Parameter estimation of compact binaries using the inspiral and ringdown waveforms

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Abstract. We analyze the problem of parameter estimation for compact binary systems that could be detected by ground-based gravitational wave detectors. So far this problem has only been dealt with for the inspiral and the ringdown phases separately. In this paper, we combine the information from both signals, and we study the improvement in parameter estimation, at a fixed signal-to-noise ratio, by including the ringdown signal without making any assumption on the merger phase. The study is performed for both initial and advanced LIGO and VIRGO detectors.

PACS numbers: 04.80.Nn, 95.55.Ym, 97.60.Gb, 07.05.Kf

1. Introduction

Coalescing compact binaries consisting of either black holes (BH) or neutron stars (NS) are among the targets of on-going searches for gravitational waves in the data of ground-based interferometric detectors such as GEO 600 [1, 2], the Laser Interferometer Gravitational Wave Observatory (LIGO) [3, 4], TAMA 300 [5], and VIRGO [6].

The coalescence of a compact binary system is commonly divided into three stages, which are not very well delimited one from another, namely the inspiral, the merger and the ringdown. Many studies so far have focused on the gravitational waves emitted during the inspiral phase because the inspiral waveform is very well understood [7–14, 15–18, 19, 20], and the event rates seem promising [15, 16, 17, 18, 19]. Gravitational waves from the merger can only be calculated using the full Einstein equations. Because of the extreme strong field nature of this epoch neither a straightforward application of post-Newtonian theory nor perturbation theory is very useful. Recent numerical work [21–24] has given some insights into the merger problem, but there are no reliable models for the waveform of the merger phase at this time. The gravitational radiation from the ringdown phase is also well known and it can be described by quasi-normal modes [25]. In spite of the importance of the ringdown there are a fewer publications on ringdown searches compared to those for inspiral searches.

Flanagan and Hughes [26] were the first in studying the contribution of the three phases to the signal-to-noise ratios both for ground-based and also space-based...
interferometers, but they did not study the problem of accuracy in the parameter estimation. This is an important problem because many efforts are now underway to detect both the inspiral and the ringdown signals using matched filtering techniques in real data [27, 29, 30, 33, 34]. In a recent paper [32], parameter estimation of inspiralling compact binaries has been revised using up to the 3.5 restricted post-Newtonian approximation, extending previous analysis [33, 34], but ignoring the other stages. The parameter estimation for the ringdown phase alone has also been studied, some time ago, for ground based detectors [35, 36, 37] as well as for LISA [38]. The aim of this paper is to discuss how parameter estimation can be improved by using information from both the inspiral and the ringdown phases combined together in matched filtering like analysis for different ground-based detectors.

This paper is organized as follows: Section 2 introduces our notation and reviews the basic concepts of signal parameter estimation in matched filtering. Section 3 provides the noise curves used in this study for initial and advanced LIGO and VIRGO. Section 4 briefly describes the waveforms that we are looking for. For the inspiral phase, we consider a non-spinning compact system with circular orbits and the waveform in the restricted post-Newtonian approximation. For the ringdown, we assume that the dominant mode has \( l = m = 2 \) and therefore the waveform is given by an exponentially decaying sinusoid. Section 5 studies the impact on the parameter estimation for coalescing binary black holes, by combining the signals from both the inspiral and ringdown phases and compares the results with the case of inspiral phase alone. The results are presented for a fixed inspiral signal-to-noise ratio of 10. Different number of parameters are used as well as different values for the ringdown efficiency. Finally section 6 concludes with a summary of our results and plans for further work.

In the Appendices we collect various technical calculations and we present an explicit analytical calculation of the Fisher matrix for the ringdown phase that has been used to compare with the numerical results.

2. Summary of parameter estimation

In this section we briefly review the basic concepts and formulas of signal parameter estimation relevant to the goal of this paper; we refer the reader to [33] for a more detailed analysis.

The output of a gravitational wave detector can be schematically represented as

\[
s(t) = h(t) + n(t),
\]

where \( n(t) \) is the noise that affects the observation and \( h(t) \) is the gravitational wave signal measured at the detector, a linear superposition of the two independent polarizations of the strain amplitude \( h_+ \) and \( h_\times \), given by

\[
h(t) = F_+(\theta, \phi, \psi) h_+(t) + F_\times(\theta, \phi, \psi) h_\times(t),
\]

where \( F_+ \) and \( F_\times \) are the antenna pattern functions, that depend on the direction of the source in the sky \( (\theta, \phi) \) and the polarization angle \( \psi \). In case of a laser interferometer detector, the expressions of \( F_+ \) and \( F_\times \) are given by [44]:

\[
F_+(\theta, \phi, \psi) = \frac{1}{2} (1 + \cos^2 \theta) \cos 2\phi \cos 2\psi - \cos \theta \sin 2\phi \sin 2\psi,
\]

\[
F_\times(\theta, \phi, \psi) = \frac{1}{2} (1 + \cos^2 \theta) \cos 2\phi \sin 2\psi + \cos \theta \sin 2\phi \cos 2\psi.
\]
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For sake of simplicity we shall made the standard assumptions that the noise \( n(t) \) has zero mean and it is stationary and Gaussian, although in realistic cases this hypothesis is likely to be violated at some level. Within this approximation, the Fourier components of the noise are statistically described by:

\[
E[\tilde{n}(f)\tilde{n}^*(f')] = \frac{1}{2}\delta(f - f')S_n(f),
\]

where \( E[] \) denotes the expectation value with respect to an ensemble of noise realization, the \(*\) superscript denotes complex conjugate, \( S_n(f) \) is the one sided noise power spectral density, and tildes denote Fourier transforms according to the convention

\[
\tilde{x}(f) = \int_{-\infty}^{\infty} e^{i2\pi ft} x(t) \, dt. 
\]

