful to rewrite eq.(4) in the momentum space. Let

$$G(y, x; y', x') = \frac{1}{(2\pi)^{d/2}} \int dp \exp(ip(x - x')) \tilde{G}(p, y, y')$$

Then, $\tilde{G}$ satisfies the equation

$$\left( - p_\mu p_\nu g^{\mu\nu}(y) \sqrt{G(y)} + \frac{\partial}{\partial y^\mu} g^{jk}(y) \sqrt{G(y)} \frac{\partial}{\partial y^k} \right) \tilde{G} = \delta(y - y')$$

Let $\tilde{G}_{\lambda, \gamma}(p; y, y') = \tilde{G}(\gamma p; \lambda y, \lambda y')$. Then, $\tilde{G}_{\lambda, \gamma}$ is a solution of the equation

$$\lambda^m \left( \lambda^{-2\alpha + m\beta + d\alpha} \gamma^2 p_\mu p_\nu g^{\mu\nu}(y) \right. + \lambda^{-2\beta + m\beta + d\alpha} \frac{\partial}{\partial y^\mu} g^{jk}(y) \sqrt{G(y)} \frac{\partial}{\partial y^k} \bigg) \tilde{G}_{\lambda, \gamma} = \delta(y - y')$$

We choose

$$\gamma = \lambda^{1 - 2\beta + \alpha}$$

Then, we obtain (similarly as in eq.(12)) from the uniqueness of the solution of eq.(17)

$$\tilde{G}(p; y, y') = \lambda^m \left( \lambda^{-2\alpha + m\beta + d\alpha} \gamma^2 p_\mu p_\nu g^{\mu\nu}(y) \right. + \lambda^{-2\beta + m\beta + d\alpha} \frac{\partial}{\partial y^\mu} g^{jk}(y) \sqrt{G(y)} \frac{\partial}{\partial y^k} \bigg) \tilde{G}_{\lambda, \gamma} = \delta(y - y')$$

Setting

$$\lambda = |p|^{\frac{1}{1 - \alpha}}$$

and $y = y' = 0$ in eq.(19) leads to the result

$$\tilde{G}(p; 0, 0) = \tilde{K}|p|^{-2\omega}$$

where

$$-2\omega = \frac{(m - 2)(1 + \beta) + d\alpha}{1 + \beta - \alpha}$$

and $\tilde{K}(\alpha, \beta, D) = \tilde{G}(\tilde{e}; 0, 0)$ ( $\tilde{e}$ is an arbitrary unit vector in $\mathbb{R}^d$).

As an example of an application of eqs.(20)-(21) we could consider de Sitter space with the metric (it of the type (1);this is Euclidean AdS$_D$ \cite{18}, the notion of the boundary in AdS and its relation to Euclidean AdS is discussed in \cite{17} \cite{18})

$$ds^2 = dt^2 + \exp(2Ht)(dx_1^2 + ... + dx_{D-1}^2) = y^{-2}(dy^2 + dx_1^2 + ... + dx_{D-1}^2)$$

where $y = H^{-1} \exp(-Ht)$.
Then, from eqs.(20)-(21)
\[ \tilde{G}(p, 0, 0) = \tilde{K}|p|^{-D+1} \] (23)
The Fourier transform of \( \tilde{G} \) in eq.(23) is infrared divergent. It can be defined as a distribution on a set of functions vanishing at \( p = 0 \). Then,
\[ \mathcal{G}_E(|x - x'|) = -K \ln|x - x'| \] (24)
Eq.(24) gives the form of the Green function for \( \sigma = 0 \) in eq.(14).

Generalizing eq.(4) we could consider a system of equations for a tensorial Green function \( \hat{G}_{\Gamma\Omega} \)
\[ \left( a_{\Gamma\Sigma}^\mu(y) \frac{\partial}{\partial x^\mu} + b_{\Gamma\Sigma}^j(y) \frac{\partial}{\partial y^j} + A_{\Gamma\Sigma}^\mu \frac{\partial}{\partial x^\mu} + B_{\Gamma\Sigma}^j \frac{\partial}{\partial y^j} + V_{\Gamma\Sigma}(y) \right) \hat{G}_{\Gamma\Omega} = \delta_{\Gamma\Omega} \] (25)
here a sum over repeated indices \( \Sigma \) is assumed, the \( \delta_{\Gamma\Omega} \) function is the product of the usual \( \delta \) function of eq.(8) and a scale invariant tensor with indices \( \Gamma\Omega \).

