Spin-isospin stability of nuclear matter

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Abstract

We calculate the density-dependent spin-isospin asymmetry energy $J(k_f)$ of nuclear matter in the three-loop approximation of chiral perturbation theory. The interaction contributions to $J(k_f)$ originate from one-pion exchange, iterated one-pion exchange, and irreducible two-pion exchange with no, single, and double virtual $\Delta$-isobar excitation. We find that the approximation to $1\pi$-exchange and iterated $1\pi$-exchange terms (which leads already to a good nuclear matter equation of state by adjusting an emerging contact-term) is spin-isospin stable, since $J(k_{f0}) \simeq 24\text{ MeV} > 0$. The inclusion of the chiral $\pi N\Delta$-dynamics, necessary in order to guarantee the spin-stability of nuclear matter, keeps this property intact. The corresponding spin-isospin asymmetry energy $J(k_f)$ stays positive even for extreme values of an undetermined short-distance parameter $J_5$ (whose possible range we estimate from realistic NN-potentials). The largest positive contribution to $J(k_f)$ (a term linear in density) comes from a two-body contact-term with its strength fitted to the empirical nuclear matter saturation point.

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In recent years a novel approach to the nuclear matter problem has emerged. Its key element is a separation of long- and short-distance dynamics and an ordering scheme in powers of small momenta. At nuclear matter saturation density $\rho_0 \simeq 0.16\text{ fm}^{-3}$ the Fermi momentum $k_{f0}$ and the pion mass $m_\pi$ are comparable scales ($k_{f0} \simeq 2m_\pi$), and therefore pions must be included as explicit degrees of freedom in the description of the nuclear many-body dynamics. The contributions to the energy per particle $\bar{E}(k_f)$ of isospin-symmetric (spin-saturated) nuclear matter as they originate from chiral pion-nucleon dynamics have been computed up to three-loop order in Refs.[1, 2]. Both calculations are able to reproduce the empirical saturation point of nuclear matter by adjusting one single parameter (either a contact-coupling $g_0 + g_1 \simeq 3.23$ [1] or a cutoff scale $\Lambda \simeq 0.65\text{ GeV}$ [2]) related to unresolved short-distance dynamics.\footnote{Fitting a cutoff scale, as done in Ref.[2], must be viewed as a short-term intermediate step before an eventual full effective field theory calculation. Cutoff independence of physical observables is in fact a primary goal of effective field theory.} The basic mechanism for saturation in this approach is a repulsive contribution to the energy per particle $\bar{E}(k_f)$ generated by Pauli-blocking in second order (iterated) pion-exchange. As outlined in Sec. 2.5 of Ref.[2] this mechanism becomes particularly transparent by taking the chiral limit $m_\pi = 0$. In that case the interaction contributions to $\bar{E}(k_f)$ are completely summarized by an attractive $k_f^3$-term and a repulsive $k_f^4$-term where the parameter-free prediction for the coefficient of the latter is very close to the one extracted from a realistic nuclear matter equation of state.

In a recent work [3] we have extended the chiral approach to nuclear matter by including systematically the effects from $2\pi$-exchange with virtual $\Delta(1232)$-isobar excitation. The physical motivation for such an extension is threefold. First, the spin-isospin-3/2 $\Delta(1232)$-resonance...
is the most prominent feature of low-energy $\pi N$-scattering. Secondly, it is well known that $2\pi$-exchange between nucleons with excitation of virtual $\Delta$-isobars generates the needed isoscalar central $NN$-attraction [4] which in phenomenological one-boson exchange models is often simulated by a fictitious scalar "$\sigma$"-meson exchange. Thirdly, the delta-nucleon mass splitting $\Delta = 293\text{ MeV}$ is of the same size as the Fermi momentum $k_{f0} \approx 2m_\pi$ at nuclear matter saturation density and therefore pions and $\Delta$-isobars should both be treated as explicit degrees of freedom. A large variety of nuclear matter properties has been investigated in this extended framework in Ref.[3]. It has been found that the inclusion of the chiral $\pi N\Delta$-dynamics is able to remove most of the shortcomings of previous chiral calculations of nuclear matter [2, 5, 6, 7]. However, there remain open questions concerning the role of yet higher orders in the small momentum expansion and its "convergence". The relation of the fitted short-distance parameters [2] to those of few-nucleon systems is not clear at this moment. Also, a rigorous power counting that justifies the perturbative chiral expansion for nuclear matter has not yet been formulated.