With a given noise spectral density for the detector, one defines the “inner product” between any two signals \( g(t) \) and \( h(t) \) by:

\[
(g|h) = 2 \int_0^{\infty} \frac{\tilde{g}^*(f)\tilde{h}(f) + \tilde{g}(f)\tilde{h}^*(f)}{S_n(f)} df.
\]

With this definition, the probability of the noise to have a realization \( n_0 \) is just:

\[
p(n = n_0) \propto e^{(n_0|n_0)/2}.
\]

The optimal signal-to-noise ratio (SNR) \( \rho \), achievable with linear methods (e.g., matched filtering the data) is given by the standard expression

\[
\rho^2 = (h|h) = 4 \int_0^{\infty} \frac{\tilde{h}(f)^2}{S_n(f)} df.
\]

In the limit of large SNR, if the noise is stationary and Gaussian, the probability that the gravitational wave signal \( h(t) \) is characterized by a given set of values of the source parameters \( \lambda = \{\lambda^k\}_k \) is given by a Gaussian probability of the form [25]:

\[
p(\lambda|h) = p^{(0)}(\lambda) \exp \left[ -\frac{1}{2} \Gamma_{jk} \Delta\lambda^j \Delta\lambda^k \right],
\]

where \( \Delta\lambda^k \) is the difference between the true value of the parameter and the best-fit parameter in the presence of some realization of the noise, \( p^{(0)}(\lambda) \) represents the distribution of prior information (a normalization constant) and \( \Gamma_{jk} \) is the so-called Fisher information matrix defined by

\[
\Gamma_{ij} = \langle \partial_i h | \partial_j h \rangle = 2 \int_0^{\infty} \frac{\partial_i \tilde{h}^*(f) \partial_j \tilde{h}(f) + \partial_i \tilde{h}(f) \partial_j \tilde{h}^*(f)}{S_n(f)} df,
\]

where \( \partial_i = \frac{\partial}{\partial \lambda^i} \).

The inverse of the Fisher matrix, known as the variance-covariance matrix, gives us the accuracy with which we expect to measure the parameters \( \lambda^k \)

\[
\Sigma^{jk} \equiv (\Gamma^{-1})^{jk} = \langle \Delta\lambda^j \Delta\lambda^k \rangle.
\]

Here the angle brackets denote an average over the probability distribution function in Eq. (10). The root-mean-square error \( \sigma_k \) in the estimation of the parameters \( \lambda^k \) can then be calculated, in the limit of large SNR, by taking the square root of the diagonal elements of the variance-covariance matrix,

\[
\sigma_k = (\langle \Delta\lambda^k \rangle^2)^{1/2} = \sqrt{\Sigma_{kk}},
\]

and the correlation coefficients \( c^{jk} \) between two parameters \( \lambda^j \) and \( \lambda^k \) are given by:

\[
c^{jk} = \frac{\langle \Delta\lambda^j \Delta\lambda^k \rangle}{\sigma_j \sigma_k} = \frac{\Sigma^{jk}}{\sqrt{\Sigma_{jj} \Sigma_{kk}}}. 
\]
3. Noise spectra of the interferometers

In this paper, we use three different noise curves to understand the effect of detector characteristics on the parameter estimation. The noise curves used are initial and advanced LIGO and VIRGO as in [32]. Those are:

For the initial LIGO

$$S_n(f) = \begin{cases} S_0 [(4.49x)^{-56} + 0.16x^{-4.52} + 0.52 + 0.32x^2], & f \geq f_s \\ \infty, & f < f_s \end{cases} \quad (15)$$

where $$x = f/f_0$$, with $$f_0 = 150$$ Hz (a scaling frequency chosen for convenience), $$f_s = 40$$ Hz is the lower cutoff frequency, and $$S_0 = 9 \times 10^{-46} \text{ Hz}^{-1}$$.

For advanced LIGO the noise curve is given by

$$S_n(f) = \begin{cases} S_0 \left[ x^{-4.14} - 5x^{-2} + \frac{111(1-x^2+x^4/2)}{(1+x^2/2)} \right], & f \geq f_s \\ \infty, & f < f_s \end{cases} \quad (16)$$

where $$f_0 = 215$$ Hz, $$f_s = 10$$ Hz and $$S_0 = 10^{-49} \text{ Hz}^{-1}$$.

Finally, for the VIRGO detector the expected noise curve is given by:

$$S_n(f) = \begin{cases} S_0 \left[(6.35x)^{-5} + 2x^{-1} + 1 + x^2\right], & f \geq f_s \\ \infty, & f < f_s \end{cases} \quad (17)$$

where $$f_0 = 500$$ Hz, $$f_s = 20$$ Hz and $$S_0 = 3.24 \times 10^{-46} \text{ Hz}^{-1}$$.

4. The gravitational-wave signal

As discussed in the introduction, the coalescence and its associate gravitational wave signal can be divided into three successive epochs in the time domain: inspiral, merger and ringdown. During the inspiral the distance between the stars diminishes and the orbital frequency sweeps up. For low-mass binary systems, the waveforms are well modeled using the post-Newtonian approximation to general relativity [7, 9, 10, 13]. Eventually the post-Newtonian description of the orbit breaks down, and the black holes cannot be treated as point particles any more. What is more, it is expected that they will reach the innermost stable circular orbit (ISCO), at which the gradual inspiral ends and the black holes begin to plunge together to form a single black hole. This is referred as the merger phase. At present, the merger phase is not well understood and no analytical reliable waveforms exist. At the end, the final black hole will gradually settle down into a Kerr black hole. The last gravitational waves will be dominated by the quasi-normal ringing modes of the black hole (see [41] and references therein) and can be treated using perturbation theory [42]. At late time, the radiation will be dominated by the $$l = m = 2$$ mode [25]. This is the so-called ringdown phase.

