THE CONCEPT OF MASS IN THE EINSTEIN YEAR *

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Abstract
Various facets of the concept of mass are discussed. The masses of elementary particles and the search for higgs. The masses of hadrons. The pedagogical virus of relativistic mass.

1 From “Principia” to Large Hadron Collider (LHC)

The term “mass” was introduced into mechanics by Newton in 1687 in his “Principia” [1]. He defined it as the amount of matter. The generally accepted definition of matter does not exist even today. Some authors of physics text-books do not consider photons – particles of light – as particles of matter, because they are massless. For the same reason they do not consider as matter the electromagnetic field. It is not quite clear whether they consider as matter almost massless neutrinos, which usually move with velocity close to that of light. Of course it is impossible to collect a handful of neutrinos similarly to a handful of coins. But in many other respects both photons and neutrinos behave like classical particles, while the electromagnetic field is the basis of our understanding of the structure of atoms. On the other hand, the so-called weak bosons $W^+, W^-, Z^0$ are often not considered as particles of matter because they are too heavy and too short-lived.

Even more unusual are such particles as gluons and quarks. Unlike atoms, nucleons, and leptons, they do not exist in a free state: they are permanently confined inside nucleons and other hadrons.

There is no doubt that the problem of mass is one of the key problems of modern physics. Though there is no common opinion even among the

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experts what is the essence of this problem. For most of particle theorists, as well as members of LHC community, the solution of the problem is connected with the quest and discovery of the higgs – scalar boson which in the Standard Model is responsible for the masses of leptons and quarks and their electroweak messengers: $W$ and $Z$. The discovery of higgs and the study of higgs sector might elucidate the problem of the pattern of hierarchy of masses of leptons and quarks: from milli electron Volts for neutrinos to about 180 GeV for $t$-quark. For many physicists it is a QCD problem: how light quarks and massless gluons form massive nucleons and atomic nuclei. Still for majority of confused students and science journalists there is no difference between mass of a body $m$ and its energy $E$ divided by $c^2$: they believe in the “most famous formula $E = mc^2$”.

If higgs exists, its discovery will depend on the funding of the particle physics. In 1993 the termination of the SSC project sent the quest for the higgs into a painful knockdown. The decision not to order in 1995 a few dozen of extra superconducting cavities prevented, a few years later, LEP II from crossing the 115 GeV threshold for the mass of the higgs.

If we are lucky and higgs is discovered around year 2010 at LHC, then the next instrument needed to understand what keeps the masses of the higgs below 1 TeV scale, is ILC (International linear collider). This machine would provide a clean environment for the study of higgs production and decays. It could also be used for discovery and study of light supersymmetric particles (SUSY). A prototype of ILC was suggested a few years ago by DESY as the project TESLA. There was no doubt that if funded, TESLA would work, but the funding was not provided by the German government. The new variant of ILC envisions increasing the maximal center of mass energy of colliding electron and positron from 0.5 TeV to 1 TeV. If everything goes well, ILC can start before 2020.

Further increase of energy, to say, 5 TeV, would call for a machine of the type of CLIC (Compact linear collider) the project of which is under discussion at CERN for more than a decade. In this machine the role of clystrons is supposed to play a low energy but very high current “decelerator” the energy of which would be pumped into the high energy accelerator part of CLIC. Unlike situation with ILC, even the mere feasibility of CLIC is not clear now. Special experimental research to ascertain the feasibility is going on at CERN.

The discussion of higgses, neutrinos and QCD in connection with the fundamental problems of mass is often accompanied and even overshadowed by a “pseudoproblem” of the so-called “relativistic mass” (see section 5).

## 2 Mass in Newtonian Mechanics

The more basic is a physical notion, the more difficult to define it in words. A good example give the 1960s editions of “Encyclopedia Britannica” where energy is defined in terms of work, while the entry “work” refers to labour and professional unions. Most people have intuitive notions of
space and time. Every physicist has intuitive notions of energy, mass, and momentum. But practically everybody has difficulties in casting these notions into words without using mathematics.

Though the definition of mass ("Definition I: The quantity of matter is the measure of the same, arising from its density and bulk conjointly") given by Newton in his "Principia" [1] was so unclear that scholars are discussing its logical consistency even today, the equations of Newtonian mechanics are absolutely self-consistent. Mass $m$ enters in the relations of velocity $\mathbf{v} = d\mathbf{r}/dt$ and momentum $\mathbf{p}$:

$$\mathbf{p} = m\mathbf{v} ,$$

as well as acceleration $\mathbf{a} = d\mathbf{v}/dt$ and force $\mathbf{F}$:

$$\mathbf{F} = d\mathbf{p}/dt = m\mathbf{a} .$$

It also enters in the equation defining the force of gravity with which a body with mass $m_1$ at point $\mathbf{r}_1$ attracts another body with mass $m_2$ at point $\mathbf{r}_2$:

$$\mathbf{F}_g = -Gm_1m_2\mathbf{r}/r^3 .$$

Here $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, $r = |\mathbf{r}|$, while $G$ is the famous Newton constant:

$$G = 6.67 \cdot 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2} .$$

The kinetic energy of a body is defined as

$$E_k = \mathbf{p}^2/2m = m\mathbf{v}^2/2 .$$

The potential gravitational energy:

$$U_g = -Gm_1m_2/r ,$$

while the total energy in this case is

$$E = E_k + U_g .$$

The total energy is conserved. When a stone falls on the earth, its potential energy decreases (becomes more negative), kinetic energy increases: so that the total energy does not change. When the stone hits the ground, its kinetic energy is shared by the ambient molecules raising the local temperature.

