CROSSING OF THE \( w = -1 \) BARRIER IN VISCOS MODIFIED GRAVITY

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Abstract

We consider a modified form of gravity in which the action contains a power \( \alpha \) of the scalar curvature. It is shown how the presence of a bulk viscosity in a spatially flat universe may drive the cosmic fluid into the phantom region \( (w < -1) \) and thus into a Big Rip singularity, even if it lies in the quintessence region \( (w > -1) \) in the non-viscous case. The condition for this to occur is that the bulk viscosity contains the power \( (2\alpha - 1) \) of the scalar expansion. Two specific examples are discussed in detail. The present paper is a generalization of the recent investigation dealing with barrier crossing in Einstein’s gravity: I. Brevik and O. Gorbunova, *Gen. Relativ. Grav.* 37, 2039 (2005).

1 Introduction

Modified versions of Einstein’s gravity continue to attract interest. The motivation for this kind of generalization has its root in the well known observations of redshifts of type Ia supernovae \([1, 2, 3]\), as well as the anisotropy of the microwave background \([4, 5, 6]\). The data may be explained by dark energy which in turn may result from a cosmological constant, a cosmic fluid with a complicated equation of state, a scalar field with quintessence or phantom-like behaviour, or perhaps by some other kind of theory. For a review of the developments up to 2003, see Ref. \([7]\); some more recent papers are Refs. \([8, 9, 10]\). A forthcoming extensive review is \([11]\).

The simplest way of dealing with the expansion of the universe mathematically, is to allow for a cosmological constant. It corresponds to putting

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the parameter $w$ equal to -1 in the equation of state for the cosmic fluid,

$$p = (\gamma - 1)\rho \equiv w\rho. \quad (1)$$

The fluid is then the same as the extreme tensile stress vacuum ”fluid” in the de Sitter universe. More general types of fluid can be envisaged: thus the region $-1 < w < -1/3$ corresponds to a quintessence fluid, whereas the region $w < -1$ corresponds to the so-called phantom fluid. It should be noted that in both these cases $\rho + 3p \leq 0$. Thus, it follows from the Friedmann equation

$$\frac{\dot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) + \frac{\Lambda}{3} \quad (2)$$

that the curve for the scale factor $a$ is always concave upwards, when drawn as function of cosmic time $t$ (it is here assumed that $\Lambda \geq 0$). The inequality $\rho + 3p < 0$ breaks the so-called strong energy condition.

Astrophysical data make it quite possible that the value of $w$ is somewhat less than -1. It was not understood until rather recently that the universe will then have the strange property of passing into a Big Rip singularity in the future [12, 13, 14]. In turn, this can lead to bizarre consequences such as negative entropy [15]. As shown in a recent paper of Nojiri et al. [16] one can now actually classify as many as four different types of the Big Rip phenomenon. It should also be noted that quantum effects which are important near Big Rip may actually act against its occurrence [17].

After providing these introductory remarks we have now come to the theme of the present work. First, we shall endow the cosmic fluid with a bulk viscosity $\zeta$. Such a viscosity is obviously compatible with spatial isotropy of the universe. In a recent paper [18] we showed how the physically natural assumption of letting $\zeta$ be proportional to the scalar expansion in a spatially flat FLRW universe can drive the fluid into the phantom region even if it starts from the quintessence region in the non-viscous case. Next, we shall investigate if this nice property of the fluid still persists if we replace Einstein’s gravity with a more complicated theory. Specifically, we shall assume the following modified gravity model:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( f_0 R^a + L_m \right). \quad (3)$$

Here $f_0$ and $\alpha$ are constants, and $L_m$ is the matter Lagrangian. This is the model recently studied by Abdalla et al. [24]. The dimension of $f_0$ is cm$^{2(\alpha-1)}$. 

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Our question thus is: does there exist a crossing of the \( w = -1 \) barrier, from \( w > -1 \) to \( w < -1 \), when the field equations are derived from the action (3)? The answer turns out to be yes, if \( \zeta \) satisfies the condition given by Eq. (38) below. This is actually a quite natural generalization of the condition found earlier for the case of Einstein’s gravity [18].