We assume that the coefficients have the following scaling properties
\[ a^{\mu\nu}(\lambda y) = \lambda^{(d-2)\alpha+m\beta} a^{\mu\nu}(y) \] (26)
\[ b^{jk}(\lambda y) = \lambda^{d\alpha+(m-2)\beta} b^{jk}(y) \] (27)
\[ A^\mu(\lambda y) = \lambda^{(m-2)\beta+d\alpha-1} A(\lambda y) \] (28)
\[ B^j(\lambda y) = \lambda^{(m-1)\beta+(d-1)\alpha-1} B^j(y) \] (29)
and
\[ V(y) = \lambda^{(m-2)\beta+d\alpha-2} V(y) \]
Then, repeating our scaling arguments of this section we could derive the results (14)-(15) for \( \mathcal{G}_{\Gamma\Omega} \) and (20)-(21) for its Fourier transform. Such results may be applicable to propagators describing an interaction of gravity on the bulk and on the brane with gauge fields on the brane as in [3][1].

3 DeWitt expansion

Let us discuss now scaling properties of the Green functions from the point of view of DeWitt expansion [19] and the Hadamard representation of the Green functions [20]. According to the DeWitt suggestion we can solve the equation (for a non-negative operator \( \mathcal{A} \))
\[ -\mathcal{A}\mathcal{G} = \delta \] (30)
by means of the heat kernel which is the fundamental solution of the heat equation
\[ \frac{d}{d\tau}\mathcal{K}_\tau = \mathcal{A}\mathcal{K}_\tau \] (31)
Namely,
\[ G = \int_0^\infty d\tau K_\tau \] (32)

Then, DeWitt makes the assumption
\[ K_\tau(X, X') = (2\pi \tau)^{-\frac{D}{2}} \exp\left(-\frac{\sigma(X, X')}{2\tau}\right)\Lambda(\tau, X, X') \] (33)

where \( \sigma(X, X') \) is the square of the geodesic distance between \( X \) and \( X' \). \( \Lambda \) has a Taylor expansion in powers of \( \tau \) for a manifold without boundary (powers of \( \sqrt{\tau} \) may appear if the manifold has a boundary). Performing the integral over \( \tau \) in eq.(32) we obtain the short distance expansion of \( G(X, X') \).

We do not know the formula for \( \sigma \) in general. We can explain the expansion (33) in the hyperbolic case (22). Then,
\[ \text{ch}(\sigma(X, X')) = 1 + (2yy')^{-1}(|(x-x')^2 + (y-y')^2) \] (34)

In the hyperbolic space the heat kernel is known exactly [21]. The DeWitt expansion holds true (\( \Lambda \) has an expansion in powers of \( \sigma \); no \( \tau \) dependence). We can see that
\[ \sigma(X, X')^{-1} \to 0 \] (35)

when \( y \to 0 \) (\( t \to \infty \) in eq.(22))and \( x \neq x' \). It follows from eq.(32) that if \( D = 2k + 1 \) is odd then
\[ G_E(x - x') = 0 \] (36)

and if \( D = 2k \) is even then
\[ G_E(x - x') = -K \ln |x - x'| \] (37)

The conclusion (37) coincides with eq.(24).