One of the greatest achievements of the XIX century was the formulation of the laws of conservation of energy and momentum in all kinds of processes.

At the beginning of the XX century it was realized that conservation of energy is predetermined by uniformity of time, while conservation of momentum — by uniformity of space.

But let us return to the notion of force. People strongly felt the force of gravity throughout the history of mankind, but only in XVII century the equations (3) and (6) were formulated.
An important notion in this formulation is the notion of gravitational potential $\varphi_g$. The gravitational potential of a body with mass $m_1$ is

$$\varphi_g = -\frac{Gm_1}{r}$$  \hspace{1cm} (8)

Thus the potential energy of a body with mass $m_2$ in a potential (8) is

$$U_g = m_2\varphi_g$$  \hspace{1cm} (9)

which coincides with eq. (6).

A century later similar equations were formulated for another long-range interaction, the electrical one:

$$F_e = e_1 e_2 r / r^3$$  \hspace{1cm} (10)

$$U_e = e_1 e_2 / r$$  \hspace{1cm} (11)

$$\varphi_e = e_1 / r$$  \hspace{1cm} (12)

$$U_e = e_2 \varphi_e$$  \hspace{1cm} (13)

In these equations, which define the Coulomb force, Coulomb potential energy, and Coulomb potential respectively, $e_1$ and $e_2$ are electrical charges of two bodies.

An important role in the theory of electricity is played by the strength of electric field $E$. Charge $e_1$ creates field with strength

$$E = e_1 r / r^3$$  \hspace{1cm} (14)

Thus

$$F_e = e_2 E$$  \hspace{1cm} (15)

As most of matter around us is electrically neutral, the electrical interaction was known for centuries only as a kind of trifle. Unlike mass $m$, which is always positive, the charge $e$ has two varieties: positive and negative. Two charges of opposite sign attract each other, while those of the same sign are repelling. Protons residing in the nucleus of an atom have positive charge, the charge of electrons, which form atomic shells, is negative. As a result the atom is electrically neutral.

Electrical interaction and its ramifications determine the main features of atoms, molecules, condensed matter, and biological cells. Gravitational interaction is too feeble to play any role at that level. To see this consider an electron and a proton. Their masses are

$$m_e = 9.1 \cdot 10^{-31} \text{ kg}$$  \hspace{1cm} (16)

$$m_p = 1.7 \cdot 10^{-27} \text{ kg}$$  \hspace{1cm} (17)

Their electric charges:

$$e_p = -e_e = e = 4.8 \cdot 10^{-10} \text{ esu}$$  \hspace{1cm} (18)

where esu denotes electrostatic unit:

$$(1 \text{ esu})^2 = 10^{-9} \text{ kg m}^3 \text{ s}^{-2}$$  \hspace{1cm} (19)
Hence
\[-e_e e_p = 2.3 \cdot 10^{-28} \text{ kg m}^3\text{s}^{-2} .\]  
(20)

On the other hand
\[Gm_em_p = 1.03 \cdot 10^{-67} \text{ kg m}^3\text{s}^{-2} ,\]  
(21)

Thus in an atom the gravitational force is \(\sim 10^{-40}\) of electric one.

The importance of gravity for our every day life is caused by the huge number of atoms in the earth, and hence by its very large mass:
\[M = 5.98 \cdot 10^{24} \text{ kg} .\]  
(22)

Taking into account the value of the radius of the earth
\[R = 6.38 \cdot 10^6 \text{ m} ,\]  
(23)

we find the gravitational force of the earth acting on a body with mass \(m\) close to its surface:
\[F_g = m g ,\]  
(24)

where \(g\) is acceleration directed towards the center of the earth:
\[g = |g| = \frac{6.67 \cdot 10^{-11} \cdot 5.98 \cdot 10^{24}}{(6.38 \cdot 10^6)^2} \text{ m s}^{-2} = 9.8 \text{ m s}^{-2} .\]  
(25)

Let us note that for gravity \(g\) plays the role of the strength of gravitational field, which is analogous to the role of \(E\) for electricity. The acceleration \(g\) does not depend on the mass or any other properties of the attracted body. In that sense it is universal. This universality was established by Galileo early in the 17th century in his experiments with balls rolling down an inclined plane. One can see this plane, with little bells ringing when a ball passes them, in Florence. (Apocryphal history tells that Galileo had discovered universality of \(g\) by dropping balls from the Tower of Pisa.)

Gravitational and electric interactions are the only long-range interactions the existence of which has been established. Two other fundamental interactions, referred to as strong and weak, have very short ranges: \(10^{-15}\) m and \(10^{-18}\) m respectively. Their first manifestations were discovered about a century ago in the form of radioactivity. Their further study has lead to new disciplines: nuclear physics and the physics of elementary particles.

