Strings in plane-fronted gravitational waves

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Abstract. Brinkmann’s plane-fronted gravitational waves with parallel rays — shortly pp-waves — are shown to provide, under suitable conditions, exact string vacua at all orders of the sigma-model perturbation expansion.


1. Introduction

The condition for the vanishing of the conformal anomaly for a bosonic string in a curved space is expressed, in σ-model perturbation theory, as $R_{\mu\nu} + \ldots = 0$, where the ellipsis denotes rank two tensors constructed from derivatives and powers of the Riemann tensor [1,2]. An example for which all such correction terms vanish is provided by ‘ordinary’ plane-fronted gravitational waves,

$$dx^2 - 2dudv + K(u,x)du^2,$$

where $x$ belongs to the $(D-2)$-dimensional flat transverse space and $u$ and $v$ are two additional light-cone coordinates. Then all higher-order correction terms vanish due to the special form of the Riemann tensor, and the vanishing anomaly condition reduces to the vacuum Einstein equation $R_{\mu\nu} = 0$ i.e. to

$$\Delta K = 0,$$

where $\Delta \equiv \Delta_{D-2}$ is the (flat) transverse Laplacian [3,4]. For exact plane waves, $K = \sum_{ij} K_{ij}(u)x^ix^j$ where $K_{ij}$ is symmetric and traceless (and for $D = 26$), the constraint $\Delta K = 0$ enforces the anomaly cancellation even non-perturbatively [3].

This result can be extended by including other massless fields, namely a dilaton, $\Phi(u)$, and an axion, $b_{\mu\nu}$, with only non-zero component $b_{iu} = \frac{i}{2}B_{ij}(u)x^j$ where $B_{ij}$ is antisymmetric. Higher-order terms vanish again, and the vanishing anomaly condition is simply [4]

$$\Delta K + \frac{1}{18}B_{ij}B^{ij} + 2\dddot{\Phi} = 0.$$

In the quadratic case the anomaly again vanishes non-perturbatively [5].

Another generalization was found by Rudd [6] who has shown the vanishing of the anomaly for a metric + dilaton system with metric

$$\sum_{i=1}^{D-2} [2\pi R_i(u)]^2(dx^i)^2 - 2dudv,$$

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The Riemann (resp. Ricci) tensors are found to be null vector \((\ell g)\), the sum of a ‘background’ metric and the vorticity \(\Theta\), Eq. (2.3) below, the Brinkmann metric behaves exactly as (1.1) and (1.4).

2. Vanishing of the anomaly in a special Brinkmann wave.

Let us start with the general Brinkmann metric (1.6). Observe first that it admits a covariantly constant null vector \((\ell^\mu)\), namely \(\partial_u\). In order to study the Weyl anomaly, let us decompose the metric (1.6) into the sum of a ‘background’ metric \(g_{\mu\nu}\) with the vector potential terms: setting \(A_u = k/2\) and \(A_v = 0\), (1.6) is re-written as

\[
(2.1) \quad \tilde{g}_{\mu\nu} = g_{\mu\nu} + 2\ell_{(\mu}A_{\nu)} \quad \text{where} \quad g_{\mu\nu}dx^\mu dx^\nu = g_{ij}(u, x)dx^i dx^j - 2du dv.
\]

The Riemann (resp. Ricci) tensors are found to be

\[
(2.2) \quad \begin{cases}
\tilde{R}_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \ell_{(\mu} \nabla_{\nu)} F_{\rho\sigma} - \ell_{(\rho} \nabla_{\sigma)} F_{\mu\nu} - \ell_{(\mu} F_{\nu)}^{\alpha} \ell_{\alpha\rho\sigma}, \\
\tilde{R}_{\mu\rho} = R_{\mu\rho} - \ell_{(\mu} \nabla^{\nu} F_{\nu\rho)} - \frac{1}{2} \ell_{\mu} F^{\nu\sigma} F_{\nu\sigma}.
\end{cases}
\]

where \(\nabla\) is covariant derivative with respect to the metric \(g_{\mu\nu}\), also \(F_{\mu\nu} \equiv 2\partial_{[\mu} A_{\nu]}\) and \(F_{\nu\rho}^{\mu} \equiv \tilde{g}^{\mu\rho} F_{\rho\nu}\).