### 4 Interacting fields on the boundary

In the model (2) we consider an interaction \( V \) concentrated on the boundary \( y = 0 \) (we do not treat here a coordinate independent geometric description of the boundary but restrict ourselves to the model (1))
\[ W = W_0 + W_I = \int dX \sqrt{G} G^{AB} \partial_A \phi \partial_B \phi + \int dX \sqrt{G} \delta(y)V(\phi) \] (38)

We define the functional measure
\[ d\mu(\phi) = Z^{-1} D\phi \exp(-W) \equiv Z^{-1} d\mu_0(\phi) \exp(-W_I) \] (39)
where the Gaussian measure is
\[
d\mu_0(\phi) = D\phi \exp(-W_0)
\]
The partition function
\[
Z = \int d\mu_0 \exp(-W_I)
\]
determines a normalization factor. The Gaussian measure is defined \[9\] by the mean
\[
\int d\mu_0(\phi)\phi(X) \equiv \langle \phi(X) \rangle
\]
and the covariance
\[
\mathcal{G}(X, X') = \int d\mu_0(\phi)(\phi(X) - \langle \phi(X) \rangle)(\phi(X') - \langle \phi(X') \rangle)
\]
(40)
In the papers on AdS-CFT correspondence \[18\][24][17]\[?\]\[?\] the choice is made \(\mathcal{G}(X, X') = G_D(X, X')\) where \(G_D\) is the Dirichlet Green function (vanishing on the boundary) and \(\langle \phi(x) \rangle = \psi(x)\) where \(\psi(x)\) is the solution of the equation
\[
A\psi = 0
\]
with a fixed boundary condition. However, the choice of the boundary field \(\psi \neq 0\) breaks the rotational and translational invariance in the \(x\) variables present in the classical action (38) with the metric (1). The approach with the classical boundary field and the Dirichlet boundary condition leads to a different quantum field theory than the one developed in refs.\[12\][16][13][14][22] (see also \[15\][23]). In our approach the QFT is determined by the choice of the Green function \(\mathcal{G}\) (40) (we set the mean \(\langle \phi \rangle = 0\)). We choose the Green function \(\mathcal{G}\) which has the symmetries of the action \(W_0\) (2). Hence, the functional measure (39) will have the symmetries of the action (38).
We can show that our approach is equivalent to a quantization with a given classical solution \(\psi\) if subsequently an average over all such solutions is performed. We assume that in the sense of bilinear forms
\[
\mathcal{G} \geq G_D
\]
(41)
Such an inequality follows from the maximum principle for elliptic operators \[25\][26]. The inequality (41) holds true also in the non-elliptic cases discussed in our earlier papers \[27\][28]. Using eq.\,(41) we may write
\[
\mathcal{G}(X, X') = G_D(X, X') + \mathcal{G}_B(X, X')
\]
where \(\mathcal{G}_B\) is a non-negative bilinear form and \(\mathcal{G}_B(0, x; 0, x') = \mathcal{G}_E(x - x')\). Then,\[9\][29]
\[
\int d\mu_0(\phi) \exp(-W_1(\phi)F(\phi)) = \int d\mu_D(\phi_D)d\mu_B(\phi_B) \exp(-W_1(\phi_D+\phi_B))F(\phi_D+\phi_B)
\]
where $\phi_D$ is a random field with the covariance $G_D$ and $\phi_B$ is the random field with the covariance $G_B$. Let us note that because $G_D$ as well as $G$ satisfy the same equation (2) then their difference satisfies the equation

$$A(X)G_B(X, X') = A(X')G_B(X, X') = 0$$

(42)

We can solve eq.(42) with the given boundary condition $G_E$

$$G_B(X, X') = \int dx \sqrt{g} \int dx' \sqrt{g} D(X, x) D(X', x') G_E(x - x')$$

(43)

where $D$ is the Green function solving the Dirichlet problem (the boundary to bulk propagator). It follows that

$$\phi_B(X) = \int dx \sqrt{g} D(X, x) \Phi(x)$$

(44)

where $\Phi$ is the Gaussian random field defined on the boundary with the covariance

$$\langle \Phi(x) \Phi(x') \rangle = G_E(x - x')$$

(45)

In the case of the de Sitter space the solution of the Dirichlet boundary problem (44) can be expressed in the form

$$\phi(X) = y^{\frac{d}{4}} \int dp \exp(ipx)|p|^\frac{d}{4} K_\nu(|p|) \tilde{\Phi}(p)$$