Quite often you might find in the literature, in the discussion of Newtonian mechanics, the terms “inertial mass” \(m_i\) and “gravitational mass” \(m_g\). The former is used in equations (1), (2), (5), defining the inertial properties of bodies. The latter is used in equations (3), (6), (24), describing the gravitational interaction. After introducing these terms a special law of nature is formulated:
\[m_i = m_g ,\]  
(26)

which is called upon to explain the universality of \(g\).
However Galileo had discovered this universality before the notion of mass was introduced by Newton, while from Newton equations (1), (2), (3) the universality of $g$ follows without additional assumptions. Thus the notions and notations $m_i$ and $m_g$ are simply redundant. As we will see later, their introduction is not only redundant, but contradicts the General Theory of Relativity, which explains why the same mass $m$ enters equations (1) - (3).

The advocates of $m_i$ and $m_g$ argue by considering the possibility that in the future the more precise experimental tests might discover a small violation of Galilean universality. But that would mean that a new feeble long-range force exists in nature. In literature this hypothetical force is often referred to as a “fifth force” (in addition to the four established ones). When and if the “fifth force” is discovered it should be carefully studied. But at present it should not confuse the exposition of well established physical laws. Especially confusing and harmful are $m_i$ and $m_g$ in the text-books for students.

At that point it is appropriate to summarize the properties of mass in Newtonian mechanics:

1. Mass is a measure of the amount of matter.
2. Mass of a body is a measure of its inertia.
3. Masses of bodies are sources of their gravitational attraction to each other.
4. Mass of a composite body is equal to the sum of masses of the bodies that constitute it; mathematically that means that mass is additive.
5. Mass of an isolated body or isolated system of bodies is conserved: it does not change with time.
6. Mass of a body does not change in the transition from one reference frame to another.

### 3  Mass in Special Relativity

Of great conceptual importance in modern science is the principle of relativity first stated by Galileo: A rectilinear motion of a physical system with constant velocity relative to any external object is unobservable within the system itself. The essence of this principle was beautifully exposed in the famous book “Dialogue Concerning the Chief World Systems – Ptolemaic and Copernican”, published in 1632 [2]:

“Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need throw it no more strongly in one
direction than another, the distances being equal: jumping with your feet together, you pass equal spaces in every direction. When you have observed all these things carefully (though there is no doubt that when the ship is standing still everything must happen in this way). Have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that, you will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. In jumping, you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than toward the prow even though the ship is moving quite rapidly, despite the fact that during the time you are in the air the floor under you will be going in a direction opposite to your jump. In throwing something to your companion, you will need no more force to get it to him whether he is in the direction of the bow or the stern, with yourself situated opposite. The droplets will fall as before into the vessel beneath without dropping toward the stern, although while the drops are in the air the ship runs many spans. The fish in their water will swim toward the front of their bowl with no more effort than toward the back, and will go with equal ease to bait placed anywhere around the edges of the bowl. Finally the butterflies and flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they will have been separated during long intervals by keeping themselves in the air. And if smoke is made by burning some incense, it will be seen going up in the form of a little cloud, remaining still and moving no more toward one side than the other. The cause of all these correspondences of effects is the fact that the ship's motion is common to all the things contained in it, and to the air also. That is why I said you should be below decks; for if this took place above in the open air, which would not follow the course of the ship, more or less noticeable differences would be seen in some of the effects noted.

Sometimes one can hear that the ship of Galileo was discussed two centuries earlier by cardinal Nicolaus Cusanus (1401 – 1464) in his book “De docta ignorantia” (“On the scientific ignorance”) published in 1440. Indeed, one can read in volume II, at the beginning of chapter XII “The properties of the earth”:

“It is clear to us that the earth is actually moving, though we do not see this, as we feel the movement only through comparison with a point at rest. Somebody on a ship in the middle of waters, without knowing that water is flowing and without seeing the shores, how could he ascertain that the ship is moving?” [3] . The relative character of motion is expressed in these lines quite clearly. But the cabin of Galileo’s ship is full of various phenomena and experiments, proving that observable effects look the same in any inertial reference frame. At this point we define an inertial reference frame, as that which moves rectilinearly with constant velocity with respect to the stars. We shall give a more accurate definition

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1I am grateful to Peter Zerwas for arousing my interest to Nicolaus Cusanus.
when considering General Theory of Relativity.

If the velocity of the ship is \( u \) and it moves along the axis \( x \), the coordinates of two inertial frames are connected by equations:

\[
t' = t , \quad (27)
\]
\[
x' = x + ut , \quad (28)
\]
\[
y' = y , \quad z' = z , \quad (29)
\]
where \( u = |u| \), the primed coordinates refer to the shore, while unprimed to the ship. From the definition of velocity \( \mathbf{v} = d\mathbf{r}/dt \) one easily sees that

\[
\mathbf{v}' = \mathbf{v} + \mathbf{u} \quad (30)
\]

and that \( \mathbf{v}', \mathbf{p}', \mathbf{a}', \mathbf{F}', \mathbf{F}_E', \mathbf{F}_s', \mathbf{U}_g', \mathbf{U}_e', \mathbf{F}_E' \) satisfy the same equations (1) - (11) as their unprimed analogues.

Galilean principle of relativity is the quintessence of Newtonian mechanics. Nevertheless the latter is called non-relativistic mechanics, as opposed to Einsteinian mechanics which is called relativistic. This is one of many examples of lack of complete consistency in the language of physics which is a natural product of its evolution. The point is that Newtonian mechanics satisfies the Galilean principle of relativity only partially. The cabin of the original Galilean ship did not contain apparatuses that were able to measure the velocity of light. This velocity was first established in 1676 by Danish astronomer O. Roemer (1644 - 1710), who deduced from the observations of the moons of Jupiter, performed by J. Cassini, that it is \( 2.4 \cdot 10^5 \) km s\(^{-1} \). Further measurements during three centuries established its present value: \( c = 3 \cdot 10^5 \) km s\(^{-1} \).