The gravitational waveform of coalescing compact binaries thus takes the form

$$h(t) = \begin{cases} h_{\text{inspiral}}(t), & -\infty < t < T_{\text{ISCO}} \\ h_{\text{merger}}(t), & T_{\text{ISCO}} < t < T_{\text{QNR}} \\ h_{\text{ringdown}}(t), & T_{\text{QNR}} < t < \infty \end{cases} \quad (18)$$

where $$T_{\text{ISCO}}$$ is the time when the system reaches the ISCO and $$T_{\text{QNR}}$$ is the time when the quasi-normal mode $$l = m = 2$$ begins to dominate the ringdown, although there is some arbitrariness in choosing $$T_{\text{ISCO}}$$ and $$T_{\text{QNR}}$$ to delimit the three epochs.
4.1. The inspiral waveform

For a non-spinning compact binary system with circular orbits, the two polarizations $h_+$ and $h_\times$ of the inspiral waveform can be well described by the post-Newtonian expansion. Thus setting $G = c = 1$, they read:

$$h_+ = 2M\eta r (M\omega)^{2/3} \left\{ H_{h+}^{(0)}(\omega) + v^{1/2} H_{h+}^{(1)}(\omega) + v^3 H_{h+}^{(2)}(\omega) + \ldots \right\},$$

$$h_\times = 2M\eta r (M\omega)^{2/3} \left\{ H_{h\times}^{(0)}(\omega) + v^{1/2} H_{h\times}^{(1)}(\omega) + v^3 H_{h\times}^{(2)}(\omega) + \ldots \right\},$$

where $v \equiv (M\omega)^{2/3}$, $\omega$ is the orbital frequency, $r$ is the distance to the source, $M = m_1 + m_2$ is the total mass, $\mu = m_1 m_2 / M$ is the reduced mass, $\eta = \mu / M$ is the symmetric mass ratio and $M = \mu^{3/5} M^{2/5} = M\eta^{3/5}$ is the chirp mass.

In what follows we consider the waveform in the restricted post-Newtonian approximation \[43\], corresponding to a frequency twice the orbital frequency, and we ignore higher order harmonics. This corresponds to the lowest terms in the series \[19\]. The functions $H^{(0)}_+, H^{(0)}_{\times}$ are given by \[10\]:

$$H^{(0)}_+ = - (1 + \cos^2 \iota) \cos \Phi(t),$$

$$H^{(0)}_{\times} = - 2 \cos \iota \sin \Phi(t),$$

with $\iota$ being the angle between the orbital angular momentum of the binary and the line of sight from the detector to the source. $\Phi$ is the phase of the gravitational wave an the instant $t$, that we consider modeled through 2nd post-Newtonian order, neglecting the higher order terms in this analysis, since they would not contribute significantly to the result.

The Fourier transform of the inspiral waveform can be computed using the stationary phase approximation \[33, 34, 39, 40\] and this yields:

$$\tilde{h}_{\text{INS}}(f) = \begin{cases} A_{\text{INS}} f^{-7/6} e^{i\Psi(f)} & f < f_{\text{ISCO}} \\ 0 & f > f_{\text{ISCO}} \end{cases},$$

with

$$A_{\text{INS}} = - \frac{\mathcal{M}^{5/6}}{r} \sqrt{\frac{5\pi}{96}} \pi^{-7/6} \sqrt{F_+^2 (1 + \iota^2)^2 + F_\times^2 4c^2},$$

$$\Psi(f) = 2\pi ft_c - \phi_c - \frac{\pi}{4} + \frac{3}{128} \sum_{k=0}^{4} A_k u^{k-5},$$

where $c = \cos \iota$, $t_c$ refers to the coalescence time, $\phi_c$ is the phase at the coalescence instant, $u = (\pi \mathcal{M} f)^{1/3}$, and the coefficients $A_k$ are given by

$$A_0 = 1$$

$$A_1 = 0$$

$$A_2 = \frac{20}{9} \left(\frac{743}{336} + \frac{11}{4} \eta \right) \eta^{-2/5}$$

$$A_3 = -16\pi \eta^{-3/5}$$

$$A_4 = 10 \left(\frac{3058673}{1016064} + \frac{5429}{1008} \eta + \frac{617}{144} \iota^2 \right) \eta^{-4/5}$$

We also consider the ISCO to take place at a separation of $6M$, corresponding to a final frequency

$$f_{\text{ISCO}} = \frac{1}{6^{3/2} \pi M}.$$
4.2. The ringdown waveform

The ringdown portion of the gravitational wave signal we consider can be described as the \( l = m = 2 \) quasi-normal mode. Therefore the gravitational radiation in the time domain is expected as the superposition of two different damped sinusoids, although one of these exponentials could be invisible in the actual waveform as discussed in [38]. In our study, we assume that the ringdown waveform can be written as in [25], corresponding to a circularly polarized wave. In this way we have

\[
h_+(t) - i h_\times(t) = \frac{A M}{r} 2 S^2_2 (\iota, \beta, a) \times \exp \left[ -i 2 \pi f_{QNR} (t - t_0) - \frac{\pi f_{QNR}}{Q} (t - t_0) + i \varphi_0 \right],
\]

where \( t_0 \) is the start time of the ringdown, \( \varphi_0 \) the initial phase, \( M \) is the total mass of the system mass (see [45] for further discussions), \( f_{QNR} \) and \( Q \) are the central frequency and the quality factor of the ringing. For this mode, a good fit to the frequency \( f_{QNR} \) and quality factor \( Q \), within an accuracy of 5\%, is

\[
f_{QNR} \approx [1 - 0.63(1 - a)^{3/10}] \frac{1}{2 \pi M}, \quad (32)
\]

\[
Q \approx 2(1 - a)^{-9/20}, \quad (33)
\]

where \( a M^2 \) is its spin, and \( a \) is the Kerr parameter that lies in the range \((0, 0.998)\).