Recent work by Bogner et al. [8] based on the universal low-momentum NN-potential $V_{\text{low-}}$ may open interesting perspectives in this direction.

Irrespective of such foundational questions it is also necessary to check various stability conditions for nuclear matter in the chiral framework. In a recent paper [9] we have analyzed spin-stability. It turned that the inclusion of the chiral $\pi N\Delta$-dynamics is essential in order to guarantee the spin-stability of isospin-symmetric nuclear matter. The truncation to fourth order terms in the small momentum expansion with interaction contributions only from $1\pi$- and iterated $1\pi$-exchange is spin-unstable [9]. This statement holds independently of the regularization scheme if the contact-terms (generating contributions linear in the nucleon density) are consistent with the empirical nuclear matter bulk properties: $E(k_{f0}) \simeq -16\text{ MeV}$ and $A(k_{f0}) \simeq 34\text{ MeV}$. Now, since a nucleon possesses four internal spin and isospin degrees of freedom one can prepare nuclear matter also in a spin-isospin mixed asymmetric configuration.

The stability of nuclear matter against such correlated spin-isospin deformations is the subject of the present paper. For recent work on generalized symmetry energy coefficients in the context of phenomenological Skyrme forces, also see Ref.[10]. Analogous earlier studies within Brueckner theory using the Reid soft-core NN-potential can be found in Ref.[11].

Let us begin with defining the spin-isospin asymmetry energy $J(k_f)$ of (infinite) nuclear matter. Consider this many-nucleon system in a state where the equal densities of the spin-up protons ($p\uparrow$) and the spin-down neutrons ($n\downarrow$) have an excess over the equal densities of the spin-down protons ($p\downarrow$) and the spin-up neutrons ($n\uparrow$). With the help of the spin- and isospin-projection operators, $(1 \pm \sigma_3)/2$ and $(1 \pm \tau_3)/2$, such a spin-isospin mixed asymmetric configuration is realized by the substitution:

$$\theta(k_f - |\vec{p}|) \rightarrow \frac{1 + \sigma_3\tau_3}{2} \theta(k_+ - |\vec{p}|) + \frac{1 - \sigma_3\tau_3}{2} \theta(k_- - |\vec{p}|),$$

in the medium insertion.\footnote{Medium insertion is a technical notation for the difference between the in-medium and vacuum nucleon propagator [2]. Effectively, it sums hole-propagation and the absence of particle-propagation below the Fermi surface $|\vec{p}| < k_f$.} Here, $k_+ = k_f(1 + \epsilon)^{1/3}$ and $k_- = k_f(1 - \epsilon)^{1/3}$ (with $\epsilon$ a small parameter) are different Fermi momenta, chosen such that the total nucleon density $\rho = (k_+^3 + k_-^3)/(3\pi^2) = 2k_f^3/3\pi^2$ stays constant. Note that Eq.(1) describes a rather peculiar asymmetric configuration of nuclear matter with equal densities of protons, neutrons, spin-up states and spin-down states: $\rho_p = \rho_n = \rho_\uparrow = \rho_\downarrow = k_f^3/3\pi^2$. The expansion of the energy per particle of spin-isospin polarized nuclear matter:

$$\bar{E}(k_+, k_-)_{3\pi-} \text{pol} = \bar{E}(k_f) + \epsilon^2 J(k_f) + \mathcal{O}(\epsilon^4), \quad k_\pm = k_f(1 \pm \epsilon)^{1/3},$$

\begin{equation}
\begin{aligned}
\theta(k_f - |\vec{p}|) \rightarrow \frac{1 + \sigma_3\tau_3}{2} \theta(k_+ - |\vec{p}|) + \frac{1 - \sigma_3\tau_3}{2} \theta(k_- - |\vec{p}|),
\end{aligned}
\end{equation}
defines the spin-isospin asymmetry energy $J(k_f)$. The obvious criterion for the spin-isospin stability of nuclear matter is then the positivity of the spin-isospin asymmetry energy: $J(k_f) > 0$. The energy per particle at fixed nucleon density $\rho$ must take on its absolute minimum value in the spin- and isospin-saturated configuration.

The first contribution to the spin-isospin asymmetry energy $J(k_f)$ comes from the kinetic energy $\sqrt{M^2 + \vec{p}^2} - M$ of a non-interacting relativistic Fermi gas of nucleons:

$$J(k_f) = \frac{k_f^2}{6M} - \frac{k_f^4}{12M^3},$$

with $M = 939$ MeV the (average) nucleon mass. The next term in this series, $k_f^6/16M^5$, is negligibly small at the densities of interest.