Of greatest importance was the discovery made two centuries later by American physicists A. Michelson and E. Morley. By using a special two arm rotating interferometer they established that the velocity of light did not depend on the angle between the light ray and the vector of velocity of the earth on its journey around the sun. In this experiment the earth itself played the role of Galilean ship. That result signalled that the simple law (30) of addition of velocities is not valid for light.

This, in its turn, meant that the coordinate transformations (27) - (30) should be changed when \( v \) and/or \( u \) (due to relative character of velocity) are of the order of \( c \).

This change had been performed by H. Lorentz (1904) [4], H. Poincare (1905) [5, 6] and A. Einstein (1905) [7, 8]. Lorentz considered deformation of electron moving through the so called ether, filling all the universe, and introduced primed spatial and time coordinates, as purely auxiliary quantities. Poincare and Einstein wrote transformations between primed and unprimed coordinates:

\[
t' = (t + ux/c^2)\gamma , \quad (31)
\]
\[
x' = (x + ut)\gamma , \quad (32)
\]
\[
y' = y , \quad z' = z , \quad (33)
\]
where
\[ \gamma = 1/\sqrt{1 - u^2/c^2} \, . \] (34)

They were called Lorentz transformations by Poincare and later by Einstein.

Poincare believed in ether and considered that the remaining problem is to understand it. Einstein simply dispensed with ether, he considered transformations (31) - (34) as a direct expressions of properties of space and time. Galilean relativity of inertial motion resulted in relativity of simultaneity, of time, and of length.

Proceeding from his article [7], Einstein came [8] to a fundamental conclusion that a body at rest has rest-energy \( E_0 \):
\[ E_0 = mc^2 \, . \] (35)

Here \( m \) is the mass of the body, while index 0 in \( E_0 \) indicates that this is the energy in the body’s rest frame.

In 1906 M. Planck explicitly wrote the expressions of total energy \( E \) and momentum \( \mathbf{p} \) of a body with arbitrary value of its velocity \( v \):
\[ E = mc^2 \gamma \, , \] (36)
\[ \mathbf{p} = m \mathbf{v} \gamma \, , \] (37)
where
\[ \gamma = (1 - v^2/c^2)^{-1/2} \, . \] (38)

These expressions can be easily derived by assuming that \( E \) and \( \mathbf{p} \) transform in the same way as \( t \) and \( \mathbf{r} \), each pair \((E, \mathbf{p}c)\) and \((t, \mathbf{r}/c)\) forming a four-dimensional vector. Indeed, by applying Lorentz transformations to a body at rest, taking into account relation (35) and writing \( v \) instead of \( u \), we come to (36) - (38). Of course, the isotropy of space should be also accounted for.

The notion of four-dimensional space-time was introduced in 1908 by G. Minkowski [10]. While four-vectors transform under Lorentz transformations (rotations in Minkowskian pseudo-Euclidian space), their squares are Lorentz-invariant:
\[ \tau^2 = t^2 - (\mathbf{r}/c)^2 \, , \] (39)
\[ m^2 c^4 = E^2 - (\mathbf{p}c)^2 \, . \] (40)
Here \( \tau \) is the so-called proper time, while \( m \), as before, is the mass of a body. But now it acquires a new meaning, which was absent in Newtonian mechanics. (Note that for a body at rest \( \mathbf{p} = 0 \) one recovers from eq. (40) the relation (35) between mass and rest-energy.)

It is impossible to discuss the concept of mass without explicitly basing the discussion on the achievements of XX century physics and especially on the notion of elementary particles such as electrons, photons and neutrinos, less elementary, such as protons and neutrons (in which quarks and gluons are confined), or composite, such as atoms and atomic nuclei. It is firmly established that all particles of a given kind (for instance all
electrons) are identical and hence have exactly the same value of mass. The same refers to protons and neutrons.

Atoms and atomic nuclei ask for further stipulations because each of these composite systems exists not only in its ground state, but can be brought to one of its numerous excited states (energy levels). For instance, a hydrogen atom is a bound system of a proton and electron, attracted to each other by Coulomb force (10). As proton is approximately two thousand times heavier than electron, one usually speaks about electron moving in the Coulomb potential (11) of proton. According to the laws of quantum mechanics this movement is quantized, forming a system of levels. The lowest level is stable, the excited ones are unstable. Electron jumps from a higher level to a lower one by emitting a quantum of light - photon. Finally it reaches the ground level.

The energy in atomic, nuclear and particle physics is measured in electron Volts: 1 eV is the energy which electron gains by traversing a potential of 1 Volt; 1 keV = 10³ eV, 1 MeV = 10⁶ eV, 1 GeV = 10⁹ eV, 1 TeV = 10¹² eV, 1 PeV = 10¹⁵ eV, 1 EeV = 10¹⁸ eV.

The binding energy of electron at the ground level of hydrogen atom is 13.6 eV. Due to relation between rest-energy and mass it is convenient to use as a unit of mass 1 eV/c². The mass of electron \( m_e = 0.511 \text{ MeV/c}^2 \), the mass of proton \( m_p = 0.938 \text{ GeV/c}^2 \). The mass of a hydrogen atom in its ground state is by 13.6 eV/c² smaller than the sum \( m_e + m_p \). This mass difference is often referred to as defect of mass.

