Tetrads in Yang-Mills geometrodynamics

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A new set of tetrads is introduced within the framework of \(SU(2) \times U(1)\) Yang-Mills field theories in curved spacetimes. Each one of these tetrads diagonalizes separately each term of the Yang-Mills stress-energy tensor. Therefore, three pairs of planes also known as blades, can be defined, and make up the underlying geometrical structure, at each point. These tetrad vectors are gauge dependent on one hand, and also in their definition, there is an additional inherent freedom in the choice of two vector fields. In order to get rid of the gauge dependence, another set of tetrads is defined, such that the only choice we have to make is for the two vector fields. A particular choice is made for these two vector fields such that they are gauge dependent, but the transformation properties of the tetrads are analogous to those already known for curved spacetimes where only electromagnetic fields are present. This analogy allows to establish group isomorphisms between the local gauge group \(SU(2)\), and the tensor product of the groups of local Lorentz tetrad transformations, either on blade one or blade two.

I. INTRODUCTION

Electromagnetic fields can be used to introduce at each point in a spacetime a local structure where a pair of blades can be defined through the use of a special tetrad \([1][2][3]\). Schouten defined what he called, a two-bladed structure in a spacetime \([3]\). These blades are planes, defined by the tetrad vectors. The timelike and one spacelike vector define what we called in \([1]\), blade one, and the other two spacelike vectors, define blade two. In turn, this tetrad is built out of the extremal field, defined through a duality rotation of the local electromagnetic field. At every point in spacetime there is a duality rotation by an angle \(−\alpha\) that transforms a non-null electromagnetic field into an extremal field,

\[
\xi_{\mu\nu} = e^{-\alpha f_{\mu\nu}} .
\]

Extremal fields are essentially electric fields and they satisfy,

\[
\xi_{\mu\nu} \ast \xi^{\mu\nu} = 0 .
\]

In \([1]\) it was proved that the local electromagnetic gauge group is isomorphic to the local group of Lorentz transformations of the tetrad vectors on blades one and two. The relation between the electromagnetic gauge transformations and the local Lorentz tetrad transformations on both blades was straightforward. The simplification in the expression of all relevant fields and equations was maximum. It was natural then, to ask if similar structures could be built for non-Abelian fields \([4]\). The Abelian nature of the electromagnetic field results in the existence of just one extremal field, \(\ast\) gauge transformations, the vectors that define blades on one and two should remain algebraically “decoupled”; but at a price. The tetrad vectors are on one hand \(SU(2)\) gauge dependent, and on the other hand, in their definition, there is an inherent freedom in the choice of two vector fields \(X^\mu, Y^\mu\). If we transform these two vectors as \(X^\mu \rightarrow X^\mu + \Lambda^\mu\), with \(\Lambda\) a scalar function, and, \(Y^\mu \rightarrow Y^\mu + \ast\Lambda^\mu\), with \(\ast\Lambda\) a scalar function, then the tetrad vectors transform in an analogous fashion to the electromagnetic Abelian case \([1]\), under \(U(1)\) transformations. \(\Lambda^\mu\) is simplifying notation for \(\xi^\mu\). The problem dwells in the \(SU(2)\) local gauge transformations, and the \(SU(2)\) local gauge dependence of the “decoupled” tetrads. We would like to find a tetrad such that the transformation properties of the tetrad vectors are analogous to the Abelian case, but under \(SU(2)\) transformations, that is, the two vectors that define blade one, remain on blade one after the \(SU(2)\) transformation, and the two that define blade two, remain on blade two after the \(SU(2)\) transformation. The question presents itself on the reason for asking such a transformation property to be fulfilled by the tetrad vectors. The answer is that the metric tensor once is built out of the tetrad vectors, must be invariant under \(SU(2)\) local gauge transformations. The requirement that for each tetrad, and under local \(SU(2)\) gauge transformations, the vectors that define blades one and two should remain on their respective blades, ensures the invariance of the metric tensor under local \(SU(2)\) gauge transformations. In order to find this new tetrad with the required \(SU(2)\) gauge transformation properties we proceed to build a new kind
of extremal field. In addition, we use a new duality rotation that involves a new complexion. Once we have this new extremal field, building the new tetrad with the acceptable \(SU(2)\) gauge transformation properties is automatic. We are going to name LB1 the group of Lorentz tetrad transformations on blade one. Analogously we name the group of rotations on blade two, LB2. Following the ideas provided in \[1\] for the Abelian case, as a general guide, it is found that a \(SU(2)\) local gauge transformation generates the composition of two transformations. A tetrad transformation, generated by a locally inertial Lorentz coordinate transformation, and a local Lorentz LB1 transformation of the tetrad vectors on blade one. Then, following again the steps in \[1\] it is proved an isomorphism between the \(SU(2)\) group of transformations, and the tensor product of three LB1 groups. As one of the transformations is generated by the group of locally inertial Lorentz coordinate transformations, the non-commutativity of the image is assured. A similar result for blade two. Through these group isomorphisms between groups of transformations, we analyze the connection between the gauge and geometrical structures.