(46)

where $\tilde{\Phi}$ is the Fourier transform of $\Phi$ and $K_\nu$ is the modified Bessel function of order $\nu$. From eq.(23)[27]

$$\langle \tilde{\Phi}(p) \tilde{\Phi}^*(p') \rangle = \delta(p - p')|p|^{-d}$$

The Schwinger functions are defined as moments of the measure $\mu$

$$\langle \phi(X_1)......\phi(X_k) \rangle = Z^{-1} \int d\mu(\phi)\phi(X_1)......\phi(X_k)$$

We calculate these Schwinger functions in the $N$-th order of the perturbation expansion

$$\langle \phi(X_1)......\phi(X_k) \rangle_N = Z^{-1} \int dX_1......dX_k \delta(y_1')......\delta(y_k')$$

(47)

$$\int d\mu(\phi(X_1)......\phi(X_k)) V(\phi(X_1'))......V(\phi(X_k'))$$

If $V(\phi)$ is a normal-ordered polynomial of order $r$ then the Schwinger functions of order $N$ are expressed by a product of at most $k$ Green functions $G(y, x; 0, x')$ and at most $(rN)!$ Green functions $G(0, x'; 0, x_k')$. It follows that if the Green functions at $y = y' = 0$ are sufficiently regular (depending on $\alpha$ and $\beta$ in
eqs.(14)-(15)) and the integration over $x$ in the interaction $W_I$ is restricted to a finite volume $\Lambda$ then the Schwinger functions (47) for non-coinciding points $X_j = (0, x_j)$ are finite and non-zero. In fact, the amputated Schwinger functions (when the propagators corresponding to the lines connecting the external points are removed) coincide with the ones for the $V(\phi)$ theory in $R^d$ calculated with the propagator (14). If in eq.(47) $X_j = (y_j, x_j)$ and we set all $y_j = 0$ then the resulting perturbative quantum field theory coincides with the $V(\phi)$ theory where the conventional propagators are replaced with $G_E$ of eq.(14). In particular, the partition function $Z$ in eq.(40) can be finite and non-zero even though the bulk theory is non-renormalizable. Note, that this result is a consequence of the singularity of the metric on the boundary. Adding the boundary interaction $W_I$ to the regular free part $W_0$ would lead to the theory with the same singularity as the free field theory in $d+1$ dimensions.

Let us write the propagator (14) in the form
\[
G_E = K|x - x'|^{-d+\rho}
\] (48)
then the effective field theory on the boundary $y = 0$ could be represented by an equivalent functional measure $\hat{\mu}$ of the form (39) with
\[
\hat{W} = \hat{W}_0 + \hat{W}_I = c_0 \int dx \phi(-\triangle)^{\frac{d}{2}}\phi + \int dx V(\phi)
\] (49)
where $\triangle$ is the Euclidean $d$-dimensional Laplacian. If the theory (49) is to be scale invariant (up to the log-terms coming from the renormalization)
\[
\phi(x) \simeq \lambda^\nu \phi(\lambda x)
\] (50)
then from the propagator (48) it follows that
\[
d - \rho = 2\nu
\] (51)
Hence, the interaction
\[
\int dx V(\phi) = \kappa \int dx \phi^r
\] (52)
is scale invariant if the order $r$ of the interaction is related to $\nu$ (and by eq.(51) to $\rho$)
\[
\nu = \frac{d}{r}
\] (53)
From eqs.(51) and (53) it follows that
\[
\rho = d(1 - \frac{2}{r})
\] (54)
If $\rho = d$ then from eq.(54) we obtain $r = \infty$. This case corresponds to the exponential potential $V(\phi) = \kappa \exp(\phi)$. In general, we obtain simple fractional
scaling dimensions (54) determined uniquely by the natural numbers \(d\) and \(r\). From eq.(15) and eq.(54)

\[ 2\nu = (d-1)\frac{1+\beta}{1-\alpha+\beta} \]  

(55)

Hence, the geometry of the surface imposes conditions on the scaling exponents and the form of the scale invariant interaction. From eq.(53) \(r = \frac{d}{\rho}\) hence the order of the scale invariant interaction is also determined by the surface geometry

\[ r = \frac{2d(1-\alpha+\beta)}{(d-1)(1+\beta)} \]  

(56)

If \(\beta = -1\) then \(\nu = 0\) and \(\rho = d\). In such a case from the scale invariant fields (24) (of dimension zero) we can form conformal fields of higher dimensions by means of exponential functions in a similar way as in two dimensional conformal field theory [31].