As \( c \) is a universal constant, it is appropriate to use it in relativistic physics as a unit of velocity and hence to put \( c = 1 \) in all above values of masses and defects of mass. In what follows we will use as units of mass eV and its derivatives: keV, MeV, GeV, etc.

One eV is a tiny unit when compared with Joule (J) or kilogram:

\[
1 \text{ J} = 6.24 \cdot 10^{18} \text{ eV} , \quad 1 \text{ eV} = 1.6 \cdot 10^{-19} \text{ J} , \quad \text{(41)}
\]

\[
1 \text{ kg} = 5.61 \cdot 10^{35} \text{ eV} , \quad 1 \text{ eV} = 1.78 \cdot 10^{-36} \text{ kg} . \quad \text{(42)}
\]

However it is four orders of magnitude larger than one degree of Kelvin (K).

\[
1 \text{ K} = 0.86 \cdot 10^{-4} \text{ eV} , \quad 1 \text{ eV} = 1.16 \cdot 10^4 \text{ K} \quad \text{(43)}
\]

(In eq. (43) we put dimensional Boltzmann factor \( k \) equal to unity, taking into account that \( kT \), where \( T \) is temperature, characterizes the mean energy of an ensemble of particles.) Let us estimate the relative change of mass in a few everyday processes.

The light from the sun is absorbed by vegetation on the earth to produce carbohydrates via reaction of photosynthesis:

\[
\text{light} + 6\text{CO}_2 + 6\text{H}_2\text{O} = 6\text{O}_2 + \text{C}_6\text{H}_{12}\text{O}_6 .
\]

The total energy of light required to produce one molecule of \( \text{C}_6\text{H}_{12}\text{O}_6 \) is about 4.9 eV. This does not mean that the photons are massive. They are massless, but the kinetic energy of photons is transformed into the rest energy of carbohydrates.
A combustion of methane in the gas burner of a kitchen stove:

\[ \text{CH}_4 + 2\text{O}_2 \rightarrow \text{CO}_2 + 2\text{H}_2\text{O} \quad (44) \]

In this reaction 35.6 MJ of heat is released per cubic meter of methane. Since the density of methane is 0.72 kg/m\(^3\) and density of oxygen is 1.43 kg/m\(^3\)

\[ \frac{\Delta m}{m} = \frac{35.6 \cdot 6.24 \cdot 10^6 \cdot 10^{18}}{(0.72 + 2 \cdot 1.43) \cdot 5.61 \cdot 10^{35}} = 1.1 \cdot 10^{-10} , \]

where in the nominator eq.(41) and in denominator eq.(42) are used.

We can look at this result differently by starting from Avogadro number:

\[ N_A = 6.022 \cdot 10^{23} \text{ mol}^{-1} \quad (45) \]

and molar volume (for ideal gas)

\[ 22.4 \cdot 10^{-3} \text{ m}^3 \cdot \text{ mol}^{-1} . \]

This means that a cubic meter of methane contains \(2.69 \cdot 10^{25}\) molecules. Thus, burning of one molecule of methane releases

\[ \frac{35.6 \cdot 6.24 \cdot 10^{24}}{2.69 \cdot 10^{25}} = 8.3 \text{ eV} \]

Now we estimate the mass of one molecule of methane plus 2 molecules of oxygen: \(16 \times 5 \cdot 0.94 \text{ GeV} = 75 \text{ GeV}\), and calculate \(\Delta m/m\) at molecular level (we use 0.94 GeV as the mass of a nucleon):

\[ 8.3 \text{ eV}/75 \text{ GeV} = 1.1 \cdot 10^{-10} . \]

Thus, we see that the sum of masses of molecules on the right hand side of eq. (44) is by 8.3 eV smaller than that on the left-hand side. This mass difference is exploited in cooking.

Another example is the melting of ice. It takes \(0.334 \cdot 10^6\) J to melt a kilogram of ice. That means that in this case the relative increase of mass \(\Delta m/m\) is (see eqs. (41) and (42)):

\[ \Delta m/m = 0.334 \cdot 10^6 \cdot 6.24 \cdot 10^{18} \cdot 1.78 \cdot 10^{-36} = 3.7 \cdot 10^{-12} . \]

If the temperature of a flat iron is increased by 200\(^\circ\) its mass increases by \(\Delta m/m = 10^{-12}\). This is readily estimated using the specific heat (25 J · mol\(^{-1}\) K\(^{-1}\) = 450 J kg\(^{-1}\) K\(^{-1}\)):

\[ \frac{\Delta m}{m} = 450(\text{J kg}^{-1}\text{K}^{-1})200 \text{ K} = 10^{-12} . \]

All these mass differences are too tiny to be measured directly. Let us note that the defect of mass in a hydrogen atom 13.5 eV is also too small to be observed directly because the mass of the proton is known with large uncertainty ±80 eV.

The tiny values of \(\Delta m\) in atomic transitions and chemical reactions were the basis for the statement that in non-relativistic physics mass is additive, and of the law of conservation of this additive mass.
However in nuclear and particle physics the defect of mass is much larger. For instance, in the case of deuteron, which is a nuclear bound state of proton and neutron, the binding energy and hence the defect of mass is 2.2 MeV, so that \( \Delta m/m \simeq 10^{-3} \).