The present paper is a continuation of a recent study of viscous cosmology in modified gravity [25]. Cf. also the very recent paper in [19] on cosmological models with viscous media.

We close this section by mentioning the following points. As discussed in [20] and [21], a dark fluid with a time dependent bulk viscosity can be considered as a fluid having an inhomogeneous equation of state. Also, the possibility of crossing the \( w = -1 \) barrier was considered in these references. Another point is that string/M theory may predict a modified gravity action with a negative power of \( R \) [22]. A consistent modified gravity theory which describes not only the cosmic acceleration but may comply with the Solar System tests was formulated in [23].

2 The fundamental formalism

We assume the spatially flat FRW metric,

\[ ds^2 = -dt^2 + a^2(t) \, dx^2, \]

(4)

and put \( \Lambda = 0 \). The four-velocity of the cosmic fluid is \( U^\mu = (U^0, U^i) \). In comoving coordinates, \( U^0 = 1, U^i = 0 \). In terms of the projection tensor \( h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu \) we can write the energy-momentum tensor as

\[ T_{\mu\nu} = \rho U_\mu U_\nu + (p - \zeta \theta) h_{\mu\nu}, \]

(5)

assuming constant temperature as well as vanishing shear viscosity in the fluid. Here the scalar expansion is \( \theta_{\mu\mu} = 3 \dot{a}/a \equiv 3H \), where \( H \) is the Hubble parameter. The effective pressure is defined as

\[ \tilde{p} = p - 3H \zeta. \]

(6)

From variation of the action (3) we obtain the equations of motion [24, 25]

\[ -\frac{1}{2} f_0 g_{\mu\nu} R^\alpha + \alpha f_0 R_{\mu\nu} R^{\alpha-1} - \alpha f_0 \nabla_\mu \nabla_\nu R^{\alpha-1} + \alpha f_0 g_{\mu\nu} \nabla^2 R^{\alpha-1} = 8\pi G T_{\mu\nu}, \]

(7)
where $T_{\mu\nu}$ corresponds to the term $L_m$ in (3). The notation is such that the values $\alpha = 1$, $f_0 = 1$ correspond to Einstein’s gravity.

We first consider the (00)-component of this equation. We calculate

$$R_{00} = -\frac{3\ddot{a}}{a}, \quad R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = 6(\dot{H} + 2H^2),$$

as well as the second order derivatives of $R^{\alpha-1}$. As $T_{00} = \rho$ we get after some algebra

$$\frac{1}{2}f_0R^\alpha - 3\alpha f_0(\dot{H} + H^2)R^{\alpha-1} + 3\alpha(\alpha - 1)f_0HR^{\alpha-2}\ddot{R} = 8\pi G\rho$$

(note that there is a printing error in a sign in the corresponding (00)-equation (16) in [25]).

Similarly, we consider the ($rr$)-component of Eq. (7). Since in a coordinate basis

$$R_{rr} = (\dot{H} + 3H^2)a^2, \quad T_{rr} = \tilde{p}g_{rr} = \tilde{p}a^2,$$

we get

$$\frac{1}{2}f_0R^\alpha - \alpha f_0(\dot{H} + H^2)R^{\alpha-1} + \alpha(\alpha - 1)f_0[2HR^{\alpha-2}\ddot{R}$$

$$+ (\alpha - 2)R^{\alpha-3}\dot{R}^2 + R^{\alpha-2}\ddot{R}] = -8\pi G\tilde{p}.$$

