The effects of quantum instantons on the thermodynamics of the $\mathbb{C}P^{N-1}$ model

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Using the $1/N$ expansion, we study the influence of quantum instantons on the thermodynamics of the $\mathbb{C}P^{N-1}$ model in 1+1 dimensions. We do this by calculating the pressure to next-to-leading order in $1/N$, without quantum instanton contributions. The fact that the $\mathbb{C}P^1$ model is equivalent to the $O(3)$ nonlinear sigma model, allows for a comparison to the full pressure up to $1/N^2$ corrections for $N = 3$. Assuming validity of the $1/N$ expansion for the $\mathbb{C}P^1$ model makes it possible to argue that the pressure for intermediate temperatures is dominated by the effects of quantum instantons. A similar conclusion can be drawn for general $N$ values by using the fact that the entropy should always be positive.

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I. INTRODUCTION

It was discovered by Belavin et al. [1] that the classical equations of motion of Euclidean QCD have topologically nontrivial solutions with finite action. Such instanton solutions and the fluctuations around them contribute to physical quantities, as first observed by 't Hooft [2]. He showed that instantons give rise to an additional source of $U(1)_A$ symmetry breaking in QCD, which e.g. is necessary to explain the relatively large mass of the $\eta'$ meson. The dependence of instanton effects on the coupling $g$, the number of colors $N_c$ and the temperature $T$, has been studied extensively afterwards. This is usually done in the dilute instanton gas approximation, which limits the conclusions to weak coupling, even though the effects of instantons are nonperturbative and typically go like $\exp(-c/g^2)$, where $c$ is a constant. In this approximation, the effects are exponentially suppressed in the limit $N \rightarrow \infty$, as discussed by Witten [3]. However, if also instantons of large size are relevant, as e.g. for the topological susceptibility, then instanton contributions can remain in the limit $N \rightarrow \infty$. See Ref. [4] for a recent discussion of instanton effects at large $N$ at $T = 0$.

Instanton solutions at nonzero temperature have also been studied. At finite temperature, bosonic field configurations, including instantons, have to satisfy periodic boundary conditions. Harrington and Shepard [5] have constructed explicit periodic classical solutions which they called calorons. Gross, Pisarski and Yaffe [6] considered their effect on the partition function of QCD, including quantum corrections at the one-loop level. Their result applies in the weak-coupling limit, i.e. at high temperature, where the temperature provides a natural cut-off on the instanton size and small-size instantons dominate. The effects of instantons at low temperatures, where the coupling is large, are very difficult to calculate. Thus studying instanton effects in theories that are less complicated than QCD may be very useful.

Especially field theories in 1+1 dimensions have been studied extensively as toy models for QCD since they share several properties. For example, the $O(N)$ nonlinear sigma model and the $\mathbb{C}P^{N-1}$ model are both asymptotically free theories and a dynamical mass is generated nonperturbatively. The $O(N)$ model has instanton solutions for $N = 3$, while the $\mathbb{C}P^{N-1}$ admits them for all $N$. Moreover, the $U(1)$ gauge symmetry of the $\mathbb{C}P^{N-1}$ model generates in the large-$N$ limit a long-range Coulomb interaction which in 1+1 dimension grows linearly, and hence is confining [7]. This is a zero-temperature result; at nonzero temperature the model is no longer confining [8].

Instantons at finite temperature were examined in detail by Affleck [9] for the $\mathbb{C}P^{N-1}$ model. He has demonstrated, by means of the large-$N$ expansion, that instead of the classical instanton solutions, rather quantum instantons (quantum calorons) are of relevance. These are stationary solutions, with quantized topological charge, of the large-$N$ quantum effective action for the $U(1)$ gauge field. In a low-temperature analysis, which allows for a derivative expansion, Affleck showed that the quantum instantons correspond to the sine-Gordon solitons, whereas at high temperature the quantum instantons coincide with the classical instantons (see also Ref. [10]).