In our study we set \( a \) to the near extremal value of 0.98 (as in [26]), although we consider \( a \) as any other independent parameter when evaluating the Fisher matrix. The function \( 2 S^2_2 \) is the spin weighted spheroidal harmonic that depends on the inclination angle of the black hole axis seen from the observer and the Kerr parameter \( a \). \( A \) is a dimensionless coefficient describing the magnitude of the perturbation when the ringdown begins. Although the value of the amplitude is uncertain, we set the amplitude of this mode by assuming that a fraction \( \epsilon \) of the system’s mass is converted into gravitational waves during the ringdown [26]

\[
E_{RD} \approx \frac{1}{8} A^2 M^2 f_{QNR} Q = \epsilon M \left( \frac{4 \mu}{M} \right)^2. \quad (34)
\]

Therefore

\[
A = \sqrt{\frac{128 \eta^2 \epsilon}{M f_{QNR} Q}}. \quad (35)
\]

The strain produced at the detector can be written as:

\[
h_{RD}(t) = A_{RD} \exp \left[ \frac{(t - t_0) \pi f_{QNR}}{Q} \right] \cos(-2 \pi f_{QNR} (t - t_0) + \gamma_0) \quad (36)
\]

where

\[
A_{RD} = \frac{A M}{r} \sqrt{F^2_+ + F^2_\times |2 S^2_2|}.
\]

The Fourier transform of the waveform becomes:

\[
\tilde{h}_{RD}(f) = \frac{A_{RD}}{2 \pi} e^{i 2 \pi f t_0} e^{i \gamma_0} \times \left( \frac{f_{QNR}}{Q} - 2i(f - f_{QNR}) \right) - \frac{f_{QNR}}{Q} - 2i(f + f_{QNR}) \right). \quad (38)
\]
5. Parameter estimation of compact binaries using the inspiral and ringdown waveforms

In this paper we want to study the impact on the parameter extraction by combining the signals from the inspiral and the ringdown epochs, neglecting all information coming from the merger epoch itself since no reliable waveforms exist so far.

Following earlier works, we choose the set of independent parameter $\lambda_{\text{INS}}$ describing the inspiral signal to be

$$\lambda_{\text{INS}} = \{\ln A_{\text{INS}}, f_0 t_c, \phi_c, \ln M, \ln \eta\},$$

while for the ringdown, the parameters could be

$$\lambda_{\text{RD}} = \{\ln A_{\text{RD}}, \ln M, \ln a, \gamma_0, t_0\}.$$  \hspace{1cm} (40)

A possible approach to this problem of parameter extraction would be to consider the two sets of parameters (39) and (40) as independent, perform matched filtering using the two template families (for the inspiral and the ringdown waveforms) and then reduce the uncertainties in the parameter estimation (in particular for the masses) by making a posterior consistency check [46]. However in this paper we follow a different approach. We consider only a single coalescing waveform, as if we were performing matched filtering with a single template family bank, as given by Eq. (18), which describes the different phases, but ignoring the information from the merger phase, and we focus our attention on how parameter estimation of the $\lambda_{\text{INS}}$ parameters can be improved as compared to the case in which the inspiral waveform is used alone. For this reason the study presented here focuses only in those mass ranges for which the inspiral signal alone could be detectable by the detector. This corresponds to a total mass of approximately $1–100 M_\odot$ for initial LIGO, $1–400 M_\odot$ for advanced LIGO, and $1–200 M_\odot$ for VIRGO.

The global waveform considered here becomes in the Fourier domain

$$\tilde{h}_{\text{GL}}(f) = \tilde{h}_{\text{INS}}(f) + \tilde{h}_{\text{RD}}(f),$$

where $\tilde{h}_{\text{INS}}(f)$ and $\tilde{h}_{\text{RD}}(f)$ are given by equations (22) and (38) respectively. This global waveform is completely determined by a set of independent parameters, given by

$$\lambda_{\text{GL}} = \{\ln A_{\text{INS}}, f_0 t_c, \phi_c, \ln M, \ln \eta, \ln a, \gamma_0, t_0\}. \quad (42)$$

Notice that we do not include $\ln A_{\text{RD}}$ as an independent parameter since for a given source location and orientation, and a given ringdown efficiency $\epsilon$, the ringdown amplitude $A_{\text{RD}}$, given by Eq. [37], is determined by the inspiral amplitude $A_{\text{INS}}$, the Kerr parameter and the masses. Instead, what we do is to find a heuristic relation between $A_{\text{RD}}$ and $A_{\text{INS}}$ by averaging over source directions and black-hole orientations, making use of the angle averages: $\langle F_{x}^2 \rangle_{\theta, \phi, \psi} = \langle F_{x}^2 \rangle_{\theta, \phi, \psi} = 1/5$, $\langle F_{x} F_{x} \rangle_{\theta, \phi, \psi} = 0$, $\langle c^2 \rangle_{i} = 1/3$, $\langle (1 + c^2)^2 \rangle_{i} = 28/15$, and $\langle |S_{i, \beta}^2|^2 \rangle_{i, \beta} = 1/4\pi$. The angle averaged root mean square (rms) values of the inspiral and ringdown amplitudes $A_{\text{INS}}, A_{\text{RD}}$, given by Eq. [20] and [38], become

$$A_{\text{INS}}^{\text{rms}} = \sqrt{\langle A_{\text{INS}}^2 \rangle} = \frac{1}{\sqrt{30\pi^{2/3}}} \frac{M^{5/6}}{r},$$

$$A_{\text{RD}}^{\text{rms}} = \sqrt{\langle A_{\text{RD}}^2 \rangle} = \frac{1}{\sqrt{10\pi}} \frac{AM}{r}.$$  \hspace{1cm} (44)
From the above equations, together with Eq. (35), we derive a relation \( A_{RD}(A_{INS}, M, \eta, a) \) through the ratio of rms of both amplitudes:

\[
A_{RD}(A_{INS}, M, \eta, a) \equiv \frac{A_{INS}^{\text{rms}}}{A_{RD}^{\text{rms}}} = \frac{\sqrt{384 \pi^{1/3} \epsilon}}{M f_{QNR} Q \eta^{2/5} M^{1/6}} A_{INS}.
\]  
(45)

Note that in this relation the product of \( M \) and \( f_{QNR} \) is just a function of \( a \) as can be seen from Eq. (32). This relation (45) will be used in calculating the SNR as well as the Fisher matrix for the global waveform.