We consider next the local conservation equation for energy. An important property of Eq. (7) is the following: taking the covariant divergence of the expression on the left hand side one obtains zero. This was shown explicitly by Koivisto [26]. Consequently, we obtain the energy-momentum conservation equation in conventional form:

$$\nabla^\nu T_{\mu\nu} = 0.$$  

Energy-momentum conservation thus occurs as a consequence of the field equations, just as in Einstein’s theory. This property strongly supports the consistency of the modified gravity theory. The conservation equation for energy is now obtained by contracting Eq. (12) with $U^\mu$:

$$\dot{\rho} + (\rho + p)3H = 9\zeta H^2,$$

which can alternatively be written as

$$\dot{\rho} = -3\gamma\rho H + 9\zeta H^2.$$
Let us next search for a differential equation for \( H = H(t) \), wherein \( \{f_0, \alpha, \gamma\} \) are given constant input parameters, and where the bulk viscosity \( \zeta(t) \) is, to begin with, taken to be an arbitrary function of time. The natural way of accomplishing this task is to differentiate the left hand side of Eq. (9) with respect to time, thereafter insert for \( \dot{\rho} \) on the right hand side the expression in Eq. (14), and finally insert for \( \rho \) again from Eq. (9). We obtain in this way
\[
\frac{3}{2} \gamma f_0 H R^\alpha + 3\alpha f_0 H[2\dot{H} - 3\gamma(\dot{H} + H^2)]R^{\alpha-1}
+ 3\alpha(\alpha-1)f_0 H[(3\gamma-1)H\dot{R} + \ddot{R}]R^{\alpha-2} + 3\alpha(\alpha-1)(\alpha-2)f_0 H \dot{R}^2 R^{\alpha-3} = 72\pi G \zeta H^2.
\]
(15)
As \( R = 6(\dot{H} + 2H^2) \), this can be regarded as a nonlinear differential equation for \( H(t) \). As the equation is complicated, its mathematical structure and physical meaning are best discussed in terms of examples. In the next section we will discuss the simplest alternative, \( f_0 = 1, \alpha = 1 \).

3 The case \( f_0 = 1, \alpha = 1 \)

This case is Einstein’s gravity. From Eqs. (9), (11) and (15) we obtain respectively
\[
3H^2 = 8\pi G \rho,
\]
(16)
\[
2\dot{H} + 3H^2 = -8\pi G \tilde{p},
\]
(17)
\[
2\dot{H} + 3\gamma H^2 = 24\pi G \zeta H,
\]
(18)
in accordance with known results (cf., for instance, Ref. [27]).

Of particular interest, as shown in [13], is to take \( \zeta \) to be proportional to the scalar expansion through a constant, here called \( \tau_1 \). Thus
\[
\zeta = \tau_1 \theta = 3\tau_1 H,
\]
(19)
On thermodynamical grounds, \( \zeta \) has to be a positive quantity. For an expanding universe, therefore, \( \tau_1 \) is also positive. If the following condition is satisfied,
\[
\chi \equiv -\gamma + 24\pi G \tau_1 > 0,
\]
(20)
the equations of motion lead to the occurrence of a future Big Rip singularity in a finite time \( t \). Thus even if we start from an initial situation where the fluid is non-viscous and being in the quintessence region \( (\gamma > 0) \), the imposition
of a sufficiently large bulk viscosity will drive the fluid into the phantom region and thereafter inevitably into the Big Rip.

On basis of the ansatz (19) the equations of motion are easily solvable as functions of $t$. We take the initial time as $t = 0$, and give a subscript zero to quantities referring to this instant. Defining the time-dependent quantity $X = X(t)$ as

$$X = 1 - \chi t \sqrt{6\pi G \rho_0}, \quad (21)$$

we can express the solutions as

$$H = \sqrt{\frac{8\pi G}{3} \rho_0 X^{-1}}, \quad (22)$$

$$\rho = \rho_0 X^{-2}, \quad (23)$$

$$p = p_0 X^{-2}, \quad \text{with} \quad p_0 = w \rho_0, \quad (24)$$

$$R = 24\pi G \left( \chi + \frac{4}{3} \right) \rho_0 X^{-2}, \quad (25)$$

showing explicitly how the singularities occur when $X \to 0$.

4 The case $\alpha = 2$

We choose this case as the next step in complexity. The choice is motivated chiefly by mathematical convenience. It corresponds to a gravitational Lagrangian density proportional to $R^2$ in the action (3).