Next, we come to interaction contributions to $J(k_f)$. The closed in-medium diagrams related to one-pion exchange (Fock diagram) and iterated one-pion exchange (Hartree and Fock diagrams) are shown in Fig. 1. Differences in comparison to the calculation of the energy per particle $\bar{E}(k_f)$ in Ref. [2] occur only with respect to the factors emerging from the spin and isospin traces over closed nucleon lines and the radii $k_{\pm} = k_f(1 \pm \epsilon)^{1/3}$ of the Fermi spheres to be integrated over. After some analytical calculation we find the following contribution to the spin-isospin asymmetry energy $J(k_f)$ from the $1\pi$-exchange Fock diagram in Fig. 1 (including its relativistic $1/M^2$-correction):

$$J(k_f) = \frac{g_\pi^3 m_\pi^3}{(4\pi f_\pi)^2} \left\{ \frac{u^3}{9} - \frac{u}{2} + \left( \frac{2u}{9} + \frac{1}{8u} \right) \ln(1 + 4u^2) \right.\right.$$  
$$\left. + \frac{m_\pi^2}{M^2} \left[ \frac{19u^3}{18} - \frac{4u^5}{9} - \frac{u^2}{2} \arctan 2u - \frac{u}{72} (1 + 18u^2) \ln(1 + 4u^2) \right] \right\}. \quad (4)$$

Here, we have introduced the abbreviation $u = k_f/m_\pi$ where $m_\pi = 135$ MeV stands for the (neutral) pion mass. As usual $f_\pi = 92.4$ MeV denotes the weak pion decay constant and we choose the value $g_A = 1.3$ of the nucleon axial-vector coupling constant in order to have a pion-nucleon coupling constant of $g_{\pi N} = g_A M/f_\pi = 13.2$. In the second and third diagram in Fig. 1 the $1\pi$-exchange interaction is iterated (once) with itself. These second order diagrams carry the large scale enhancement factor $M$ (the nucleon mass). It stems from an energy denominator that is equal to a difference of small nucleon kinetic energies. With a medium insertion at each of two equally oriented nucleon propagators we obtain from the three-loop Hartree diagram in Fig. 1 the following contribution to the spin-isospin asymmetry energy:

$$J(k_f) = \frac{\pi g_\pi^4 M m_\pi^4}{6(4\pi f_\pi)^8} \left\{ \left( 15u + \frac{7}{2u} \right) \ln(1 + 4u^2) - 14u - 16u^2 \arctan 2u \right\}. \quad (5)$$
The right Fock diagram of iterated $1\pi$-exchange (see Fig. 1) with two medium insertions on non-neighboring nucleon propagators gives rise on the other hand to a contribution to the spin-isospin asymmetry energy of the form:

$$J(k_f) = \frac{\pi g_\lambda^4 M_m^4}{9(4\pi f_\pi)^4} \left[ \frac{21u}{5} - \frac{64u^3}{15} - \left( \frac{9 + 16u^2 + 64u^4}{15} \right) \arctan u \right. \\
+ \left( \frac{33}{10u} + \frac{14u}{3} \right) \ln(1 + u^2) - \left( \frac{3}{u} + 2u \right) \ln(1 + 4u^2) \\
\left. + (9 - 4u^2) \arctan 2u + (9 + 4u^2) \int^u_0 \arctan x - \arctan 2x \right] \frac{dx}{u(1 + 2x^2)} \right) . \quad (6)$$

This expression does not include the contribution of a linear divergence $\int_0^\infty dl$ of the momentum-space loop integral. In dimensional regularization such a linear divergence is set to zero, whereas in cut-off regularization it is equal to a momentum space cut-off $\Lambda$. The additional term specific for cut-off regularization will be given in Eq.(13). An in-medium diagram with three medium insertions represents Pauli-blocking effects in intermediate NN-states induced by the filled Fermi sea of nucleons. The unequal filling of the $(p\uparrow, n\downarrow)$ and $(p\downarrow, n\uparrow)$ Fermi seas shows its consequences in the spin-isospin asymmetry energy. After some extensive algebraic manipulations we end up with the following double-integral representation of the contribution to the spin-isospin asymmetry energy $J(k_f)$ from the Hartree diagram in Fig. 1 with three medium insertions:

$$J(k_f) = \frac{g_\lambda^4 M_m^4}{(4\pi f_\pi)^4 u^3} \int^u_0 dx \int^1_0 dy \left[ \left[ \frac{2uxy(3u^2 - 5x^2 y^2)}{(u^2 - x^2 y^2)} - (u^2 + 5x^2 y^2) \right] \\
\times \left[ \frac{2s^2 + s^4}{1 + s^2} - 2 \ln(1 + s^2) \right] + \frac{4u^2Hs^5(8s' - 9s)}{9(1 + s^2)^2} \right. \\
\left. + \left[ 2uxy + (u^2 - x^2 y^2) \right] \right] \\
\times \left[ (5 + s^2)(9s^2 - 16s's' + 16s^2) + 8s(1 + s^2)(2s'' - 10s' + 9s) \right] \frac{s^4}{9(1 + s^2)^3} \right) \right) . \quad (7)$$

where we have introduced several auxiliary functions:

$$H = \ln \frac{u + xy}{u - xy}, \quad s = xy + \sqrt{u^2 - x^2 + x^2 y^2}, \quad s' = u \frac{\partial s}{\partial u}, \quad s'' = u^2 \frac{\partial^2 s}{\partial u^2} . \quad (8)$$

Note that Eq.(7) stems from a nine-dimensional principal-value integral over the product of three Fermi spheres of varying radii $k_\pm = k_f(1 \pm \epsilon)^{1/3}$ which has been differentiated twice with respect to $\epsilon$ at $\epsilon = 0$. Of similar structure is the contribution to $J(k_f)$ from the iterated $1\pi$-exchange Fock diagram with three medium insertions. Because of the two different pion propagators in the Fock diagram one ends up (partially) with a triple-integral representation for its contribution to the spin-isospin asymmetry energy:

$$J(k_f) = \frac{g_\lambda^4 M_m^4}{72(4\pi f_\pi)^4 u^3} \int^u_0 dx \left[ G(9G_{20} + 2G_{11} + 9G_{02} - 16G_{01} - 9G) \\
+ 9G_{10}^2 + 2G_{01}G_{10} - 5G_{02}^2 + 4x^2 \int^1_0 dy \int^1_{-1} dz \frac{yz \theta(y^2 + z^2 - 1)}{yz} \right. \\
\times \left[ \frac{2s^3t^3(16s't - 9st - 12s't')}{(1 + s^2)(1 + t^2)} + \frac{s^2(t^2 - \ln(1 + t^2)}{(1 + s^2)^2} \\
\times \left[ (3 + s^2)(16ss' - 9s^2 - 16s^2) + 4s(1 + s^2)(12s' - 9s - 4s'') \right] \right] \right) . \quad (9)$$
Here, we have split into factorizable and non-factorizable parts. These two pieces are distinguished by whether the (remaining) nucleon propagator in the three-loop Fock diagram can be canceled or not by terms from the product of $\pi N$-interaction vertices. The factorizable terms can be expressed through the auxiliary function:

$$G = u(1 + u^2 + x^2) - \frac{1}{4x}[1 + (u + x)^2][1 + (u - x)^2] \ln \frac{1 + (u + x)^2}{1 + (u - x)^2},$$

and its partial derivatives for which we have introduced a (short-hand) double-index notation:

$$G_{ij} = x^i u^j \frac{\partial^j G}{\partial x^i \partial u^j}, \quad 1 \leq i, j \leq 2.$$  

For the presentation of the nonfactorizable terms one needs also copies of the quantities $s$ and $s'$ defined in Eq.(8) which depend (instead of $y$) on another directional cosine $z$:

$$t = xz + \sqrt{u^2 - x^2 + x^2 z^2}, \quad t' = u \frac{\partial t}{\partial u}. $$

In the chiral limit $m_\pi = 0$ the fourth order contributions in Eqs.(5-9) sum up to a negative $k_f^4$-term of the form: $J(k_f)|_{m_\pi=0} = -(g_A k_f/4\pi f_\pi)^4(M/405)(32\pi^2 + 741 + 1848 \ln 2)$. Finally, we give the expression for the linear divergence specific to cut-off regularization:

$$J(k_f) = \frac{10g_A^4 M\Lambda}{3(4\pi f_\pi)^3} k_f^3,$$

to which only the iterated $1\pi$-exchange Fock diagram (with two medium insertions) has contributed. In the case of the Hartree diagram the linear divergence drops out after taking the second derivative with respect to $\epsilon$. One observes that the term in Eq.(13) is just $-1/3$ of the corresponding contribution to the energy per particle $\bar{E}(k_f)$ (see Eq.(15) in Ref.[2]). In this context it is interesting to note that for terms linear in density $\rho$ the relation $3J(k_f)_{\text{lin}} = -\bar{E}(k_f)_{\text{lin}}$ holds generally. It is a consequence of the spin-isospin structure $3 - \vec{\sigma}_1 \cdot \vec{\sigma}_2 \vec{\tau}_1 \cdot \vec{\tau}_2$ of a Fierz-antisymmetric NN-contact interaction (see e.g. Eq.(35) in Ref.[12]).

Now we can turn to numerical results. In Fig. 2 we show the spin-isospin asymmetry energy $J(k_f)$ of nuclear matter as a function the nucleon density $\rho = 2k_f^3/3\pi^2$. The solid line corresponds to a calculation up to fourth order in small momenta. It includes besides the kinetic energy term Eq.(3) the contributions from static $1\pi$-exchange and iterated $1\pi$-exchange. For reasons of consistency we have dropped the small relativistic $1/M^2$-correction in Eq.(4) since it is of fifth order in the small momenta $k_f$ and $m_\pi$. The cut-off scale $\Lambda = 0.61$ GeV has been adjusted to the nuclear matter saturation point $\rho_0 = 0.173$ fm$^{-3}$ and $E(k_f) = -15.3$ MeV. The resulting value of the nuclear matter compressibility $K = k_f^2 E''(k_f) = 252$ MeV is consistent with a recent extrapolation from giant monopole resonances of heavy nuclei [13], which gave $K = (260 \pm 10)$ MeV. One can read off from Fig.2 a positive value of the spin-isospin asymmetry energy at saturation density: $J(k_{f_0}) = J(2m_\pi) = 23.9$ MeV. It indicates the spin-isospin stability of nuclear matter in this approximation. The largest positive contribution to $J(2m_\pi) = 23.9$ MeV comes from the term, Eq.(13), linear in density and amounts to $9.2$ MeV at saturation density $k_{f_0} = 2m_\pi$. Compared to that the largest negative contribution is $-32.8$ MeV and it stems from the iterated $1\pi$-exchange Fock diagram with two medium insertions, Eq.(6). The remaining numerically smaller contributions cancel each other to a large extent. It must

$^3$This provisional procedure introduces a model-dependence that lies outside effective field theory.
The spin-isospin asymmetry energy $J(k_f)$ of nuclear matter versus the nucleon density $\rho = \frac{2k_f^3}{3\pi^2}$. The solid line shows the result of a calculation up to fourth order in small momenta including $1\pi$-exchange and iterated $1\pi$-exchange. The cut-off scale $\Lambda = 0.61\,\text{GeV}$ has been adjusted to the saturation point: $\rho_0 = 0.173\,\text{fm}^{-3}$, $\bar{E}(k_f0) = -15.3\,\text{MeV}$. The positive values of $J(k_f)$ indicate the spin-isospin stability of nuclear matter in this approximation.

However, it should be noted that at this level of approximation, with interaction terms only from $1\pi$-exchange and iterated $1\pi$-exchange, nuclear matter is spin-unstable [9]. The inclusion of higher order terms (in particular $2\pi$-exchange with virtual $\Delta$-isobar excitation) is mandatory in order to achieve spin-stability of nuclear matter.

Therefore, we turn now to contributions to $J(k_f)$ of fifth order in the small momentum expansion. At three-loop order these terms are generated by (irreducible) two-pion exchange between nucleons. The corresponding one-loop diagrams for elastic NN-scattering are shown in Fig. 3. Since we are counting the delta-nucleon mass splitting $\Delta = 293\,\text{MeV}$ (together with $k_f$ and $m_\pi$) as a small momentum scale the diagrams with single and double virtual $\Delta(1232)$-isobar excitation shown in Fig. 4 belong to the same order. By closing the two open nucleon lines of the one-loop diagrams in Figs. 3,4 to either two or one ring one gets (in diagrammatic representation) the Hartree or Fock contribution to the energy density. The Hartree contri-

![Figure 2: The spin-isospin asymmetry energy $J(k_f)$ of nuclear matter versus the nucleon density $\rho = \frac{2k_f^3}{3\pi^2}$. The solid line shows the result of a calculation up to fourth order in small momenta including $1\pi$-exchange and iterated $1\pi$-exchange. The cut-off scale $\Lambda = 0.61\,\text{GeV}$ has been adjusted to the saturation point: $\rho_0 = 0.173\,\text{fm}^{-3}$, $\bar{E}(k_f0) = -15.3\,\text{MeV}$. The positive values of $J(k_f)$ indicate the spin-isospin stability of nuclear matter in this approximation.