The SNR values for equal-mass black hole binaries are shown in figure 1. This figure indicates that there is a range of masses (different for the different noise curves) for which both the inspiral and the ringdown signals could be detectable and one could search for both portions of the signal in order to improve the SNR and the accuracy in parameter estimation.

It is clear that both in the time domain, as well as, in the frequency domain, the inspiral signal is decoupled from the ringdown one. The inspiral waveform \( h_{INS}(f) \) ranges from the lower cut-off frequency \( f_s \) to \( f_{ISCO} \), while the ringdown \( h_{RD}(f) \) is centered around \( f_{QNR} \) with a certain bandwidth, that in the literature is considered to be smaller than \( \Delta f/f_{QNR} = 0.5 \), although in our numerical simulations for the ringdown signal we use the same lower cut-off frequency \( f_s \) and a higher cut-off frequency of 5000 Hz. This justifies that the Fisher matrix, defined in Eq. (11), of the global waveform can be computed as

\[
\Gamma_{ij} = (\partial_i h_{GL}|\partial_j h_{GL}) = (\partial_i h_{INS}|\partial_j h_{INS}) + (\partial_i h_{RD}|\partial_j h_{RD}),
\]  
(46)

neglecting the cross elements \( (\partial_i h_{INS}|\partial_j h_{RD}) \). Therefore the Fisher matrix can be computed as the sum of Fisher matrix of the inspiral waveform plus the Fisher matrix of the ringdown

\[
\Gamma_{GL} = \Gamma_{INS} + \Gamma_{RD},
\]  
(47)

where we just need to be consistent in computing the elements corresponding to the same parameter set. Also the total SNR is given by

\[
\rho_{GL}^2 = \rho_{INS}^2 + \rho_{RD}^2.
\]  
(48)

The way we proceed is to analyze first the well known case of the inspiral signal alone, and then we compare the results with those when using the inspiral and ringdown waveforms together. In order to separate the effects of increasing the number of parameters from the fact we are using a more complex waveform, we study two different cases:

i. The case in which only the five inspiral parameters are considered. This is equivalent to have no uncertainties in the spin of the final black hole, nor in the initial phase and time of the ringdown signal. This of course, would not be realistic in a search, but it provides the optimal improvement in parameter estimation one could expect from the fact that we added the ringdown waveform.

‡ The calculation of the SNR for the ringdown waveform is computed differently from what was done by Flanagan and Hughes in [26]. Instead of taking \( |t - t_0| \) in the damped exponential, integrating over \( t \) over \( -\infty \) to \( +\infty \) and dividing the result by \( \sqrt{2} \) to compensate for the doubling, we assume that the waveform \( h_{RD}(t) \) vanishes for \( t < t_0 \) and integrate only over \( t > t_0 \).

§ Note that the distance between \( f_{QNR} \) and \( f_{ISCO} \) is larger than the bandwidth of the ringdown signal \( h_{RD}(f) \). In particular if we consider the value \( a = 0.98 \) then \( (f_{QNR} - f_{ISCO})/f_{QNR} = 0.833 \) that suggests no overlap between the inspiral and the ringdown signal.
Figure 1. The averaged signal-to-noise ratio for equal-mass black hole coalescences detected by ground-based interferometers at a luminosity distance of 1 Gpc. The solid line is the SNR curve for the inspiral, and the dash and dash-dotted lines for the ringdown portion of the signal assuming a value of $\alpha = 0.98$ and $\epsilon$ equals to 1.5% and 0.5%, respectively. The top panel corresponds to initial LIGO, the middle panel to advanced LIGO and the bottom one to VIRGO.
ii. The more realistic case in which all the eight independent parameters given in (42) are considered.

In Appendix A the reader can find the explicit calculations of all the waveform derivatives necessary to compute the Fisher matrix which is then computed numerically.

In our analysis, we set the Kerr parameter $a = 0.98$ (as in [26]), and we consider two different values of $\epsilon$: a more optimistic value of 1.5%, (half the value used in [26]), and a more pessimistic one of 0.5%, that are more consistent with recent numerical simulations [22]. With these parameters we study a range of masses, analyzing both the equal-mass and unequal-mass cases, for three ground-based detectors: initial LIGO, advanced LIGO and VIRGO, using the noise curves described in section 3. All the errors are computed at a fixed value of inspiral SNR of 10.

For the equal-mass case the results are presented in figures 2, 3 and 4 corresponding to initial LIGO, advanced LIGO and VIRGO, respectively. The errors of $t_c$, $M$ and $\eta$ and some of the associated correlation coefficients for the inspiral signal alone, for different pairs of masses, can be found in table 1. In tables 2 and 3 one can find the comparison of errors and correlation coefficients for the different cases we have analyzed. In particular table 2 refers to the case (i) in which only the five inspiral parameters are considered and table 3 refers to the case (ii) in which we use the eight global parameters. In all cases the errors improve, and the improvement is higher for larger masses for which the ringdown signal contribution to the SNR increases. This improvement could be explained by the greater structure and variety of the global waveform but also by the variation of some of the correlation coefficients, although this is not fully assessed in this paper. We have just noticed that the correlations coefficients relative to the masses decrease when the ringdown signal is added, as can be seen in the tables. We also notice that the improvement is very significant for massive systems with very large errors for the inspiral waveform alone. These large errors are associated to the small number of useful cycles of the inspiral signal of these systems [22]. Therefore the effect induced in parameter estimation due to the inclusion of the ringdown signal could be understood in terms of additional number of gravitational wave cycles accumulated. Although, from the present analysis, it is not clear which of these considerations is the dominant aspect to completely understand the variation in parameter estimation observed with the global waveform.