Of main interest is now Eq. (15), in order to make the appropriate determination of the time dependence of the bulk viscosity. The equation becomes

$$\frac{3}{2} \gamma f_0 R^2 - 6 f_0 [(3\gamma - 2) \dot{H} + 3\gamma H^2] R$$

$$+ 6 f_0 [(3\gamma - 1) H \dot{R} + \ddot{R}] = 72\pi G \zeta H. \quad (26)$$

In analogy with Eq. (22) we make the following ansatz for $H(t)$:

$$H = \frac{H_0}{1 - BH_0 t}, \quad (27)$$

where $B$ is a nondimensional constant. In order for a Big Rip to occur, $B$ has to be positive. Let us now take $\zeta(t)$ to be proportional to the cube of
the scalar expansion. Thus, denoting the proportionality constant by $\tau_2$, we put

$$\zeta = \tau_2\theta^3 = 27\tau_2 H^3. \quad (28)$$

The important point is that by inserting these expressions into Eq. (26) we find the time-dependent factors to drop out. What remains is an algebraic equation determining the value of $B$ in terms of the given initial conditions:

$$B^3 + \left(2 + \frac{3}{4}\gamma\right) B^2 + \frac{3}{2}\gamma B - \frac{9\pi G\tau_2}{f_0} = 0 \quad (29)$$

(note that the dimension of $f_0$ in this case is $\text{cm}^{2(\alpha-1)} = \text{cm}^2$).

The general analysis of this cubic equation leads to rather unwieldy expressions. The structure of the equation is most conveniently discussed by means of examples. We define for convenience the quantity

$$K = \frac{9\pi G\tau_2}{f_0}. \quad (30)$$

### 4.1 Non-viscous fluid

We consider this case for checking purposes. We now have $\tau_2 = 0$. From Eq. (29) we obtain, when the trivial solution $B = 0$ is disregarded,

$$B^3 + \left(2 + \frac{3}{4}\gamma\right) B^2 + \frac{3}{2}\gamma B = 0 \quad (31)$$

for any value of the constant $\gamma$. Thus both for a "normal" fluid ($\gamma \geq 1$) and for a quintessence fluid ($0 < \gamma < 2/3$) the two solutions of Eq. (31) are both negative (note that the product of the roots is equal to $3\gamma/2$). There is thus no Big Rip solution in the non-viscous case when $\gamma \geq 0$, as we would expect.

### 4.2 Inclusion of viscosity, $\tau_2 > 0$. Vacuum fluid

Consider first the vacuum fluid case, $\gamma = 0$. Equation (29) yields

$$B^3 + 2B^2 - K = 0. \quad (32)$$

If the left hand side of this expression is drawn as a function of $B$, we see that there is a local maximum at $B = -4/3$ and a local negative minimum at $B = 0$, irrespective of the magnitude of $K$. There exists thus one single
positive root of the equation, for all positive $K$. This root is caused by the viscosity, and leads to the future singularity. Let us assume that $K$ (or $\tau_2$) increases from $K = 0$ upwards; then we first encounter a parameter region in which there exist three real roots [28]. Let us assume this region in the following, and introduce an angle $\phi$ in the interval $0 < \phi < 180^0$ such that

$$\cos \phi = -\left(1 - \frac{27}{16}K\right).$$

(33)

The solutions can then be written in the form

$$B = \begin{cases} 
-\frac{2}{3} + \frac{4}{3}\cos \frac{\phi}{3}, \\
-\frac{2}{3} + \frac{4}{3}\cos \left(\frac{\phi}{3} + 120^0\right), \\
-\frac{2}{3} + \frac{4}{3}\cos \left(\frac{\phi}{3} + 240^0\right).
\end{cases}$$

(34)

As an example, we choose the value $K = 8/27$, corresponding to $\phi = 120^0$. Then, $B = \{-1.9196, -0.4351, 0.3547\}$, where the last positive solution describes the viscosity-generated Big Rip phenomenon.