![Figure 3: One-loop diagrams of irreducible two-pion exchange between nucleons.](image-url)
Figure 4: One-loop two-pion exchange diagrams with single and double $\Delta(1232)$-isobar excitation. Diagrams for which the role of both nucleons is interchanged are not shown.

bution to the spin-isospin asymmetry energy $J(k_f)$ vanishes identically because the relevant $2\pi$-exchange NN T-matrix in forward direction is spin-independent [4, 12]. The Fock contribution on the other hand is obtained by integrating the spin- and isospin-contracted T-matrix over the product of two Fermi spheres of radii $k_{\pm} = k_f(1 \pm \epsilon)^{1/3}$. We separate regularization dependent short-range contributions to the T-matrix (originating from the ultra-violet divergences of the one-loop diagrams in Figs. 3,4) from the unique long-range terms with the help of a twice-subtracted dispersion relation. The occurring subtraction constants give rise to a contribution to the spin-isospin asymmetry energy of the form:

$$J(k_f) = -B_3 \frac{k_f^3}{3M^2} + J_5 \frac{k_f^5}{M^4}. \quad (14)$$

The dimensionless parameters $B_3 = -7.99$ has been adjusted in Ref.[3] to the saturation minimum $\bar{E}(k_{f0}) = -16$ MeV. Again, we recognize in the first part of Eq.(14) the relation $3J(k_f)_{\text{lin}} = -E(k_f)_{\text{lin}}$ for terms linear in the density $\rho = 2k_f^3/3\pi^2$. The other subtraction constant $J_5$ in front of the $k_f^5/M^4$-term is (a priori) not constrained by any empirical (ground-state) property of nuclear matter. The long-range parts of the $2\pi$-exchange (two-body) Fock diagrams can be expressed as a dispersion-integral:

$$J(k_f) = \frac{1}{6\pi^3} \int_{2m_\pi}^{\infty} d\mu \left\{ \text{Im}(3W_C + 2\mu^2 V_T + 4\mu^2 W_T) \frac{k_f^3}{3} \left[ \frac{4k_f^2}{\mu} \frac{k_f^4}{\mu^3} - \mu \ln \left(1 + \frac{4k_f^2}{\mu^2}\right) \right] + \text{Im}(V_C + 3W_C + 2\mu^2 V_T + 6\mu^2 W_T) \left[ \frac{\mu k_f^3}{2} - \frac{k_f^3}{3\mu^3} + \frac{8k_f^5}{8k_f} \ln \left(1 + \frac{4k_f^2}{\mu^2}\right) \right] \right\} \quad (15)$$

where $\text{Im}V_C$, $\text{Im}W_C$, $\text{Im}V_T$ and $\text{Im}W_T$ are the spectral functions of the isoscalar and isovector central and tensor NN-amplitudes, respectively. Explicit expressions of these imaginary parts for the contributions of the triangle diagram with single $\Delta$-excitation and the box diagrams with single and double $\Delta$-excitation can be easily constructed from the analytical formulas given in Sec. 3 of Ref.[4]. The $\mu$- and $k_f$-dependent weighting functions in Eq.(15) take care that at low and moderate densities this spectral-integral is dominated by low invariant $\pi\pi$-masses $2m_\pi < \mu < 1$ GeV. The contributions to the spin-isospin asymmetry energy $J(k_f)$ from irreducible $2\pi$-exchange (with only nucleon intermediate states, see Fig. 3) can also be cast into the form Eq.(15). The corresponding non-vanishing spectral functions read [12]:

$$\text{Im}W_C(i\mu) = \sqrt{\mu^2 - 4m_\pi^2} \left[ 4m_\pi^2(1 + 4g_A^2 - 5g_A^4) + \mu^2(23g_A^4 - 10g_A^2 - 1) + \frac{48g_A^4m_\pi^4}{\mu^2 - 4m_\pi^2} \right], \quad (16)$$
Figure 5: Hartree and Fock three-body diagrams related to $2\pi$-exchange with single virtual \( \Delta \)-isobar excitation. They represent interactions between three nucleons in the Fermi sea. The combinatoric factor is 1 for each diagram.