Let us note that the theory with an interaction on the boundary can be considered as a scaling limit of the one with an interaction in the bulk. For this purpose define \(V_\lambda(\phi(y,x)) = V(\lambda y, x))\), calculate the Schwinger functions (47) perturbatively and at the end take the limit \(\lambda \to 0\).

5 \(\mathcal{M}^{d+1} \to \mathbb{R}^d\) reduction

If \(m = 1\) then we may change coordinates in eq.(4) introducing a new coordinate \(z\) instead of \(y\) in such a way that

\[ \frac{dy}{dz} = g^{DD}(y)\sqrt{G(y)} \]  

(57)

Then, eq.(4) reads

\[ \left(g^{DD}(y)G(y)g^{\mu\nu}(y)\frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\nu} + \partial_z^2\right)G = g^{DD}\sqrt{G}\delta(y-y')\delta(x-x') = \delta(z-z')\delta(x-x') \]  

(58)

The action (38) takes the form

\[ W = \int dydx\sqrt{G}\left(g^{DD}\partial_y\phi\partial_y\phi + g^{\mu\nu}\partial_y\phi\partial_y\phi + \kappa_0\delta(y)V(\phi)\right) \]

\[ = \int dzdx\partial_z\phi\partial_z\phi + \int dzdxg^{DD}\partial_z\phi\partial_z\phi + \kappa_0 \int dzdx\sqrt{G}\delta(z)V(\phi) \]  

(59)

In scale invariant models \(G(0)\) is either zero or infinite. Then, we must renormalize the interaction defining \(\kappa_0 = \kappa_0\sqrt{G(0)}\). We have studied the behaviour of the Green functions \(G\) of eq.(58) in refs.[27][28]. Assuming that

\[ g^{\mu\nu}g^{DD}G \approx \delta^{\mu\nu}|z|^{2\gamma} \]  

(60)
for a small $z$ we have shown (this is the same result as the one in eq.(14)) that for small $|x-x'|$ 
$$G_E(x-x') \simeq |x-x'|^{-d+\frac{1}{1+\gamma}}$$  \hspace{1cm} (61)  

As discussed in sec.4 the theory at $z = 0$ is the same as the one with the interaction $V$ and the propagator (48). Hence, depending on the value of $\gamma \geq -1 + \frac{1}{d}$ the boundary field theory can be much more regular than the bulk field theory. When $\gamma = -1 + \frac{1}{d}$ (as in the case of de Sitter space) then the model (59) becomes superrenormalizable for polynomial interactions.

It is interesting to consider $d = 2$ and $\rho = 2$ with the exponential interaction $\exp \phi$ on the boundary. Then, the boundary correlation functions are conformal invariant (the Liouville model [30]) whereas the bulk correlation functions are not renormalizable.

The other interesting case is the boundary of the Euclidean (anti)de Sitter space [18](eqs.(22)-(24)) with an exponential interaction. Again the boundary correlation functions of exponentials can be conformal invariant [31] whereas the bulk correlation functions are non-renormalizable.