The numerical results for the inspiral and the ringdown waveforms separately have been verified by comparing with those existing in the literature (for different masses and noise curves). Moreover, for the ringdown case alone we have also found a good agreement with an analytical approximation as described in Appendix B.

6. Conclusions

We have carried out a study to understand the implications of adding the ringdown to the inspiral signal on parameter estimation of non-spinning binaries using the covariance matrix. We have compared the results using three different noise curves corresponding to initial LIGO, advanced LIGO and VIRGO.

The result of our study is that the parameter estimation of $t_c$, $\phi_c$, $M$ and $\eta$ improves significantly, as expected, by employing the extra information that comes from the ringdown, for those systems with a total mass such that both the inspiral and the ringdown signal could be detectable by the detectors. Naturally the improvement
Figure 2. In this figure we compare the errors in the estimation of $t_c$, $M$ and $\eta$ for equal-mass black hole coalescences by the initial LIGO interferometers at a fixed inspiral SNR of 10. The solid line corresponds to the inspiral signal only and the others to the combined inspiral plus ringdown waveforms. The dashed line corresponds to the case in which only the five independent inspiral parameters \( \epsilon = 1.5\% \), while the dot-dashed lines correspond to the cases in which we consider all the independent global parameters \( \epsilon = 1.5\% \) and \( \epsilon = 0.5\% \), respectively.
Figure 3. Same as figure 2 for advanced LIGO.
Figure 4. Same as figure 2 for VIRGO
Table 1. Measurements of errors and some of the associated correlation coefficients using the 2nd post-Newtonian binary inspiral waveform at a SNR of 10. For each of the three detector noise curves the table presents $\Delta t_c$ (in msec), $\Delta \phi_c$ (in radians), $\Delta M/M$ and $\Delta \eta/\eta$ (in percentages). The cases considered here correspond to NS-BH and BH-BH binaries of different masses.

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is larger in the case of considering a smaller number of parameters, but in both cases, the five parameter case and the eight parameter case, the improvement is significant. The study is performed at a fixed inspiral SNR of 10, therefore the error in $\Delta A_{\text{INS}}/A_{\text{INS}}$ would be of 10% for the inspiral signal alone. This is also improved by adding the ringdown.

In this work we have made a number of simplifying assumptions, ignoring the merger phase, considering only the 2nd post-Newtonian inspiral phase formula instead of the 3.5 that is already known, using only a single mode for the ringdown signal, and ignoring angular dependencies (because of the angle averages we use). For this reason the results obtained here should be considered just as an indication of which could be the real effect in the parameter estimation by combining the inspiral with the ringdown signal. The preliminary results obtained here seem to be very encouraging. Therefore it would be interesting to extend the analysis to a more realistic case and also for different data analysis techniques (different from matched filtering). Although we have focused on ground-based detectors a similar study could be performed for LISA.
Table 2. Measurements of errors and associated correlation coefficients using the 2nd post-Newtonian binary inspiral waveform at a SNR of 10 together with the ringdown waveform, using a set of five parameters \(\{\ln A_{\text{INS}}, f_0 t_c, \phi_c, \ln M, \ln \eta\}\), excluding \(\{\ln a, \gamma_0, t_0\}\).

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Initial LIGO

Advanced LIGO

VIRGO
Table 3. Measurements of errors and associated correlation coefficients using the 2nd post-Newtonian binary inspiral waveform at a SNR of 10 together with the ringdown waveform, using a set of eight parameters \{ln $A_{INS}$, $f_0$, $\phi_c$, ln $M$, ln $\eta$, ln $a$, $\gamma_0$, $t_0$\}.

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Acknowledgments

The authors gratefully acknowledge the support of the Spanish Ministerio de Educación y Ciencia research project FPA-2004-03666.

Appendix A. The waveform derivatives

In order to calculate the Fisher matrix with respect to the \{\ln A_{INS}, f_0 t_c, \phi_c, \ln M, \ln \eta, \ln a, \gamma_0, t_0\} basis, we need to compute first the waveform derivatives. For the inspiral waveform \(\tilde{h}_{INS}(f)\) these are the following

\[
\frac{\partial \tilde{h}_{INS}}{\partial \ln A_{INS}} = \tilde{h}_{INS} \tag{A.1}
\]

\[
\frac{\partial \tilde{h}_{INS}}{\partial f_0 t_c} = i 2\pi (f/f_0) \tilde{h}_{INS} \tag{A.2}
\]

\[
\frac{\partial \tilde{h}_{INS}}{\partial \phi_c} = -i \tilde{h}_{INS} \tag{A.3}
\]

\[
\frac{\partial \tilde{h}_{INS}}{\partial \ln M} = i \frac{1}{128} \sum_{k=0}^{4} A_k (k-5) u^{k-5} \tilde{h}_{INS} \tag{A.4}
\]

\[
\frac{\partial \tilde{h}_{INS}}{\partial \ln \eta} = i \left( \frac{3}{128} \sum_{k=0}^{4} B_k u^{k-5} \tilde{h}_{INS} \right) \tag{A.5}
\]

where the parameter \(A_k\) are given by equations (25)-(29) and \(B_k\) are

\[
B_0 = \frac{\partial A_0}{\partial \ln \eta} = 0 \tag{A.6}
\]

\[
B_1 = \frac{\partial A_1}{\partial \ln \eta} = 0 \tag{A.7}
\]

\[
B_2 = \frac{\partial A_2}{\partial \ln \eta} = \left( \frac{-743}{378} + \frac{11}{3} \right) \eta^{-2/5} \tag{A.8}
\]

\[
B_3 = \frac{\partial A_3}{\partial \ln \eta} = \frac{48}{5} \pi \eta^{-3/5} \tag{A.9}
\]

\[
B_4 = \frac{\partial A_4}{\partial \ln \eta} = \left( -\frac{3058673}{127008} + \frac{5429}{504} \eta + \frac{617}{12} \eta^2 \right) \eta^{-4/5} \tag{A.10}
\]