Quantum instanton solutions may also affect thermodynamic quantities. When one performs a $1/N$ expansion around the constant stationary point $A_\mu = 0$, one restricts to configurations with zero winding number $Q$. In this way, one may leave out important contributions arising from quantum instantons. In this article we will...
demonstrate, by including $1/N$ corrections, that neglecting quantum instantons even leads to unphysical results, such as a negative entropy. Since the contributions of the configurations with nonzero winding number are difficult to include, we will obtain in an indirect way the combined effect of these configurations. In order to show this, we exploit the equivalence between the $\mathbb{CP}^1$ model and the $O(3)$ nonlinear sigma model. The equivalence at the classical level was first pointed out by Eichenherr [13], while the quantum equivalence was shown by Banerjee [14]. In Ref. [15] we have obtained strong indications that the $1/N$ expansion at next-to-leading order (NLO) yields a good approximation to the exact pressure for the $O(N)$ nonlinear sigma models for all finite values of $N$, down to $N = 4$. For our present purposes we checked that this also applies to $N = 3$, which means that a well-behaved pressure is obtained that differs from the $N \to \infty$ pressure by order $1/N$. In that calculation of the pressure of the $O(3)$ nonlinear sigma model, one implicitly integrates over all (quantum) instanton configurations. Hence, the NLO pressure should be a good approximation to the exact pressure up to order $1/N^2$ corrections. Because of the equivalence to the $\mathbb{CP}^1$ model, the pressure for the latter should be the same as that of the $O(3)$ nonlinear sigma model, upon inclusion of all quantum instantons. The difference between the pressure of the $Q = 0$ sector of the $\mathbb{CP}^1$ model and the pressure of the $O(3)$ nonlinear sigma model should thus give the contribution to the pressure of the $\mathbb{CP}^1$ model from the topological configurations with $Q \neq 0$. Since the $1/N$ expansions in the $O(N)$ nonlinear sigma model and the $\mathbb{CP}^{N-1}$ model are different, the NLO results of the pressure of the $O(3)$ nonlinear sigma model and that of the $\mathbb{CP}^1$ model do not necessarily coincide. But if we assume that like in the $O(N)$ nonlinear sigma model the $1/N$ expansion does not break down in the $\mathbb{CP}^{N-1}$ model for small values of $N$ (even for $N = 2$ in this case) we can make a meaningful comparison between the pressure of the $O(3)$ nonlinear sigma model and the $\mathbb{CP}^1$ model to NLO in $1/N$. That allows us to estimate the size of the contribution of topological contributions with $Q \neq 0$ to the pressure for $N = 2$. In addition, for general values of $N$ we can derive a lower bound on the contribution of the topological configurations to the pressure, by using the fact that the entropy should always be positive. In this way, we find strong indications that the topological configurations with $Q \neq 0$ give a large contribution to the pressure and other thermodynamical quantities for intermediate temperatures.

The article is organized as follows. In Sec. II the essentials of the $\mathbb{CP}^{N-1}$ model are reviewed. We discuss the relevant details of the quantum effective action in Sec. III. The calculation of the effective potential is explained in Sec. IV. In Sec. V the results of the calculation of the pressure are presented. Finally a summary and conclusions are given in Sec. VI.

II. THE $\mathbb{CP}^{N-1}$ MODEL

The $\mathbb{CP}^{N-1}$ model is described by the following Lagrangian which is invariant under local U(1) and global SU(N) transformations

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi_i^* \partial^\mu \phi_i + \mathcal{L}_{\text{int}}, \quad \phi_i^* \phi_i = N/g_b^2, \quad i = 1 \ldots N, \]

where $\phi(x)$ is a complex scalar field and $g_b$ is the bare coupling constant. The interaction part of the Lagrangian is given by

\[ \mathcal{L}_{\text{int}} = \frac{g_b^2}{2N} (\phi_i^* \partial_\mu \phi_i) (\phi_j^* \partial^\mu \phi_j). \]

The Lagrangian can also be written in terms of a U(1) gauge field $A_\mu$

\[ \mathcal{L} = \frac{1}{2} |D_\mu \phi_i|^2, \quad \phi_i^* \phi_i = N/g_b^2. \]

where $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative. By solving the equations of motion for $A_\mu$ and inserting this expression into (2), the original Lagrangian Eq. (1) is recovered.