### 6 Semi-infinite statistical systems

In this section we relate the model of sec.4 to some models of statistical physics [4] [5] [6]. Let us consider Ginzburg-Landau Lagrangians with a scalar field $\phi(X)$ where $X \in \mathbb{R}^D$, $X = (z, x)$ with $x \in \mathbb{R}^d$ and $z > 0$. The boundary is at $z = 0$. We consider the action

$$W(\phi) = \int dz \int dx \left( c_1 \sum_{\mu=1}^d \partial_{\mu} \phi \partial_{\mu} \phi + c_0 \delta(z) \sum_{\mu=1}^d \partial_{\mu} \phi \partial_{\mu} \phi + c_3 \partial_z \phi \partial_z \phi + \kappa_1 V_1(\phi) + \kappa_0 \delta(z) V_0(\phi) \right)$$  \hspace{1cm} (62)  

The free propagator for a perturbation expansion around the Gaussian theory is determined by the equation

$$\left( c_1 \sum_{\mu=1}^d \partial_{\mu} \partial_{\mu} + c_0 \delta(z) \sum_{\mu=1}^d \partial_{\mu} \partial_{\mu} + c_3 \partial_z \partial_z \right) G = \delta$$  \hspace{1cm} (63)  

This model is related to the models of secs.4 and 5. We have studied the propagator (63) in refs.[27][28]. If in the equation (59) $a^{\mu \nu} = g^{\mu \nu} g^{DD} G = \delta^{\mu \nu} |z|^{-1}$ then the scaling properties of $a^{\mu \nu}$ are the same as the ones of $\delta(z)$ (i.e., $\delta(\lambda z) = \lambda^{-1} \delta(z)$). In such a case the propagator (63) has the same short distance behaviour as the one for $a^{\mu \nu}(y) = \delta^{\mu \nu} |y|^{-1}$ and $b = 1$ in eqs.(26)-(27) (if $c_1 = 0$ then these propagators coincide). Now, in eq.(26) $(d-2)\alpha + m\beta = -1$ and $(m-2)\beta + d\alpha = 0$. Hence, in eq.(14) $\sigma(d-1) = d-2$ and if $c_1 = 0$ then we have exactly
\[ G_E(|x - x'|) = K|x - x'|^{-d+2} \]  

As a consequence the propagator in \((d + 1)\) dimensions behaves as the one in \(d\) dimensions. In particular, if \(d + 1 = 3\) then with the gradient term on the boundary \((\alpha_0 \neq 0)\) we obtain the logarithmic short distance behaviour of boundary correlation functions. In such a case the model on the boundary \((\kappa_1 = 0)\) has the properties of the two-dimensional Euclidean field theory. If \(V_0(\phi) = \phi^6\) then we obtain a renormalizable field theory in the bulk which on the boundary reduces to the well-studied model of a tricritical phase transition. If \(d + 1 = 4\) and \(V_0(\phi) = \phi^4\) then we have a typical superrenormalizable Landau-Ginzburg model. The model is superrenormalizable because the truncated Green functions are the same as in the threedimensional theory. The model in \(d + 1 = 5\) with the \(\phi^4\) interaction on the boundary is still renormalizable. In terms of the lattice approximation our semi-infinite model with \(c_1 = 0\) describes spins whose interaction in the bulk is restricted to the lines perpendicular to the boundary. It is still surprising that although the system is \(D\) dimensional the correlation functions behave like the ones of an \(d\)-dimensional system in spite of the opportunity for the interaction to spread into the \(D\)-th dimension.

7 Scattering theory with an interaction concentrated on the boundary

We could approach the models of secs.4-6 in the conventional Hamiltonian framework. We consider the scattering theory for the field equations

\[ A\phi = -\delta(z)V'(\phi) \]  

We could treat \(z\) either as a spatial coordinate or as a time. Let us concentrate here on the latter interpretation. Then, for \(z \to -\infty\) the interaction is switched off and \(\phi \to \phi^{in}\) where

\[ A\phi^{in} = 0 \]  

We consider here the real time \(z\) and the wave operator \(A = -\partial_z^2 - A^2\) where \(A^2\) is a positive operator. We quantize the field \(\phi^{in}\) and construct the interaction Hamiltonian \(H_I(z) = \delta(z) \int dxV(\phi^{in})\). We can derive the S-matrix describing an interaction \(V(\phi)\) of the field \(\phi^{in}\) by means of the conventional reduction formulas [16] [33]. We assume that the geometry is fixed and therefore the asymptotic states are constructed on a given gravitational background. As usual the S-matrix is determined by the time-ordered correlation functions

\[ \langle T (\phi(X_1)\ldots\phi(X_n)) \rangle = \langle T (\phi^{in}(X_1)\ldots\phi^{in}(X_n) \exp(-i \int dz dx \delta(z) \sqrt{g}V(\phi^{in}))) \rangle \]  