Of special pedagogical interest is the reaction of annihilation of electron and positron into two photons (two \( \gamma \)-quanta). Photons are massless particles, which always move with velocity \( c \). The latter statement follows from eqs. (36), (37), (40):

\[
\frac{v}{c} = \frac{|p|}{E} = 1 \quad \text{if} \quad |p| = E .
\]

Depending on their energy, photons are referred to as quanta of radio waves, visible and invisible light, X-rays, \( \gamma \)-quanta.

The reaction of annihilation is

\[
e^+ + e^- \rightarrow \gamma + \gamma .
\]

Let us consider the case when electron and positron annihilate at rest. Then their total energy is \( E = E_0 = 2m_e c^2 \), while the total momentum \( p = 0 \). Due to conservation of energy and momentum the two photons will fly with opposite momenta, so that each of them will have energy equal to \( m_e c^2 \). The rest frame of \( e^+ + e^- \) will be obviously the rest frame of two photons. Thus, the rest energy of the system of two photons will be \( 2m_e c^2 \) and hence the mass of this system will be \( 2m_e \), in spite of the fact that each of the photons is massless. We see that mass in relativity theory is conserved, but not additive.

In general case the system of two free particles with energies and momenta \( E_1, p_1 \) and \( E_2, p_2 \) has total energy

\[
E = E_1 + E_2 ,
\]

and total momentum

\[
p = p_1 + p_2 .
\]

These equations follow from additivity of energy and momentum. The mass of the system is defined as before by eq. (40). Hence

\[
m^2 = (E_1 + E_2)^2 - (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2E_1E_2(1 - v_1v_2) .
\]

It follows from eq. (50) that the mass of a system of two particles depends not only on masses and energies of these particles, but also on the angle between their velocities. Thus for two photons \( m \) is maximal when this angle is \( \pi \) and vanishes when it is zero.

The mass of system has lost its Newtonian meaning of an amount of substance, its main characteristic being now rest energy (in units, where \( c = 1 \)). Newtonian equations can be obtained from relativistic ones in the limiting case of low velocities (\( v/c \ll 1 \)). In that case \( \gamma \) given by eq. (38) becomes

\[
\gamma = (1 - v^2/c^2)^{-1/2} \simeq 1 + \frac{v^2}{2c^2} ,
\]

(51)
so that for one particle we get:

\[ E = mc^2 + \frac{mv^2}{2} = E_0 + E_{\text{kin}}, \quad (52) \]

\[ p \simeq mv. \quad (53) \]

For a system of two particles in the limit of vanishing \( v \) we get from eq. (50)

\[ m^2 \simeq (m_1 + m_2)^2. \quad (54) \]

Thus the approximate additivity of mass is restored.

We started the description of Newtonian mechanics by consideration of static gravitational and electric interactions, in particular, their potentials (8) and (12). For particles at rest these potentials do not depend on time. The situation is drastically changed when the velocity of particles is not negligible. Let us start with electrodynamics. First, in addition to the scalar potential \( \varphi \) we have now vector potential \( \mathbf{A} \), so that \( \varphi, \mathbf{A} \) form a four vector. Second, because of finite velocity \( c \) of propagation of electromagnetic perturbations, both \( \varphi \) and \( \mathbf{A} \) are retarded:

\[ \varphi(t_2, r_2) = \frac{e}{r - \frac{\mathbf{v}}{c}}, \quad (55) \]

\[ \mathbf{A}(t_2, r_2) = \frac{ev}{c(r - \frac{\mathbf{v}}{c})}, \quad (56) \]

where

\[ r = r_2(t_1) - r_1(t_1), \quad (57) \]

while

\[ \mathbf{v} = \mathbf{v}(t_1, r_1). \quad (58) \]

Thus defined \( \varphi \) and \( \mathbf{A} \) allow one to calculate the strengths of electric and magnetic fields:

\[ \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \text{grad} \, \varphi, \quad (59) \]

\[ \mathbf{H} = \text{rot} \, \mathbf{A}, \quad (60) \]

where the differentiation is performed with respect to \( t_2, r_2 \).

In a four-dimensional form the six components of antisymmetric tensor of the strength of electromagnetic field \( F_{ik} \) are expressed in terms of derivatives of four-dimensional potential:

\[ F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}. \quad (61) \]

The lower indices are referred to as covariant, while the upper ones as contravariant.

\[ A_i = (\varphi, -\mathbf{A}), \quad x^i = (ct, \mathbf{r}), \quad i = 0, 1, 2, 3 \quad (62) \]

The components of \( \mathbf{E} \) and \( \mathbf{H} \) are expressed in terms of components of \( F_{ik} \):

\[ E_x = F_{11}, \quad E_y = F_{02}, \quad E_z = F_{03}, \quad (63) \]

13
\[ H_x = F_{23}, \quad H_y = F_{31}, \quad H_z = F_{12}. \] (64)

The Lorentz invariant products of four-vectors are constructed by using the so-called metric tensor \( \eta_{ik} \), which in an inertial reference frame is given by a diagonal \( 4 \times 4 \) matrix:

\[ \eta^{ik} = \eta_{ik} = \text{diag}(1, -1, -1, -1) \] (65)

with vanishing non-diagonal elements. Multiplication of a covariant vector by \( \eta^{ik} \) gives a contravariant vector, e.g.:

\[ p^i = \eta^{ik} p_k, \] (66)

where summation over index \( k \) is presumed.