A look at Eq. (2.2) confirms that the higher order terms in the perturbation expansion do not vanish in general [10]. For example, \(\tilde{R}_{\rho\sigma\lambda} \tilde{R}^{\rho\sigma\lambda} \neq 0\), etc. The clue of Horowitz and Steif to overcome this is to express the Riemann tensor using the covariantly constant null vector and demand that it contain two \(\ell_{\mu}\)'s [4,10]. We show below that this is satisfied by the following special choice:

\[
(2.3) \quad \tilde{g}_{\mu\nu}dx^\mu dx^\nu = g_{ij}(u)dx^i dx^j - 2du \left[ dv + \frac{1}{2} \epsilon_{ij}(u)x^j dx^i \right] + k(u, x)du^2,
\]

where \(\epsilon_{ij}\) is an \(u\)-dependent matrix. Firstly, the background Riemann tensor,

\[
(2.4) \quad R_{\mu\nu\rho\sigma} = -2\ell_{(\mu} \tilde{g}_{\nu\sigma)}^{\rho} \ell_{\rho)} + g^{\alpha\beta} \ell_{(\mu} \tilde{g}_{\nu\sigma)}^{\alpha} \ell_{\alpha\beta\rho)}^{\beta},
\]

is proportional to two \(\ell_{\mu}\)'s as required. Secondly, the two middle terms in the Riemann tensor in Eq. (2.2) will contain two \(\ell_{\mu}\)’s if no triple transverse indices arise,

\[
(2.5) \quad \nabla_i F_{jk} = 0 \quad \text{for all} \quad i, j, k = 1, \ldots, D - 2,
\]

which is automatically satisfied by the choice (2.3). Then the argument of Horowitz and Steif [4] shows that all higher-order terms in the perturbation expansion vanish:

(a) In those terms with at least two \(R_{\mu\nu\rho\sigma}\)’s at least one of the null vectors \(\ell_{\mu}\)’s are contracted, and vanish therefore.
(b) Terms of the form $\bar{\nabla}^\mu \bar{\nabla}^\nu \bar{R}_{\mu\nu\rho\sigma}$ are related to the covariant derivative of the Ricci tensor by the Bianchi identity and hence vanish also.

(c) Terms containing $\bar{R}_{\mu\nu\rho\sigma}$’s as well as its covariant derivatives, one again has to contract at least one $\ell$ on an index of either the curvature tensor or its covariant derivative. In both cases, one gets zero.

Explicitly, the only non-vanishing components of the background Riemann (resp. Ricci) tensors are

\[
\begin{aligned}
R_{iuvj} &= \frac{1}{2} \tilde{g}_{ij} - \frac{1}{4} g^{mn} \tilde{g}_{mi} \tilde{g}_{nj}, \\
R_{uu} &= \left( \frac{1}{2} \tilde{g}_{ij} - \frac{1}{2} g^{mn} \tilde{g}_{mi} \tilde{g}_{nj} \right) g^{ij}.
\end{aligned}
\]

Thus, the vacuum Einstein equations $\bar{R}_{\mu\nu} = 0$ require

\[
\bar{R}_{uu} = \left( \frac{1}{2} \tilde{g}_{ij} - \frac{1}{2} g^{mn} \tilde{g}_{mi} \tilde{g}_{nj} \right) g^{ij} - \nabla^i F_{ui} - \frac{1}{4} F_{ij} F^{ij} = 0,
\]

and we conclude that if (2.7) holds, then we get an exact string solution at all orders in sigma-model perturbation theory.

3. Reduction to ordinary plane wave.

Now we explain why the special Brinkmann metric (2.3) works. We prove in fact that (2.3) can be brought into the simple form (1.1) by a sequence of coordinate transformations. At each step, $x, u, v$ (resp. $X, U, V$) denote the old (resp. new) coordinates.

**Step 1.** The positive transverse metric $g_{ij} = g_{ij}(u)$ is, by assumption, a function of $u$ only. There exists therefore a (time-dependent) matrix $C = (C_i^a) \in GL(D - 2, \mathbb{R})$ such that

\[
g_{ij}(u) = \delta_{ab} C_i^a(u) C_j^b(u).
\]

Such a $C$ is unique up to an orthogonal matrix. Then, introducing the inverse matrix $D = C^{-1}$, the coordinate transformation

\[
X = Cx, \quad U = u, \quad V = v
\]

flattens out the transverse metric while preserving the form of the other terms, i.e. yields

\[
dX^2 - 2dU \left[ dV + \frac{1}{2} E_{ij} X^j dX^i \right] + K(U, X) dU^2,
\]

where

\[
E = (E_{ij}) = -2 D^{-1} \dot{D} + D^T e D, \quad e = (e_{ij}),
\]

\[
K = k + X^T \left[ \dot{D}^T (D^{-1})^T D^{-1} \dot{D} - \dot{D}^T e D \right] X,
\]

the superscript ‘T’ denoting transposition. This generalizes a result of Gibbons [11].

Let us now decompose the matrix $E_{ij}(u)$ into the sum of a symmetric and an antisymmetric matrix, $E_{ij} = S_{ij} + A_{ij}$.