(67)
The expectation value on the rhs is calculated in the vacuum for the free field \( \phi^{in} \) defined on the curved manifold (there may be many vacuum states and it belongs to the theory to choose one of them). In perturbation theory (in the Euclidean region) the correlation functions are expressed by the Green functions (47). The S-matrix is determined by \( \tau \)-functions

\[
S = 1 + \sum_n \frac{(-i)^n}{n!} \int dX_1...dX_n \tau(X_1,...,X_n) :\phi^{in}(X_1)....\phi^{in}(X_n) : + ...
\]

where

\[
\tau(X_1,...,X_n) = \mathcal{A}(X_1)....\mathcal{A}(X_n)\langle T\left(\phi(X_1)...\phi(X_n)\right) \rangle
\]

Hence, from eq.(47) the \( \mathcal{A} \) operators cancel the external propagators and the \( \tau \) functions depend on Green functions defined at \( z = 0 \).

Let us consider some examples. First, the case when in eq.(58)

\[
g^{DD}G_{\mu\nu}(y) = \delta^{\mu\nu}|z|^{-1}
\]

(this is an analogue of eq.(63)). Then

\[
\mathcal{A} = |z|^{-1}\Delta - \partial_z^2
\]

The solutions of eq.(69) have the form

\[
\phi^{in}(z,x) = \sqrt{z} \int dp \left( a(p)H^{(1)}_\nu(C|p|\sqrt{z}) + a^+(p)\overline{H^{(1)}_\nu(C|p|\sqrt{z})} \right) \exp(ipx)
\]

where \( H^{(1)}_\nu \) is the Hankel function of order \( \nu \) [32] and \( C \) is a positive constant.

A quantization of \( a \) and \( a^+ \) (as creation and annihilation operators) leads to a quantum scattering theory described by an expansion of the S-matrix in asymptotic fields \( \phi^{in} \). Note, that at \( z = 0 \) the Feynman propagator

\[
\mathcal{G}_E(x - x') = K \int dp \exp[ip(x - x')]|p|^{-2}
\]

for \( \mathcal{A} \) in eq.(69) is equal to the one for a massless Euclidean free field in \( d \) dimensions.

As a second example we consider a \( V \) interaction of particles in the hyperbolic space (22) (whose analytic continuation \( y \rightarrow iz \) gives de Sitter space). In such a case eq.(66) has the solution (at the real time \( y \rightarrow z^2 \))

\[
\phi^{in}(y,x) = y^\omega \int dp \left( a(p)H^{(1)}_\nu(C|p|y) + a^+(p)\overline{H^{(1)}_\nu(C|p|y)} \right) \exp(ipx)
\]

where \( \omega = \frac{d}{2} \).

If the interaction is of the form \( \delta(y)V \) then the propagators are logarithmic and the model will be superrenormalizable for (normal ordered) polynomial \( V \). There will be no ultraviolet divergencies in the S-matrix.

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8 Summary

We find it as a remarkable property of some singular second order differential operators in $D$ dimensions that the Green function restricted to the boundary can be less singular than $|x - y|^{-D+2}$. When we put an interaction on the boundary then the regularity of the model can be extended to models with a non-trivial scattering. Models with an interaction on the boundary are certainly of physical relevance in statistical physics. It is still unclear whether such models are acceptable in high energy physics as, e.g., the brane theories. We suggest however that restricting an interaction to a brane can give a promising way of avoiding ultraviolet divergencies in quantum field theory. Our construction of the quantum field theory with a boundary (which preserves the symmetries of the boundary) is different than the AdS-CFT approach. However, there appear some interesting relations between both approaches if we treat the boundary value $\phi_0$ as random (quantum) and average over $\phi_0$.

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