4.3 Inclusion of viscosity, $\tau_2 > 0$. The fluid almost a vacuum fluid

As measurements show that the parameter $\gamma$ is close to unity, it is of physical interest to make a perturbative expansion around $\gamma = 0$. Let us consider the formalism to the first order in $\gamma$, assuming

$$|\gamma| \ll 1.$$  

(35)

The basic equation is Eq. (29), as before. Drawing the left hand side of the equation versus $B$, we see that the rightmost extremal point is a local minimum, with coordinates $(-\frac{3}{4}\gamma, -K)$. Therefore, the equation must have one positive root. Choosing again the value $K = 8/27$ as an example, we can as before define an angle $\phi$ in the region $0 < \phi < 180^0$; it is now determined by

$$\cos \phi = -\frac{1}{2} \left(1 - \frac{9}{16}\gamma\right).$$

(36)

The positive root is then found to be

$$B = -\frac{2}{3} \left(1 + \frac{3}{8}\gamma\right) + \frac{4}{3} \left(1 - \frac{3}{16}\gamma\right) \cos \frac{\phi}{3},$$

(37)

leading to the viscosity-generated Big Rip in this particular case. If $\gamma = 0$, the first member of Eq. (34) is recovered.
5 Remarks on the general case

For arbitrary values of $\alpha$ we can no longer give complete solutions, but it turns out that the most important part of the problem, namely to determine which form of viscosity leads to a Big Rip, can easily be dealt with. First, we assume the same form of $H = H(t)$ as before; cf. Eq. (27), where the value of the parameter $B$ depends on $\alpha$. This means that $R = 6(H + 2H^2)$ has the same form as before. In analogy with Eqs. (19) and (28) we assume next that $\zeta$ is proportional to the scalar expansion raised to the power $(2\alpha - 1)$. Denoting the proportionality constant by $\tau_\alpha$, our basic ansatz thus reads

$$\zeta = \tau_\alpha \theta^{2\alpha - 1} = \tau_\alpha (3H)^{2\alpha - 1}. \quad (38)$$

Upon insertion of the expressions for $H, R$ and $\zeta$ into Eq. (15) we see that the time-dependent factors again drop out, and we remain with the following equation determining the value of the constant $B$:

$$(B + 2)^{\alpha - 1} \{9(2 - \alpha)\gamma + 3[\alpha + 3\gamma + \alpha(2\alpha - 3)(3\gamma - 1)]B$$

$$+ 6\alpha(\alpha - 1)(2\alpha - 1)B^2\} - \frac{144}{f_0} \left(\frac{3}{2}\right)^\alpha \pi G \tau_\alpha = 0. \quad (39)$$

Because of its complexity this equation has to be analyzed in each specific case. Any positive root for $B$ leads to a Big Rip singularity, as follows from the form (27) for $H(t)$.

If $\alpha = 2$, Eq. (39) reduces to our previous Eq. (29). It is of interest to reconsider in more detail the case $\alpha = 1$: then Eq. (39) yields the solution

$$B = -\frac{3}{2} \gamma + 36\pi G \tau_1. \quad (40)$$

Since in this case

$$H_0 = \sqrt{\frac{8\pi G}{3} \rho_0} \quad (41)$$

according to Eq. (22), it follows upon insertion that

$$X = 1 - BH_0t = 1 - \chi t \sqrt{6\pi G \rho_0}, \quad (42)$$

where $\chi$ is defined by Eq. (20). This is agreement with our previous expression (21) for $X$ when $\alpha = 1$. 

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6 Summary

Our treatment was based on the form (3) for the action. Then assuming the cosmic fluid to possess a bulk viscosity $\zeta$ varying with the scalar expansion as in Eq. (38), we found how the fluid could in principle be driven into the Big Rip singularity, even if it stayed in the quintessence region in the non-viscous case. This is a property following directly from Eq. (27) for $H$, whenever the constant $B$ is positive. In general, $B$ is determined by Eq. (39). We worked out the solutions in reasonable detail when $\alpha = 1$ (Einstein’s gravity), and also when $\alpha = 2$, assuming a vacuum fluid ($\gamma \equiv 1 + w = 0$) as well as almost a vacuum fluid ($|\gamma| \ll 1$).

Finally, we mention that it would be of interest to incorporate the above formalism in the modified Gauss-Bonnet gravity interpreted as dark energy [29, 30].

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References


[26] T. Koivisto, gr-qc/0505128