The $\mathbb{CP}^1$ model is equivalent to the $O(3)$ nonlinear sigma model [13, 14]. The correspondence can be made explicit by writing the $O(3)$ nonlinear sigma fields $\chi(x)$ as

\[ \chi_a(x) = \sqrt{g_b^2/N} \phi^a_i(x) (\sigma_a)_{ij} \phi_j(x), \quad a = 1 \ldots 3, \]

where $\sigma_a$ are Pauli matrices. Using Eq. (4), the Lagrangian for the $O(3)$ nonlinear sigma model, $\mathcal{L} = (\partial_\mu \chi_a)^2/2$, with the constraint $\chi_a \chi_a = N/g_b^2$ turns into the $\mathbb{CP}^1$ Lagrangian, Eq. (1), with the corresponding constraint.

As mentioned the $\mathbb{CP}^{N-1}$ model allows instanton solutions for all $N$. This follows from the fact that $\mathbb{CP}^{N-1} \cong \text{SU}(N)/\text{U}(N-1)$, such that $\pi_2(\mathbb{CP}^{N-1}) = \mathbb{Z}$. For the $O(N)$ nonlinear sigma models on the other hand, the relevant coset is $O(N)/O(N-1) \cong S^{N-1}$, where $\pi_2(S^{N-1}) \neq 0$ for $N = 3$ only. Since one also has a correspondence between the $\mathbb{CP}^{N-1}$ model and the $O(2N)$ nonlinear sigma model in the limit $N \to \infty$, one can conclude that the instantons of the $\mathbb{CP}^{N-1}$ model disappear in the limit $N \to \infty$, which coincides with the fact that they have infinite action in this limit ($S = \pi N|Q|/g_b^2$ for the classical instantons).

III. EFFECTIVE ACTION

The constraint in Eq. (1) can be implemented by introducing an auxiliary field $\alpha$. The Lagrangian Eq. (1) can then be written as

\[ \mathcal{L} = \frac{1}{2} |D_\mu \phi_i|^2 - \frac{i}{2} \alpha (\phi_i^* \phi_i - N/g_b^2). \]
Since the Lagrangian is quadratic in the $\phi$’s, they can be integrated out exactly, resulting in the following effective action

$$S_{\text{eff}} = N \text{Tr} \ln \left( -D_\mu D^\mu - i\alpha \right) + i \frac{N}{2g_b^2} \int_X \alpha(x),$$

where the subscript $X$ indicates integration over two-dimensional Euclidean space. The vacuum expectation value of the $\alpha$ field is purely imaginary and can therefore be written as $\alpha = im^2 + \tilde{\alpha}/\sqrt{N}$, where $(\alpha) = im^2$ and $\tilde{\alpha}$ a quantum fluctuating field. The scaling of the quantum fluctuating field with a factor $1/\sqrt{N}$ is merely a convenient way of implementing the $1/N$ expansion and has no effect on the final results. This yields

$$S_{\text{eff}} = N \text{Tr} \ln \left[ -\partial^2 + m^2 - i \{\partial_\mu, A^\nu\} + A_\mu A^\mu \right] - i \frac{\tilde{\alpha}}{\sqrt{N}} - \frac{N}{2g_b^2} \int_X \left( m^2 - \frac{i}{\sqrt{N}} \tilde{\alpha}(x) \right).$$

Affleck showed that $S_{\text{eff}}$ has stationary solutions $A_\mu$ at finite temperature that have a quantized topological charge. Since these solutions are stationary points of an action in which quantum effects are incorporated, they are called ‘quantum instantons’. Such instantons need to be considered in a full calculation of the pressure. As a first step to investigate the relevance of the quantum instantons, we take into account fluctuations around the trivial vacuum $A_\mu = 0$. We will do this in a way consistent with the $1/N$ expansion by scaling the gauge fields with a factor $1/\sqrt{N}$ as well. In the $1/N$ expansion around the stationary point $A_\mu = 0$, quantum instantons do not arise, as their nonzero boundary values (in the $A_1 = 0$ gauge, these are $A_0(x_0, \pm \infty) = 2\pi n \pm T$, where $Q = n_+ - n_-$) will not be achieved. To next-to-leading (NLO) order in $1/N$, one obtains