This manuscript is organized as follows. In section II, a set of three tetrads is introduced by just studying the diagonalization of each term in the stress-energy tensor. In section III new tetrads are introduced such that their transformation properties under \(SU(2)\) gauge transformations follow the same geometrical pattern than the Abelian electromagnetic ones. In section IV, the transformation properties of the tetrads introduced in the previous section are analyzed, as well as the group isomorphisms associated with them. Throughout the paper we use the conventions of \[1\] \[2\]. In particular we use a metric with sign conventions \(++++\), and \(f^k_{\mu\nu}\) are the geometrized Yang-Mills field components, \(f^k_{\mu\nu} = (G^{1/2}/c^2) E^k_{\mu\nu}\).

II. COMPLEXIONS FOR THE DECOUPLED TETRAD

The stress-energy tensor for the \(SU(2)\) Yang-Mills field can be written as \[5\],

\[
T_{\mu\nu} = f^k_{\mu\lambda} f^k_{\nu} \lambda + \ast f^k_{\mu\lambda} \ast f^k_{\nu} \lambda ,
\]

where the summation convention on the internal index \(k\) is applied, and \(f^k_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\sigma\tau} f^k_{\sigma\tau}\) is the dual tensor of \(f^k_{\mu\nu}\). The duality rotation given by equation (59) in \[2\], can be written separately for each internal index \(k\) as,

\[
f^{(k)}_{\mu\nu} = \cos \alpha_k \hat{\xi}^{(k)}_{\mu\nu} + \sin \alpha_k \ast \hat{\xi}^{(k)}_{\mu\nu} ,
\]

or,

\[
\hat{\xi}^{(k)}_{\mu\nu} = \cos \alpha_k f^{(k)}_{\mu\nu} - \sin \alpha_k \ast f^{(k)}_{\mu\nu} ,
\]

where the summation convention for the index \((k)\) between parenthesis is not applied. We can also follow the same procedure as in \[2\] for each internal index \(k\) and define the three complexions by imposing,

\[
\hat{\xi}^{(k)}_{\mu\nu} \ast \hat{\xi}^{(k)\mu\nu} = 0 .
\]

As a result,

\[
\tan(2\alpha_k) = - f^{(k)}_{\mu\nu} \ast f^{(k)\mu\nu} / f^{(k)}_{\lambda\rho} f^{(k)\lambda\rho} .
\]

Then, it is straightforward to express the stress-energy tensor in terms of the extremal field “decoupled” internal components,

\[
T_{\mu\nu} = \sum_{k=1}^{3} \left( \hat{\xi}^{(k)}_{\mu\lambda} \hat{\xi}^{(k)\lambda\nu} + \ast \hat{\xi}^{(k)}_{\mu\lambda} \ast \hat{\xi}^{(k)\lambda\nu} \right) = \sum_{k=1}^{3} T^{(k)}_{\mu\nu} .
\]

Following the Abelian ideas we can define as many sets of tetrad vectors at every point in spacetime, as generators has the gauge group,

\[
V^{(k)}_{\mu} = \hat{\xi}^{(k)}_{\mu\lambda} X^\lambda \quad (9)
\]

\[
V^{(k)}_{(1)} = \sqrt{-Q^{(k)}} / 2 \hat{\xi}^{(k)}_{\mu\lambda} X^\lambda \quad (10)
\]

\[
V^{(k)}_{(2)} = \sqrt{-Q^{(k)}} / 2 \ast \hat{\xi}^{(k)}_{\mu\lambda} Y^\lambda \quad (11)
\]

\[
V^{(k)}_{(3)} = \ast \hat{\xi}^{(k)}_{\mu\lambda} Y^\lambda \quad (12)
\]