**Step 2.** The symmetric part yields a gradient, $S_{ij} x^j = \partial_i \Lambda$ with $\Lambda(u, x) = \frac{1}{2} S_{ij}(u) x^i x^j$, and can therefore be gauged away as

\[
X = x \quad U = u \quad V = v + \frac{1}{2} \Lambda(u, x),
\]
brings the metric (3.3) into the form
\[
dX^2 - 2dU \left[ dV + \frac{1}{2} A_{ij} X^i dX^j \right] + K(U, X) dU^2,
\]
with
\[
K = k(U, X) + \partial_u \Lambda = k + \frac{1}{2} \delta_{ij} X^i X^j.
\]

**Step 3.** The freedom in choosing the matrix \( C \) can be used to transform away the antisymmetric part \( A_{ij} \): there exists an *orthogonal* time-dependent matrix \((O_{ij}) \in O(D-2)\) such that
\[
A \equiv (A_{ij}) = -2O^{-1}\dot{O},
\]
and the coordinate transformation
\[
X = O\hat{x} \quad U = u \quad V = v
\]
gives the metric (3.6) the simple form (1.1) with
\[
K = k + \frac{1}{4} A_{ij} A_{ij} X^i X^j.
\]

The absence of the conformal anomaly follows therefore from those results proved in Refs [3-5] for ordinary plane waves, provided Eq. (1.2) is satisfied.

At this stage, we can include axions and dilatons: first, the metric is brought into the ordinary plane wave form (1.1), and then the condition for the vanishing of the anomaly is simply derived from the Horowitz-Steif condition (1.3).

Observe that, at each step, the additional term added to the coefficient of \( k \) is quadratic in the transverse variable. Thus, if the function \( k \) in Eq. (2.3) we started with was quadratic, we would end up with an exact plane wave and the vanishing of the anomaly would follow non-perturbatively from Refs. [3] and [5].

**4. Examples.**

i) Consider first the special Brinkmann metric
\[
\phi(u) \, dx^2 - 2du \left[ dv + \frac{1}{2} A_{ij}(u) x^i dx^j - \frac{1}{2} \phi(u) \delta_{ij} x^i dx^j \right] + k(u, x) du^2,
\]
where \( A_{ij} \) is antisymmetric and \( \phi > 0 \) [12,9]. As explained in Section 2, this yields an exact string vacuum as soon as the vacuum Einstein equation \( \tilde{R}_{uu} = 0 \), i.e.
\[
\Delta k - \frac{A_{ij} A^{ij}}{2\phi} + \frac{D-2}{2} (\log \phi)^\prime \phi = 0,
\]
is satisfied. Another way of obtaining this result is to carry out the coordinate transformations indicated in Section 3,
\[
\begin{align*}
1. \quad &X = \sqrt{\phi} \hat{x}, \quad U = u, \quad V = v \\
2. \quad &X = \hat{x}, \quad U = u, \quad V = v + \frac{1}{2} \phi^{-1} \dot{\phi} x^2 \\
3. \quad &X = O\hat{x}, \quad U = u, \quad V = v
\end{align*}
\]
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where \(-2O^T \dot{O} = (A_{ij})\). This results in expressing (4.1) as the ordinary plane wave (1.1) with

\[
K(U, X) = k(u, \frac{X}{\sqrt{\phi}}) - \frac{1}{4\phi^2}A_{ik}A_j^k X^i X^j,
\]

and then Eq. (4.2) follows simply from (1.2), \(\Delta K = 0\).

Interestingly, the Ansatz (4.1) is built from the same ingredients as the metric + axion + dilaton system studied by Horowitz and Steif [4,5,10], and the condition (4.2) is, up to the sign of the quadratic term and to some re-definition of the fields, the same as the condition (1.3) of Horowitz and Steif [4]. Adding to the metric (4.1) an axion and a dilaton, \(B_{ij}\) and \(\Phi(u)\), the Horowitz-Steif condition (1.3) is generalized to

\[
\Delta k + \left[ D - \frac{2}{2} \right] (\log \phi)^{-1} \phi + 2\dot{\phi} + \frac{1}{8} B_{ij} B^{ij} - \frac{A_{ij} A^{ij}}{2\phi} = 0.
\]

Let us point out that Eq. (4.5) hints also to a possible cancellation between the vector potential and the axion. Using the non-symmetric connection-approach [13], we have shown recently [14] that this is indeed the case. For \(\phi = 1, \Phi = 0\) for example, we recover a recent result presented in Ref. 14.