$$S_{\text{eff}} = N \text{Tr} \ln \left( -\partial^2 + m^2 \right) - \frac{Nm^2}{2g_b^2} \beta V + \frac{1}{2} \int_X \tilde{\alpha}(x) \left( \frac{1}{g_b^2} - \int_P \frac{1}{P^2 + m^2} \right) + \frac{1}{2} \int_{X,Y} \tilde{\alpha}(x) \Gamma(x - y) \tilde{\alpha}(y) + \frac{1}{2} \int_{X,Y} A^\mu(x) \Delta_{\mu\nu}(x - y) A^\nu(y),$$

where $\beta = 1/T$ is equal to the inverse temperature and $V$ is the volume of the one-dimensional space. Equation shows that although a kinetic term for the gauge fields is absent in the classical action, such a term is generated by quantum fluctuations. Its tensorial structure at finite temperature is the same as at zero temperature, which is specific to 1 + 1 dimensions. One finds

$$\Gamma(P) = \frac{1}{2\pi \sqrt{P^2(P^2 + 4m^2)}} \ln \left( \frac{\sqrt{P^2 + 4m^2} + \sqrt{P^2}}{\sqrt{P^2 + 4m^2} - \sqrt{P^2}} \right) + \Gamma_T(P),$$

$$\Delta_{\mu\nu}(P) = \left( \delta_{\mu\nu} - \frac{P_\mu P_\nu}{P^2} \right) \Delta_{\mu\nu}(P),$$

where $P = (p, p_0)$ and

$$\Delta_{\mu\nu}(P) = \frac{1}{2\pi} \left[ \frac{P^2 + 4m^2}{P^2} \ln \left( \frac{\sqrt{P^2 + 4m^2} + \sqrt{P^2}}{\sqrt{P^2 + 4m^2} - \sqrt{P^2}} \right) - 2 \right] + (P^2 + 4m^2) \Gamma_T(P),$$

$$\Gamma_T(P) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dq}{\omega_q} \left( P^2 + 2pq + 4p_0, \omega_q \right) n(\omega_q).$$

Here $n(\omega_q) = (\exp(\beta\omega_q) - 1)^{-1}$ is the Bose-Einstein distribution function and $\omega_q = \sqrt{q^2 + m^2}$.

### IV. EFFECTIVE POTENTIAL

The leading-order (LO) contribution to the effective potential can be read off directly from Eq. 8. The next-to-leading order corrections, is obtained by carrying out a Gaussian integration over the fluctuations $\tilde{\alpha}$ and $A_\mu$. In order to do so, one has to fix a gauge, and throughout the paper we employ the generalized Lorentz gauge. Including the contribution from the ghost, we obtain the contribution to the effective potential from the gauge field:

$$\mathcal{V}_\text{gauge}(m^2) = \frac{1}{2} \sum_P \ln P^2 - \frac{1}{2} \sum_P \ln \det \left( \Delta_{\mu\nu} + \frac{1}{\xi} P_\mu P_\nu \right) = \frac{1}{2} \sum_P \ln P^2 - \frac{1}{2} \sum_P \ln \Delta_{\mu\nu},$$

where the sum-integral is defined as

$$\sum_{P_0 = 2\pi \mu T} \frac{dp}{2\pi}.$$

We emphasize that Eq. 18 is independent of the gauge-fixing condition. From Eq. 9 and the results above, we obtain the following finite temperature effective potential up to next-to-leading order in $1/N$.