\[
\text{Im}V_T(i\mu) = -\frac{6g_A^4\sqrt{\mu^2 - 4m^2}}{\mu(4f_\pi)^4}. \tag{17}
\]

Next, we come to the additional $2\pi$-exchange three-body terms which arise from Pauli blocking of intermediate nucleon states (i.e. from the $(1 \pm \sigma_3 \tau_3)\theta(k_\pm - |\vec{p}|)$ terms in the in-medium nucleon propagators [2]). The corresponding closed Hartree and Fock diagrams with single virtual $\Delta$-excitation are shown in Fig. 5. The contribution of the left three-body Hartree diagram to the spin-isospin asymmetry energy $J(k_f)$ has the following analytical form:

\[
J(k_f) = \frac{g_A^4m_\pi^6u^2}{27\Delta(2\pi f_\pi)^4}\left[\left(\frac{27}{4} + 8u^2\right)\ln(1 + 4u^2) + 2u^4(1 - 9\zeta) - 22u^2 - \frac{5u^2}{1 + 4u^2}\right]. \tag{18}
\]

The delta propagator shows up in this expression merely via the (reciprocal) mass-splitting $\Delta = 293$ MeV. Furthermore, we have already inserted in Eq.(18) the empirically well-satisfied relation $g_{\pi N\Delta} = 3g_{\pi N}/\sqrt{2}$ for the $\pi N\Delta$-coupling constant. The parameter $\zeta = -3/4$ has been introduced in Sec.2 of Ref.[3] in order to reduce a too strongly repulsive $\rho^2$-term in the energy per particle $\bar{E}(k_f)$. It controls the strength of a three-nucleon contact interaction $\sim (\zeta g_A^4/\Delta f_\pi^4)(\bar{N}N)^3$ which has the property that it contributes equally but with opposite sign to the energy per particle $\bar{E}(k_f)$ and the spin-isospin asymmetry energy $J(k_f)$. The contribution of both three-body Fock diagrams in Fig.5 to the spin-isospin asymmetry energy $J(k_f)$ can be represented as:

\[
J(k_f) = \frac{g_A^4m_\pi^6}{108\Delta(4\pi f_\pi)^4u^3}\int_0^u dx\left\{ -4G_{S01}G_{S10} - 10G_{S01}^2 - 18G_{S10}^2 \\
+ 2G_S(9G_S + 16G_{S01} - 9G_{S02} - 2G_{S11} - 9G_{S20}) - 2G_{T01}G_{T10} \\
- 17G_{T10}^2 - 9G_{T10}^2 + G_T(9G_T + 16G_{T01} - 9G_{T02} - 2G_{T11} - 9G_{T20}) \right\}, \tag{19}
\]

with the two auxiliary functions:

\[
G_S = \frac{4ux}{3}(2u^2 - 3) + 4x\left[ \arctan(u + x) + \arctan(u - x) \right] \\
+ (x^2 - u^2 - 1)\ln\frac{1 + (u + x)^2}{1 + (u - x)^2}, \tag{20}
\]

\[
G_T = \frac{ux}{6}(8u^2 + 3x^2) - \frac{u}{2x}(1 + u^2)^2 \\
+ \frac{1}{8}\left[ \frac{(1 + u^2)^3}{x^2} - x^4 + (1 - 3u^2)(1 + u^2 - x^2) \right]\ln\frac{1 + (u + x)^2}{1 + (u - x)^2}. \tag{21}
\]
Figure 6: The spin-isospin asymmetry energy \( J(k_f) \) of nuclear matter versus the nucleon density \( \rho = 2k_f^3/3\pi^2 \). In comparison to Fig. 2 the effects from 2\( \pi \)-exchange with single and double virtual \( \Delta \)-isobar excitation are now included. The solid, dashed, and dashed-dotted curves correspond to the choices \( J_5 = 0 \), \(-0.95 \), and \(-19 \) of the short-range parameter \( J_5 \) introduced in Eq.(14). The positive values of \( J(k_f) \) ensure the spin-isospin stability of nuclear matter.

The double-indices on \( G_S \) and \( G_T \) have the same meaning as explained in Eq.(11) for the function \( G \).