ii) Things work similarly in Rudd’s toroidal case (1.4). The metric is plainly of the form (2.3) and provides therefore an exact string vacuum as soon as Einstein’s equation, \(\tilde{R}_{uu} = \sum_i \tilde{R}_i = 0\), is satisfied. The more general condition (1.5) can be derived by ‘straightening out’ the transverse metric following Step 1 and then by gauge-transforming,

\[
\begin{align*}
1. & \quad X' = 2\pi R_i(u)x^i, \quad U = u, \quad V = V \\
2. & \quad X = x, \quad U = u, \quad V = v + \frac{1}{2} \sum_i \tilde{R}_i (x^i)^2
\end{align*}
\]

which takes (1.4) into the exact plane wave

\[
dx^2 - 2dudv + \left( \sum_i \tilde{R}_i (x^i)^2 \right) du^2.
\]

The associated Einstein equation \(\Delta K = 0\) is plainly the same as \(\tilde{R}_{uu} = 0\) above. Adding a dilaton, \(\Phi(u)\), condition (1.5) follows from the Horowitz-Steif condition (1.3). The vanishing of the anomaly follows from the results in Ref. [4] perturbatively, and from [5] non-perturbatively. (Note that one could also add an axion in the same way.)

iii) The two previous Ansätze, (1.4) and (4.1), can be unified by considering rather

\[
\phi(u) R_i^2(u)(dx^i)^2 - 2du \left[ dv + \frac{1}{2} A_{ij}(u)x^j dx^i - \frac{1}{4} \dot{\phi}(u) R_i^2(u) \delta_{ij} x^i dx^j \right] + k(u, x) du^2.
\]

Repeating the previous calculation we get that if

\[
\Delta k - \frac{A_{ij} A^{ij}}{2\phi} + \sum_i \tilde{R}_i \phi + \frac{D - 2}{2} (\log \phi)^{-1} \phi = 0,
\]

then the anomaly vanishes at all orders in sigma-model perturbation theory.

5. Discussion.

In the ‘one-coupling case’ considered in the first example of Section 4, i.e. for \(g_{ij} = \phi(u) \delta_{ij}\), Step 1 can be replaced by a conformal rescaling. Indeed, if \(\tilde{\eta}_{\mu\nu} = \phi(u) \eta_{\mu\nu}\), then (see, e.g. [15])

\[
\tilde{R}_{uu} = \tilde{R}_{uu} + \frac{D - 2}{2} \left[ (\log \phi)^{-1} + \frac{1}{2} \left( \frac{\phi}{\phi} \right)^2 \right].
\]
Calculating the Ricci tensor of the rescaled metric and using (5.1) we, again, get the constraint (4.2).

Our Ansatz (2.3) is the most general pp-wave in \( D = 4 \) dimensions \([7]\). In higher dimensions this is no longer true, however, and (1.6) is indeed more general than the ordinary plane wave (1.1).

In suitable coordinates all Brinkmann metrics \([7]\) can be written as \( g_{ij}(u, x)dx^i dx^j - 2dudv \), so that all information is encoded in the transverse metric. In Section 3 we have ‘flattened out’ the transverse metric under the assumption that this latter is a function of \( u \) only. This is, however, not the most general case when higher-order correction terms vanish. Consider, for example, the time \( u \) and space \( i \) dependent metric

\[
\sum_{i=1}^{D-2} \cosh^2 \left( \sqrt{\epsilon_i w_i (x^i + u)} \right) \left( dx^i \right)^2 - 2dudv,
\]

where \( \epsilon_i = \pm 1 \) and \( w_i = \text{const.} \) Introducing

\[
\begin{align*}
X^i &= \frac{1}{\sqrt{\epsilon_i w_i}} \sinh \left( \sqrt{\epsilon_i w_i (x^i + u)} \right), \\
V &= v + \frac{1}{2} \sum_i \left( \frac{1}{2\sqrt{\epsilon_i w_i}} \sinh \left( 2\sqrt{\epsilon_i w_i (x^i + u)} \right) + x^i + u \right), \\
U &= u,
\end{align*}
\]

then (5.2) is turned into the exact plane wave (1.1) with \( K = \sum_i \left( 1 + \epsilon_i w_i^2 (X^i)^2 \right) \), which is an exact string vacuum as soon as the vacuum Einstein equation \( \sum_i \epsilon_i w_i^2 = 0 \) is satisfied.

More generally, Step 1 can be implemented whenever the transverse metric is conformally flat, which requires the transverse Weyl tensor to vanish for each fixed value of \( u \).

At last, it would be interesting to know (i) precisely when can a general Brinkmann metric (1.6) be brought into the ordinary plane-wave form (1.1) and (ii) if this is indeed necessary for the vanishing of all higher-order terms.

\textbf{Note added.} After this paper was published, anomaly cancellation has been proved non-perturbatively in a Wess-Zumino context \([16]\).

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\textbf{References}


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