$$\mathcal{V}(m^2) = N\mathcal{V}_\text{LO}(m^2) + \mathcal{V}_\text{NLO}(m^2),$$

where

$$\mathcal{V}_\text{LO}(m^2) = \frac{m^2}{2g_b^2} - \sum_P \ln (P^2 + m^2),$$

$$\mathcal{V}_\text{NLO}(m^2) = - \frac{1}{2} \sum_P \ln \Gamma(P) - \frac{1}{2} \sum_P \ln \Delta_{\mu\nu}(P) + \frac{1}{2} \sum_P \ln P^2.$$
The effective potential is ultraviolet divergent. To regulate the divergences, we introduce an ultraviolet momentum cutoff $\Lambda$. After subtracting $T$ and $m$-independent infinite constants, we obtain

$$V_{\text{LO}}(m^2) = \frac{m^2}{2g_s^2} - \frac{m^2}{4\pi} \left[ 1 + \ln \left( \frac{\Lambda^2}{m^2} \right) \right] + \frac{1}{4\pi} T^2 J_0(\beta m),$$

where

$$J_0(\beta m) = \frac{8}{T^2} \int_0^\infty \frac{dq q^2}{\omega_q} n(\omega_q).$$

The minimum of the leading-order effective potential obeys the following gap equation

$$\frac{1}{\mu_0^2} = \frac{1}{2\pi} \ln \left( \frac{\Lambda^2}{m_e^2} \right) + \frac{1}{2\pi} J_1(\beta m),$$

where

$$J_1(\beta m) = 4 \int_0^\infty \frac{dq}{\omega_q} n(\omega_q).$$

In order to calculate the NLO order contribution to the effective potential, we write $V_{\text{NLO}}$ as a sum of divergent ($D$) and finite parts ($F$) in the following way

$$V_{\text{NLO}}(m^2) = -\frac{1}{2} (D_1 + D_2 + F_1 + F_2 + F_3 + F_4) - \frac{\pi T^2}{3},$$

where the divergent and finite quantities are defined through the following relations

$$D_1 + F_1 = \int P \ln \tilde{\Gamma}(P), \quad F_3 = \oint \ln \tilde{\Gamma}(P),$$

$$D_2 + F_2 = \int P \ln \tilde{\Delta}_\mu^\nu(P), \quad F_4 = \oint \ln \tilde{\Delta}_\mu^\nu(P).$$

Here $\tilde{\Gamma}(P) = 2\pi \sqrt{P^2(P^2 + 4m^2)} \Gamma(P)$, $\tilde{\Delta}_\mu^\nu(P) = 2\pi \sqrt{P^2/(P^2 + 4m^2)} \Delta_\mu^\nu(P)$, and

$$\oint P = \oint P - \int P.$$  

The functions $D_1$ and $D_2$ contain the ultraviolet divergences of the NLO order effective potential. In order to isolate these divergences, the high-momentum limit of $\tilde{\Gamma}(P)$ and $\tilde{\Delta}_\mu^\nu(P)$ are needed. In the high-momentum approximation ($|p| \gg T$), we obtain

$$\tilde{\Gamma}(P) \approx \ln \left( \frac{P^2}{m_e^2} \right) + \frac{2m^2}{P^2} + \frac{2m^2 J_1(\beta m)}{P^2} \left( 1 - \frac{2\beta m}{P^2} \right),$$

$$\tilde{\Delta}_\mu^\nu(P) \approx \ln \left( \frac{P^2}{m_e^2} \right) + \frac{6m^2}{P^2} + \frac{2m^2 J_1(\beta m)}{P^2} \left( 1 - \frac{2\beta m}{P^2} \right),$$

where $m^2 = m^2 \exp[-J_1(\beta m)]$ and $m^2_e = m^2 \exp[2 - J_1(\beta m)]$. The divergences $D_1$ and $D_2$ can be obtained by integrating $\tilde{\Gamma}$ and $\tilde{\Delta}_\mu^\nu$ over spatial momenta and we find

$$D_1 = \frac{1}{4\pi} \left[ \Lambda^2 \ln \ln \left( \frac{\Lambda^2}{m_e^2} \right) - m_2^2 \ln \left( \frac{\Lambda^2}{m_e^2} \right) \right] + 2m^2 \ln \ln \left( \frac{\Lambda^2}{m_e^2} \right),$$