The inspiral waveform has no dependency on \(\ln a, \gamma_0\) and \(t_0\). Therefore the remaining derivatives vanish

\[
\frac{\partial \tilde{h}_{INS}}{\partial \ln a} = \frac{\partial \tilde{h}_{INS}}{\partial \gamma_0} = \frac{\partial \tilde{h}_{INS}}{\partial t_0} = 0. \tag{A.11}
\]

The derivatives of the ringdown waveform \(\tilde{h}_{RD}(f)\) can be computed by taking into account the implicit dependencies of \(A_{RD}(A_{INS}, M, \eta, a), f_{QNR}(M, \eta, a), \) and \(Q(a)\). We get

\[
\frac{\partial \tilde{h}_{RD}}{\partial \ln A_{INS}} = \tilde{h}_{RD} \tag{A.12}
\]

\[
\frac{\partial \tilde{h}_{RD}}{\partial \ln M} = \frac{1}{6} \tilde{h}_{RD} - \frac{\partial \tilde{h}_{RD}}{\partial \ln f_{QNR}} \tag{A.13}
\]
Parameter estimation of compact binaries

\[ \frac{\partial \tilde{h}_{RD}}{\partial \ln \eta} = \frac{2}{5} \tilde{h}_{RD} + \frac{3}{5} \frac{\partial \tilde{h}_{RD}}{\partial \ln f_{QNR}} \]  
(A.14)

\[ \frac{\partial \tilde{h}_{RD}}{\partial \ln a} = \frac{9(-100 + 21(1-a)^{3/10}a)}{40(-100 + 63(1-a)^{3/10})(-1 + a)} \tilde{h}_{RD} + 
\quad + \frac{189a}{20(1-a)} \frac{\partial \tilde{h}_{RD}}{\partial \ln f_{QNR}} + 
\quad + \frac{9a}{20(1-a)} \frac{\partial \tilde{h}_{RD}}{\partial \ln Q} \]  
(A.15)

where

\[ \frac{\partial \tilde{h}_{RD}}{\partial \ln f_{QNR}} = A_{RD}e^{2if\pi t_0}f_{QNR} \]  
(A.16)

\[ \frac{\partial \tilde{h}_{RD}}{\partial \ln Q} = A_{RD}e^{2if\pi t_0}f_{QNR} \]  
(A.17)

The remaining derivatives are

\[ \frac{\partial \tilde{h}_{RD}}{\partial \gamma_0} = \frac{A_{RD}e^{2if\pi t_0}}{2\pi} \]  
(A.18)

\[ \frac{\partial \tilde{h}_{RD}}{\partial t_0} = 2if\pi \tilde{h}_{RD} \]  
(A.19)

\[ \frac{\partial \tilde{h}_{RD}}{\partial f_{0tc}} = \frac{\partial \tilde{h}_{RD}}{\partial \phi_c} = 0 \]  
(A.20)

Appendix B. Analytical analysis of the Fisher matrix for the ringdown waveform

In what follows we are interested in finding an analytical approximation to the Fisher matrix for the ringdown waveform in order to compare and verify the numerical results obtained with a Fortran code. For this comparison let us focus with the simpler case with five parameters, in which we are interested in computing the Fisher matrix with respect to the basis \((\ln A_{INS}, f_{0tc}, \phi_c, \ln M, \ln \eta)\), thus assuming that there are no uncertainties in the parameters \((\ln a, \gamma_0, t_0)\). Of course, the ringdown signal does not depend on \(f_{0tc}\) and \(\phi_c\), therefore the problem is reduced to three parameters, although the signal would depend only on two independent ones, e.g., \((\ln A_{RD}, \ln f_{QNR})\).

The way we proceed is to compute first the Fisher matrix of the ringdown signal with respect to \((\ln A_{RD}, \ln f_{QNR})\). Assuming constant noise over the bandwidth of the
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signal\|, and taking the δ-function approximation as in \[35\], the elements $\Gamma_{\ln A \ln A}$ and $\Gamma_{\ln f_{\text{QNR}} \ln f_{\text{QNR}}}$ can be computed analytically using Mathematica. For $\gamma_0 = 0$ we get\footnote{The approximation that the noise is constant over the bandwidth of the signal is a good approximation for all the detectors considered here when $a \geq 0.9$ corresponding to $\Delta f / f_{\text{QNR}} \leq 0.5$ as explained in \[35\]. In this paper we consider only the case in which $a = 0.98$.},

$$\Gamma_{\ln A \ln A} = \frac{2A_{\text{RD}}^2 Q (1 + 2Q^2)}{\pi (1 + 4Q^2) f_{\text{QNR}} S_n(f_{\text{QNR}})},$$  \hspace{1cm} (B.1)' \\
and

$$\Gamma_{\ln f_{\text{QNR}} \ln f_{\text{QNR}}} = \frac{A_{\text{RD}}^2 Q (1 + 4Q^2 + 8Q^8)}{\pi (1 + 4Q^2) f_{\text{QNR}} S_n(f_{\text{QNR}})}. \hspace{1cm} (B.2)$$

For the cross term $\Gamma_{\ln A \ln f_{\text{QNR}}}$, Finn’s approximation \[35\] can no longer be employed, because $\partial \tilde{h}_{\text{RD}} / \partial \ln f_{\text{QNR}}$ is not a symmetric function around $f_{\text{QNR}}$. In order to compute this term we will consider the following properties we have derived.