2
where \( Q^{(k)} = \xi^{(k)}_\mu \hat{\xi}^{(k)\mu
u} \). \( Q^{(k)} \) is assumed not to be zero. We are free to choose the vector fields \( X^\lambda \) and \( Y^\lambda \), as long as the four vector fields (9-12) are not trivial. Two identities in the extremal field are going to be used extensively in this work, in particular, to prove that tetrad (9-12) diagonalizes the stress-energy tensor \( k \) component. Using the general identity for two antisymmetrical fields,

\[
A_{\mu\sigma} B^{\nu\sigma} - *B_{\mu\sigma} *A^{\nu\sigma} = \frac{1}{2} \delta^{\nu}_{\mu} A_{\sigma\tau} B^{\sigma\tau},
\]

(13)

the first identity results from equation (6), which is in fact the algebraic equation or condition imposed, in order to define the \( \alpha_k \) complexion,

\[
\hat{\xi}^{(k)}_{\mu\sigma} * \hat{\xi}^{(k)\mu
u} = 0.
\]

(14)

Using again (13), we can find the second identity for each internal value of \( k \),

\[
\hat{\xi}^{(k)}_{\mu\sigma} \xi^{(k)\nu\sigma} - *\hat{\xi}^{(k)\mu\sigma} *\xi^{(k)\nu\sigma} = \frac{1}{2} \delta^{\nu}_{\mu} \xi^{(k)\sigma\tau} \xi^{(k)\sigma\tau}.
\]

(15)

When we make iterative use of (14) and (15) we find,

\[
V^{(k)\sigma}_{(1)} T^{(k)\tau}_{(1)} = \frac{Q^{(k)}}{2} V^{(k)\tau}_{(1)}
\]

(16)

\[
V^{(k)\sigma}_{(2)} T^{(k)\tau}_{(2)} = \frac{Q^{(k)}}{2} V^{(k)\tau}_{(2)}
\]

(17)

\[
V^{(k)\sigma}_{(3)} T^{(k)\tau}_{(3)} = \frac{-Q^{(k)}}{2} V^{(k)\tau}_{(3)}
\]

(18)

\[
V^{(k)\sigma}_{(4)} T^{(k)\tau}_{(4)} = \frac{-Q^{(k)}}{2} V^{(k)\tau}_{(4)}.
\]

(19)

In [2] the stress-energy tensor for the Abelian field was diagonalized through the use of a Minkowskian frame in which the equation for this tensor was given in (34) and (38). In this work, for non-Abelian fields we provide the explicit expression for the tetrad in which the stress-energy tensor was given in (34) and (38). In this work, for non-Abelian fields we provide the explicit expression for the tetrad in which the stress-energy tensor \( k \) component is diagonal. The freedom we have to choose the vector fields \( X^\mu \) and \( Y^\mu \), represents available freedom that we have to choose the tetrad. If we make use of equations (14) and (15), it is straightforward to prove that (9-12) is a set of orthogonal vectors. If transformations of the vector field \( X^\mu \rightarrow X^\mu + \Lambda^\mu \), with \( \Lambda \) a scalar function, are introduced in an analogous fashion to [1], regarding these transformations in the general sense explained in [1], in the section “general tetrad”, then, we can carry over each of the “decoupled” tetrads, the same conclusions reached in [1] regarding isomorphisms, for instance. Even though these transformations do not work exactly as in the Abelian case, because the “decoupled” fields \( \hat{\xi}^{(k)\mu\nu} \) are not considered to transform, as transformations that affect only the vector field \( X^\mu \), they behave as in the Abelian case. The “decoupled” tetrads are fundamentally providing information about the number of independent pairs of blades we can build at each point, but their transformation properties do not satisfy the requirement for the invariance of the metric tensor under \( SU(2) \) local gauge transformations.