Up to now we considered point-like particles. If the charge is smeared over a finite volume with density \( \rho \), the total charge of a particle is given by integral:

\[ e = \int \rho(r) dr. \] (67)

Similarly:

\[ ev = \int v(r) \rho(r) dr. \] (68)

The four-vector \( j^i = (c \rho, v \rho) \) (69) describes the density of the four-current. (In the case of a point-like particle \( \rho = e \delta(r - r_1) \).)

The famous Maxwell’s equations of classical electrodynamics have the form:

\[ \frac{\partial F_{ik}}{\partial x^k} = -\frac{4\pi}{c} j^i, \] (70)

\[ \frac{\partial F_{ik}}{\partial x^k} = 0, \] (71)

Here

\[ \tilde{F}_{ik} = \varepsilon^{iklm} F_{lm}, \] (72)

where \( \varepsilon^{iklm} \) is Lorentz-invariant antisymmetric tensor. We see that current \( j^i \) is the source of electromagnetic field.

### 4 Mass in General Relativity

Let us now consider relativistic gravity. The role of gravitational potentials is played by 10 components of symmetric metric tensor \( g_{ik}(x^i) \), four of them being diagonal, while six off-diagonal. What is very important is that in the case of gravity the ten components of \( g^{ik} \) are functions of space-time points \( x^i \): they change from one point to another. The source of gravitational field, the analogue of vector \( j^i \), is the density of energy-momentum tensor \( T^{ik} \). \( T^{ik} \) is symmetric and conserved

\[ \frac{\partial T^{ik}}{\partial x^k} = 0. \] (73)
The total 4-momentum of a system
\[ p^i = \frac{1}{c} \int T^{0i}(t) dt \]   \hspace{1cm} (74)

Hence \( T^{00} \) is the density of energy, while \( T^{10}/c, T^{20}/c \) and \( T^{30}/c \) represent the density of momentum. For a point-like particle with mass \( m \) the density of mass \( \mu \) is given by
\[ \mu = m \delta (r - r_1) \]   \hspace{1cm} (75)

\[ T^{ik} = \mu c \frac{dx^i}{ds} \frac{dx^k}{dt} = \mu c u^i u^k \frac{ds}{dt}, \]   \hspace{1cm} (76)

where \( u^i \) is contravariant velocity, while \( ds \) is an invariant interval:
\[ u^i = \frac{dx^i}{ds} \]   \hspace{1cm} (77)

\[ ds^2 = g_{ik} dx^i dx^k \]   \hspace{1cm} (78)

\[ ds = c d\tau = \sqrt{g_{00}} dx^0. \]   \hspace{1cm} (79)

Hence
\[ \tau = \frac{1}{c} \int \sqrt{g_{00}} dx^0, \]   \hspace{1cm} (80)

where \( \tau \) is the proper time for a given point in space.

The connection between \( u^i \) and ordinary 3-velocity \( v \) is
\[ u^i = (\gamma, \frac{v}{c\gamma}). \]   \hspace{1cm} (81)

Thus
\[ u_i u^i = 1. \]   \hspace{1cm} (82)

The most important conclusion is that the source of gravitational field is proportional to the mass of a particle.

The equation for gravitational potential \( g_{ik} \), derived by Einstein in 1915, is more complex than the Maxwell equation for \( A^i \):
\[ R_{ik} - \frac{1}{2} g_{ik} R = 8\pi G T_{ik}. \]   \hspace{1cm} (83)

Here \( R_{ik} \) is the so-called Ricci tensor, while \( R \) is scalar curvature:
\[ R = g^{ik} R_{ik}. \]   \hspace{1cm} (84)

The role of electromagnetic field strength \( F_{ik} \) is played in gravity by the affine connection:
\[ \Gamma^l_{ki} = \frac{1}{2} g^{lm} \left( \frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right), \]   \hspace{1cm} (85)

while the role of derivative \( \partial_k F^{ik} \) is played by the left-hand side of eq. (83), where the Ricci tensor is given by:
\[ R_{ik} = g_{in} g^{rs} \left( \frac{\partial \Gamma^a_{ks}}{\partial x^r} - \frac{\partial \Gamma^a_{kr}}{\partial x^s} + \Gamma^a_{ps} \Gamma^p_{ks} - \Gamma^a_{ps} \Gamma^p_{kr} \right). \]   \hspace{1cm} (86)
The drastic difference of gravidynamics from electrodynamics is the nonlinearity of Einstein equation (83): it contains products of affine connections. This nonlinearity manifests itself at low values of $v/c$ as a tiny effect in the precession of perihelia of planets (Mercury). However it is very important for strong gravitational fields in such phenomena as black holes.

From principal point of view of highest priority is the dual role of the tensor $g_{ik}$, which is both dynamical and geometrical. Dynamically $g_{ik}$ represents the potential of gravitational field. On the other hand $g_{ik}$ and its derivatives determine the geometry of space-time. Einstein gave to his theory of relativistic gravity the name of General Theory of Relativity (GTR).

It is clear from the above equations of GTR that in non-relativistic limit $v/c \ll 1$ the gravitational interaction is determined by only one mass $m$, while notations $m_g$ and $m_i$ are redundant and misleading. Even more useless are concepts of active and passive gravitational mass often considered by some authors.