The reader should notice that for any set of parameters ($\ln A, \{\lambda^i\}$) and any waveform of the form

$$\tilde{h}(A, \{\lambda^i\}, f) = A \tilde{H}(\{\lambda^i\}, f), \hspace{1cm} (B.3)$$

the elements of the Fisher matrix satisfy the relations

$$\Gamma_{\ln A \ln A} = \frac{1}{2} \frac{\partial \Gamma_{\ln A \ln A}}{\partial \lambda^i}, \hspace{1cm} (B.4)$$

$$\Gamma_{ij} = \partial_i \Gamma_{\ln A \lambda^i} - (h | \partial_i h). \hspace{1cm} (B.5)$$

Then using the standard definition of SNR given by Eq. (9) we have

$$\Gamma_{\ln A \ln A} = \rho^2, \hspace{1cm} (B.6)$$

and consequently

$$\Gamma_{\ln A \ln A} = \frac{1}{2} \partial_i \rho^2. \hspace{1cm} (B.7)$$

These relations hold true for both the inspiral and the ringdown signals when considering $A$ to be the amplitude of the signal. In case of the ringdown signal, using equations \[B.1\], \[B.0\] and \[B.7\] we get

$$\rho_{\text{RD}}^2 = \frac{2A_{\text{RD}}^2 Q (1 + 2Q^2)}{\pi (1 + 4Q^2) f_{\text{QNR}} S_n(f_{\text{QNR}})}, \hspace{1cm} (B.8)$$

$$\Gamma_{\ln A \ln f_{\text{QNR}}} = \frac{1}{2} \rho_{\text{RD}}^2 \left(1 + S\right), \hspace{1cm} (B.9)$$

where

$$S = \frac{1}{S_n(f_{\text{QNR}})} \frac{d S_n(f_{\text{QNR}})}{d \ln f_{\text{QNR}}}. \hspace{1cm} (B.10)$$

Note that the δ-function approximation in this case is equivalent to consider $S = 0$, but this term is not negligible. For example if we consider initial LIGO and a total mass of 10, 20 or 100 $M_\odot$, the corresponding $S$ value would be 1.989, 1.959 and 1.264 respectively.

\footnote{If instead of using $\gamma_0 = 0$ we take $\gamma_0 = \pi/2$ then $\Gamma_{\ln A \ln A} \ln A_{\text{RD}}$ becomes $4A_{\text{RD}}^2 Q^3 / [\pi (1 + 4Q^2) f_{\text{QNR}} S_n(f_{\text{QNR}})]$ equivalent to Finn’s result \[35\].}
The Fisher matrix with respect to the basis \((\ln A_{\text{INS}}, \ln M, \ln \eta)\) (which naturally will be degenerate) can be easily be computed by taking into account the amplitude relation given by Eq. \ref{eq:RD} and considering

\[
\begin{pmatrix}
\frac{\partial \tilde{h}_{RD}}{\partial \ln A_{\text{INS}}} \\
\frac{\partial \tilde{h}_{RD}}{\partial \ln M} \\
\frac{\partial \tilde{h}_{RD}}{\partial \ln \eta}
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
1/6 & -1 & 0 \\
2/5 & 3/5 & 0
\end{pmatrix} \begin{pmatrix}
\frac{\partial \tilde{h}_{RD}}{\partial \ln A_{\text{RD}}} \\
\frac{\partial \tilde{h}_{RD}}{\partial \ln f_{\text{QNR}}} \\
\frac{\partial \tilde{h}_{RD}}{\partial \ln f_{\text{QNR}}}
\end{pmatrix}. \tag{B.11}
\]

If we define the constant matrix

\[
\mathcal{C} = \begin{pmatrix}
1 & 0 & 0 \\
1/6 & -1 & 0 \\
2/5 & 3/5 & 0
\end{pmatrix}, \tag{B.12}
\]

let \(\Gamma\) be the Fisher matrix with respect to \((\ln A_{\text{RD}}, \ln f_{\text{QNR}})\) and \(\hat{\Gamma}\) the Fisher matrix with respect to \((\ln A_{\text{INS}}, \ln M, \ln \eta)\), in this particular case, \(\Gamma\) and \(\hat{\Gamma}\) are related in the following way

\[
\hat{\Gamma} = \mathcal{C} \Gamma \mathcal{C}^T, \tag{B.13}
\]

where the superscript \(T\) indicates transposed matrix. The matrix \(\hat{\Gamma}\) has the elements:

\[
\hat{\Gamma}_{\ln A_{\text{INS}} \ln A_{\text{INS}}} = \rho_{\text{RD}}^2,
\]

\[
\hat{\Gamma}_{\ln A_{\text{INS}} \ln M} = \rho_{\text{RD}}^2 \frac{4 + 3S}{6},
\]

\[
\hat{\Gamma}_{\ln A_{\text{INS}} \ln \eta} = \rho_{\text{RD}}^2 \frac{1 - 3S}{10},
\]

\[
\hat{\Gamma}_{\ln M \ln M} = \rho_{\text{RD}}^2 \frac{2 \left(72Q^2 + 6S + 43\right) Q^2 + 6S + 25}{72Q^2 + 36},
\]

\[
\hat{\Gamma}_{\ln M \ln \eta} = \rho_{\text{RD}}^2 \frac{1}{60} \left(-72Q^2 + 9S + \frac{18}{2Q^2 + 1} + 13\right),
\]

\[
\hat{\Gamma}_{\ln \eta \ln \eta} = \rho_{\text{RD}}^2 \frac{1}{50} \left(36Q^2 - 12S + \frac{9}{2Q^2 + 1} - 4\right),
\]

and trivially

\[
\hat{\Gamma}_{f_{\text{QNR}} \lambda} = \hat{\Gamma}_{\phi_{\text{QNR}} \lambda} = 0. \tag{B.20}
\]

The analytical approximation and the numerical results are compared in Table \ref{table:B1} for the initial LIGO detector and two different values of the total mass.

Bibliography

Table B1. Comparison of the analytical approximation and the numerical results, as described in the text, for the elements of the ringdown Fisher matrix for the initial LIGO detectors assuming $a = 0.98$ and $\gamma_0 = 0$.

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[38] E. Berti, V. Cardoso, and C.M. Will, gr-qc/0512160