$$D_2 = \frac{1}{4\pi} \left[ \Lambda^2 \ln \ln \left( \frac{\Lambda^2}{m_e^2} \right) - m_2^2 \ln \left( \frac{\Lambda^2}{m_e^2} \right) \right] + 6m^2 \ln \ln \left( \frac{\Lambda^2}{m_e^2} \right).$$

where $\ln(x)$ is the logarithmic integral defined by

$$\ln(x) = \mathcal{P} \int_0^x dt \frac{1}{\ln t}.$$  

Here $\mathcal{P}$ denotes the principal-value prescription. From $D_1$ and $D_2$ it can be seen that (through the dependence on $m^2$ and $m^2_e$) the effective potential contains temperature-dependent divergences. They cannot be eliminated in a temperature-independent way. See Ref. [16] for a detailed discussion of the occurrence of these divergences. However, they become temperature-independent at the minimum of the effective potential (see Sec. V). The finite functions $F_1$ and $F_2$ will be obtained numerically. In order to calculate these functions, we write the divergences partly in terms of an integral. This prevents subtracting large quantities which can give rise to big numerical errors. The functions $F_1$ and $F_2$ are calculated using the following expressions

$$F_1 = \mathcal{P} \int P \ln \left[ \frac{\tilde{\Gamma}(P)}{\ln(P^2/m_e^2)} \right] - \frac{2m^2}{4\pi} \ln \left( \frac{\Lambda^2}{m_e^2} \right),$$

$$F_2 = \mathcal{P} \int P \ln \left[ \frac{\tilde{\Delta}_\mu^\nu(P)}{\ln(P^2/m_e^2)} \right] - \frac{6m^2}{4\pi} \ln \left( \frac{\Lambda^2}{m_e^2} \right).$$

At zero temperature it was found that $F_1 \approx m^2 c_1/(2\pi)$ and $F_2 \approx m^2 c_1/(2\pi)$, where $c_1 \approx 0.611671457\ldots$. For convenience the finite-temperature parts of $F_1$ and $F_2$ are defined as $F_1 = F_1 - m^2 \gamma_{E}/(2\pi)$ and $F_2 = F_2 - m^2 c_1/(2\pi)$. These functions divided by $T^2$ depend on $\beta m$ only and are displayed in Fig. 1. In the limit $\beta m \to \infty$, these functions go to zero because the temperature-dependent parts of the inverse propagators are exponentially suppressed compared to the zero-temperature contribution. For small $\beta m$, these functions also go to zero as can be inferred from the limit $\beta m \to 0$ of $\Gamma(P)$. This limit can be found by first performing a momentum integration and then noting that the dominant contribution arises from the zeroth Matsubara mode. This yields

$$\Gamma(P) \approx \frac{1}{\beta m} \frac{P^2}{P^4 + 4m^2 p^2}.$$
of the zero-temperature inverse propagators can be used
noting that for large \( \beta m \), the temperature-dependent
part of the inverse propagator does not contribute to \( F_3 \)
and \( F_4 \). Furthermore, the dominant contribution to the
difference of a sum-integral and an integral arises from
the low-momentum modes. Thus the large-\( m \) behavior
of the zero-temperature inverse propagators can be used
to obtain a large-\( \beta m \) approximation for \( F_3 \) and \( F_4 \):

\[
F_3 \approx \frac{1}{2} \int_0^\beta \ln P^2 = -\frac{\pi}{6} T^2, \quad F_4 \approx \frac{3}{2} \int_0^\beta \ln P^2 = -\frac{\pi}{2} T^2. 
\]  
(33)

As can be seen in Fig. 2 this is in agreement with the
numerical calculations. The small-\( \beta m \) limit of \( F_3/T^2 \)
and \( F_4/T^2 \) can be obtained too, using the small-\( \beta m \) limit
of \( \Gamma(P) \). The result \( F_3 \approx 0 \) and \( F_4 \approx 0 \) is in agreement with
the numerical calculations displayed in Fig. 2.