5 The pedagogical virus of “relativistic mass”

The “famous formula $E = mc^2$” and the concept of “relativistic mass” increasing with velocity, which follows from it, are historical artifacts, contradicting the basic symmetry of Einstein’s Special Relativity, the symmetry of 4-dimensional space-time. The relation discovered by Einstein is not $E = mc^2$, but $E_0 = mc^2$, where $E_0$ is the energy of a free body at rest introduced by Einstein in 1905. The source of the longevity of the “famous formula” is the irresponsible attitude of relativity theory experts to the task of explaining it to the non-experts.

The notion of “relativistic mass” presents a kind of pedagogical virus which very effectively infects new generations of students and professors and shows no signs of decline. Moreover in the Year of Physics it threatens to produce a real pandemia.

I published my first articles [11,12] against the “Einstein famous equation $E = mc^2$” in 1989. The subject seemed important to me because it concerned the proper teaching of special relativity at high schools, colleges and universities and explaining its genuine meaning to a wide audience of non-physicists, the so-called “pedestrians” in popular science magazines and books.

The task looked also not absolutely formidable because a consistent presentation of relativity existed for a long time in the world-wide accepted textbook by Landau and Lifshitz [13], which was the basis of my own understanding, and in some other textbooks.

The existence of countless texts, in which the essence of relativity was mutilated (or semi-mutilated) had two sides. On one hand, it looked discouraging, especially because among the authors of these texts there were
many famous physicists, the fathers and greatest authorities of modern physics. On the other hand, it was a challenge. So I tried to explain clearly to the readers the beauty of four-dimensional space-time approach and the ugliness and inconsistency of “relativistic mass”, an illegitimate child of relativistic and non-relativistic equations.

My optimism had increased when in 1992 Taylor and Wheeler in the second edition of their influential and popular “Spacetime Physics” [14] included a “Dialog: Use and Abuse of the Concept of Mass”, in which they supported my articles [11, 12]. A copy of this book is in my bookcase with a postcard sent to me in October 1991 by John Archibald Wheeler. The postcard has a photo of the famous Albert Einstein Memorial in front of the building of the National Academy of Sciences, Washington, DC. The bronze sculpture of Einstein includes a copybook with $E = mc^2$ on an open page.

Since that time I received hundreds of letters from physicists (both professors and students) stating their adherence to the four-dimensional formulation of relativity and to the Lorentz invariant concept of mass. In a few cases I helped the authors to correct erroneous explanations of the concept of mass in preparing new editions of their textbooks. However the number of proponents of relativistic mass seemed not to decrease.

A leading role in promoting the relativistic mass have played the books by Max Jammer [15, 16]. Especially aggressive the proponents of relativistic mass became in connection with the World Year of Physics, which marks the 100th anniversary of fundamental articles published by Einstein in 1905.

The campaign started by the September 2004 issue of “Scientific American”, full with “relativistic mass” equal to $m_0/\sqrt{1 - v^2/c^2}$, where $m_0$ is rest mass, and “the most famous equation $E = mc^2$”. A letter to the editors, defending the four-dimensional approach and invariant mass had been rejected by the editor G. Collins who in April 2005 wrote: “Most important, we believe that tackling the issue head-on in the manner you and your coauthors want in the letters column of Sci. Am. would be very confusing to our general audience and it would make the subject seen all the more mysterious and impenetrable to them”. Thus to avoid “head-on” collision of correct and false arguments the editors of Sci. Am. preferred to hide from the readers the correct viewpoint.

P. Rodgers – the Editor of European “Physics World” wrote in January 2005 in editorial [17]: “... $E = mc^2$ led to the remarkable conclusion that mass and energy are one and the same”. Unlike G. Collins, P. Rodgers published a letter criticizing this statement and partly agreed with the criticism [18].

In September 2005 the bandwagon of relativistic mass was joined by “The New York Times”, which published an article by B. Green [19]. The journalists were supported by renowned scientists, such as R. Penrose, who in a new thousand pages thick book had written [20]:

“In a clear sense mass and energy become completely equivalent to one another according to Einstein’s most famous equation $E = mc^2$.”
How many students, teachers and journalists will be infected by this sentence? How many readers had been infected by the famous book by S. Hawking [21], the second edition of which appeared in 2005? On the very first page of it Hawking wrote:

“Someone told me that each equation I included in the book would halve the sales. I therefore resolved not to have any equations at all. In the end, however, I did put in one equation, Einstein’s famous equation $E = mc^2$. I hope that this will not scare off half of my potential readers.”

I am sure that the usage of $E = mc^2$ had doubled the sales of his book, the buyers being attracted by the famous brand. But is it possible to estimate the damage done to their understanding of relativity theory and to the general level of the literature on relativity incurred by this case of spreading the virus?

Two recent preprints by Gary Oas [22, 23] written in the framework of Educational Program for Gifted Youth at Stanford University were devoted to the use of relativistic mass. The author “urged, once again, that the use of the concept at all levels to be abandoned” [22]. The manuscript has been submitted for publication to the “American Journal of Physics”, but was rejected as being “too lengthy” (it contains 12 pages!). A lengthy bibliography (on 30 pages) of books referring to special and/or general relativity is provided in ref. [23] to give a background for discussions of the historical use of the concept of relativistic mass. It is easy to forecast the aggressive reaction of the virus infected community to this attempt to cure it.

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