In Fig.6 we show again the spin-isospin asymmetry energy \( J(k_f) \) of nuclear matter as a function of the nucleon density \( \rho = 2k_f^3/3\pi^2 \). The solid line includes all the contributions from chiral 1\( \pi \)- and 2\( \pi \)-exchange written down in Eqs.(3-9,14-19). The (yet undetermined) short-range parameter \( J_5 \) has been set to zero, \( J_5 = 0 \). We note as an aside that the term linear in the density and the cut-off \( \Lambda \), Eq.(13), is now not counted extra since the parameter \( B_3 = -7.99 \) \[3\] collects all such possible terms. Numerically, these two terms linear in density are anyhow almost identical. One observes in Fig. 6 a positive spin-isospin asymmetry energy \( J(k_f) \) which rises monotonically with the density \( \rho \). The inclusion of the chiral \( \pi N \Delta \)-dynamics does therefore not disturb the spin-isospin stability of nuclear matter. It is also interesting to look at numerical values of \( J(k_f) \) and their decomposition. At a Fermi momentum of \( k_f = 2m_\pi \) (corresponding to \( \rho = 0.173 \text{ fm}^{-3} \)) the spin-isospin asymmetry energy is now \( J(2m_\pi) = 69.5 \text{ MeV} \) (setting \( J_5 = 0 \)). The most significant changes in comparison to the previous fourth order calculation come from the two-body Fock and three-body Hartree contributions Eqs.(15,18) which amount together to \( 30.6 \text{ MeV} + 20.7 \text{ MeV} = 51.3 \text{ MeV} \). About one third thereof (namely \( 16.6 \text{ MeV} \)) stems from the three-body contact interaction proportional to \( \zeta = -3/4 \).

The size of the short-distance parameter \( J_5 \) in Eq.(14) is still open and large negative values could endanger the spin-isospin stability. In order to get an estimate of \( J_5 \) we bring into play the complete set of four-nucleon contact-couplings written down in Eqs.(3,4) of Ref.[14]. This set represents the most general short-range NN-interaction quadratic in momenta and it involves seven low-energy constants \( C_1, \ldots , C_7 \). After computing the spin-isospin asymmetry energy
from the corresponding contact-potential in Hartree-Fock approximation we find:

\[ J_5 = \frac{M^4}{18\pi^2} (C_2 - 4C_1) = \frac{M^4}{144\pi^3} \left[ 3C(3P_1) + C(3P_0) + 3C(3P_1) + 5C(3P_2) \right]. \quad (22) \]

In the second line of Eq.(22) we have reexpressed the relevant linear combination of \( C_{1,2} \) through the so-called spectroscopic low-energy constants which characterize the short-range part of the NN-potential in the spin-singlet and spin-triplet \( S \)- and \( P \)-wave states. In that representation we obtain from the entries of table IV in Ref.[14] for the three NN-potentials \(^4\) CD-Bonn, Nijm-II, and AV-18 the numbers: \( J_5 = -1.34, -0.57, \) and \(-0.94\). The dashed line in Fig.6 shows the spin-isospin asymmetry energy \( J(k_f) \) which results from taking their average value \( J_5 = -0.95\). The corresponding reduction of the spin-isospin asymmetry energy is negligible. The dashed-dotted curve in Fig.6 corresponds to the extreme choice \( J_5 = -19\). One can see that even with such a large negative \( J_5 \)-value the spin-isospin stability of nuclear matter remains still preserved. We can therefore conclude that spin-isospin stability is a robust property of the chiral approach to nuclear matter (at least in the three-loop approximation). This is a important finding.

In summary we have investigated in this work the spin-isospin stability of nuclear matter in the framework of chiral perturbation theory. For that purpose we have calculated the density-dependent spin-isospin asymmetry energy \( J(k_f) \) of nuclear matter to three-loop order. The interaction contributions to \( J(k_f) \) originate from \( 1\pi \)-exchange, iterated \( 1\pi \)-exchange, and (irreducible) \( 2\pi \)-exchange with no, single, and double virtual \( \Delta \)-isobar excitation. We have found that the approximation to \( 1\pi \)- and iterated \( 1\pi \)-exchange terms is spin-isospin stable, since \( J(k_f) > 0 \). The inclusion of the chiral \( \pi N \Delta \)-dynamics (necessary to ensure the spin-stability [9] of nuclear matter) keeps this property intact. The largest positive contribution to \( J(k_f) \) comes from a two-body contact interaction with its strength fitted to the empirical nuclear matter saturation point.

References


\(^4\)The short-distance structure of realistic NN-potentials and effective field theory could be very different. The idea here is simply to explore the extreme possible range of \( J_5 \).