V. CONTRIBUTION OF QUANTUM INSTANTONS TO THE PRESSURE

In Sec. IV the effective potential was evaluated and it
was found that it contains temperature-dependent ultra-
violet divergences. At the minimum these temperature-
dependent divergences will disappear as will be discussed
now. To calculate the effective potential at the minimum,
one only needs to solve the leading-order gap equation
as was shown by Root [19]. As a result, the LO and
NLO order contributions to the pressure are given by

\[
P_{LO} = \mathcal{V}_{LO}(m_T^2) - \mathcal{V}_{LO}(m_0^2), 
\]  
(34)

\[
P_{NLO} = \mathcal{V}_{NLO}(m_T^2) - \mathcal{V}_{NLO}(m_0^2), 
\]  
(35)

where \( m_T^2 \) is the solution of the leading-order gap equation
at temperature \( T \). By using the leading-order gap equation, it can be shown that at the minimum the
divergent terms \( D_1 \) and \( D_2 \) become

\[
D_1 = \frac{\Lambda^2}{4\pi} \left[ \ln \left( \frac{2\pi}{g_0} \right) - \exp \left( -\frac{2\pi}{g_0} \right) \right] \exp \left( \frac{2\pi}{g_0} \right) 
+ \frac{2m_0^2}{4\pi} \ln \ln \left( \frac{\Lambda^2}{m_T^2} \right), 
\]  
(36)

\[
D_2 = \frac{\Lambda^2}{4\pi} \left[ \ln \left( \frac{2\pi}{g_0} - 2 \right) - \exp \left( \frac{2-2\pi}{g_0} - 2 \right) \right] \exp \left( \frac{2\pi}{g_0} - 2 \right) 
+ \frac{6m_0^2}{4\pi} \ln \ln \left( \frac{\Lambda^2}{m_T^2} \right). 
\]  
(37)

Hence, the temperature-dependent quadratic divergence
and the divergence proportional to \( \ln \ln(x) \) become temperature-independent at the minimum of the effective
potential. As a result these divergences can be eliminated by counterterms that are independent of temperature.
Furthermore, the divergences proportional to \( \ln \ln(x) \) can be eliminated by the coupling-constant renormalization,
which amounts to the substitution \( g_0^2 \rightarrow g_0^2 - Z_3^2 g_0^2 \), where

\[
\frac{1}{Z_3^2} = 1 + \frac{g_0^2}{2\pi} \ln \left( \frac{\Lambda^2}{\mu^2} \right) + \frac{2 g_0^2}{N} \ln \ln \left( \frac{\Lambda^2}{\mu^2} \right). 
\]  
(38)

Using the results above, it follows that the leading and
next-to-leading order contributions to the pressure are given by

\[
P_{LO} = \frac{m_T^2}{2g_0^2} - \frac{m_T^2}{4\pi} \left[ 1 + \ln \left( \frac{\mu^2}{m_T^2} \right) \right] + \frac{T^2}{4\pi} J_0(3m_T) 
+ \frac{m_0^2}{4\pi}, 
\]  
(39)

\[
P_{NLO} = -\left[ 1 \right] \left[ \tilde{F}_1(m_T) + \tilde{F}_2(m_T) + F_3(m_T) \right] + \frac{1}{4\pi} (\gamma_E + c_1)(m_T^2 - m_0^2) - \frac{\pi}{3} T^2. 
\]  
(40)

The results of the calculation of the pressure are displayed in Fig. 3 for the arbitrary choice \( g_0^2(\mu = 500) = 10 \)
and different values of $N$. As one can see, for low temperatures and all finite values of $N$ the pressure first decreases with increasing $T$. A decreasing pressure implies that the entropy becomes negative. Clearly, this is in conflict with the third law of thermodynamics, which states that the entropy is minimal at zero temperature. As we will remind the reader below, we have strong indications that the 1/$N$ expansion itself is not the reason for the negative pressure, therefore, it is likely the fact that the contribution to the pressure in the region where the pressure increases with increasing $T$ vanishes in the limit $N \to \infty$.

Moreover, the problem disappears as the temperature increases, which is consistent with the fact that at weak coupling the effects of instantons become highly suppressed as function of $T$.

Using the equivalence with the O(3) nonlinear sigma model, the contribution from the quantum instantons with nonzero winding number to the pressure of the $\mathbb{C}P^1$ model can be estimated. Because the integration over all scalar-field configurations is done exactly in the O(3) nonlinear sigma model, including those with $Q \neq 0$, the effects of all quantum instantons are automatically included in the large-$N$ quantum effective potential for $\phi$. In Fig. 4 the result of the NLO order calculation of the pressure of the O($N$) nonlinear sigma model for $N = 3$ is compared to the contribution of the configurations with $Q = 0$ to the pressure of the $\mathbb{C}P^1$ model. Fig. 4 shows that for very low and high temperatures the pressures coincide, while for intermediate temperatures they are very different. This difference is displayed in Fig. 5 and is a strong hint that quantum instantons give a sizable contribution to the pressure in the region where the pressure increases considerably.

As we have already mentioned in the introduction, in Ref. [15] we have obtained strong indications that the 1/$N$ expansion yields trustworthy results for the O($N$) nonlinear sigma models for all finite values of $N$, down to $N = 4$. The $N = 3$ pressure presented here is in full agreement with the results obtained earlier and there is no reason to believe that the 1/$N$ expansion for $N = 3$ is not to be trusted. The NLO corrections for the O($N$) nonlinear sigma model are of the expected order 1/$N$. The pressure of O(3) nonlinear sigma model evaluated to NLO in the 1/$N$ expansion includes the effects of all quantum instantons (up to 1/$N^2$ corrections). Therefore, we believe that we have obtained a good approximation to the exact $\mathbb{C}P^1$ model pressure. As said, this is up to order 1/$N^2$ corrections, which cannot solve the discrepancy with the pressure of the $Q = 0$ sector of the $\mathbb{C}P^1$ model.

Since the entropy has to be positive, for general values of $N$ we can estimate a lower bound on the contribution of the quantum instantons with $Q \neq 0$ to the pressure. The lower bounds turn out to be almost independent of $N$ and have a similar shape with a somewhat lower maximum than the estimated contribution for $N = 3$.
VI. SUMMARY AND CONCLUSIONS

In this article the effect of quantum instantons on thermodynamical quantities of the $\mathbb{C}P^{N-1}$ model was investigated. We expanded the effective potential of the $\mathbb{C}P^{N-1}$ model around the trivial vacuum, and calculated it to NLO order in $1/N$. It was shown that the effective potential contains temperature-dependent divergences which only can be renormalized at the minimum of the effective potential. Hence thermodynamic quantities can be rendered finite as in the (non)linear sigma model [15, 18].

We found that for finite $N$, the contribution from the vacuum with $Q = 0$ gives rise to a negative pressure for intermediate temperatures where the leading-order pressure increases rapidly. Since this is unphysical, it indicates that quantum instantons contribute significantly to the pressure in this temperature range. In agreement with the disappearance of the instantons in the limit $N \to \infty$, the problem of the negative pressure becomes less severe for large values of $N$.

For the $\mathbb{C}P^1$ model, we found the contribution of the quantum instantons by using its (quantum) equivalence to the $O(3)$ nonlinear sigma model. In the $1/N$ approximation to the $O(3)$ nonlinear sigma model, one implicitly integrates over all quantum instantons and finds a well-behaved, increasing pressure at next-to-leading order in $1/N$ that should be a good approximation to the exact pressure, up to $1/N^2$ corrections, even for $N = 3$ [13]. Assuming that the $1/N$ expansion does not break down in the $\mathbb{C}P^{N-1}$ model for small values of $N$ as well, we have compared the next-to-leading order calculation of the pressure of the $\mathbb{C}P^1$ model without quantum instantons to the pressure of the $O(3)$ nonlinear sigma model. This comparison allowed us to estimate the contribution of quantum instantons with non-zero winding number to the pressure for $N = 2$. For general values of $N$ we were able to estimate a lower bound on the contribution of the quantum instantons by using the fact that the entropy should be positive. Calculating explicitly the contribution from quantum instantons with winding number $Q = 1$ would be a useful step to get a deeper understanding of this issue.