### III. EXTREMAL FIELD IN \( SU(2) \) GEOMETRODYNAMICS

One of our goals is to build a “non-decoupled” tetrad such that \( SU(2) \) generated tetrad transformations on blade one, leave the tetrad vectors that generate this blade, on blade one, and similarly for \( SU(2) \) generated rotations on blade two. This property is fundamental to ensure the invariance of the metric tensor under \( SU(2) \) local gauge transformations, and is going to be used when proving the existence of morphisms between the local \( SU(2) \) gauge group and the local LB1, LB2 groups. The “decoupled” tetrads clearly do not have this property. Let us define then, a “non-decoupled” extremal field as,

\[
\zeta_{\mu\nu} = \cos \beta \ f_{\mu\nu} - \sin \beta \ *f_{\mu\nu},
\]

(20)

In order to define the complexion \( \beta \), we are going to impose the \( SU(2) \) invariant condition,

\[
Tr[\zeta_{\mu\nu} * \zeta^{\mu\nu}] = \zeta^k_{\mu\nu} * \zeta^{k\mu\nu} = 0,
\]

(21)
where the summation convention was applied on the internal index \( k \). The \( SU(2) \) invariant complexion \( \beta \) turns out to be,

\[
\tan(2\beta) = -f^k_{\mu\nu} \ast f^{k\mu\nu} / f^p_{\lambda\rho} f^{p\lambda\rho},
\]

where again the summation convention was applied on both \( k \) and \( p \).

Now we would like to consider gauge covariant derivatives. For instance, the gauge covariant derivatives of the three “non-decoupled” extremal field internal components,

\[
(22)
\]

\[
(23)
\]

where \( \epsilon_{klp} \) is the completely skew-symmetric tensor in three dimensions with \( \epsilon_{123} = 1 \), and \( g \) is the coupling constant. The symbol “;” stands for the usual covariant derivative associated with the metric tensor \( g_{\mu\nu} \). If we consider for instance the Yang-Mills vacuum field equations,

\[
f^k_{\mu\nu} = 0 \quad (24)
\]

\[
* f^{k\mu\nu} = 0 \quad (25)
\]

then, two potentials, \( A_{k\mu} \) and \( *A_{k\mu} \) do exist for the Yang-Mills field, in addition to the two potentials \( A_{\mu} \) and \( *A_{\mu} \) for the electromagnetic field, that is also present in the space-time under consideration [1]. With all these elements, we can proceed to define the antisymmetric field,

\[
(26)
\]

where \( Z_{\mu\nu\sigma\tau} \) could be for instance \( \zeta^p_{\sigma\tau} \zeta^p_{\mu\nu} - \zeta^p_{\mu\sigma} \zeta^p_{\tau\nu} \), or \( \zeta^p_{\sigma\tau} \zeta^p_{\mu\nu} \). The summation convention on the internal index \( k \) as well as \( p \) was applied. It is clear that (26) is invariant under \( SU(2) \) local gauge transformations. If our choice for an antisymmetric field is (26), then the duality rotation we perform next, in order to obtain the new extremal field, is the duality rotation that we have available on the \( Z_{\mu\nu\sigma\tau} \) tensor,

\[
(27)
\]

As always we choose this complexion to be defined by the condition,

\[
(28)
\]

which implies that,

\[
(29)
\]

This new kind of local \( SU(2) \) gauge invariant extremal tensor \( \epsilon_{\mu\nu} \), allows in turn for the construction of the new tetrad,

\[
(30)
\]

\[
(31)
\]

\[
(32)
\]

\[
(33)
\]

where \( Q_{ym} = \epsilon_{\mu\nu} \epsilon^{\mu\nu} \). It is straightforward using (13), to prove that they are orthogonal.
IV. GAUGE GEOMETRY

The question remains about the choice we can make for the two vector fields \( X^\rho \) and \( Y^\rho \) in (30-33) such that we can reproduce in the \( SU(2) \) environment, the tetrad transformation properties of the Abelian environment.

One possible choice could be \( X^\rho = Tr[\Sigma^{\alpha\beta} E_\alpha^\rho E_\beta^\lambda * \xi_{\rho\sigma} * \xi_{\lambda T} A^T] \) and \( Y^\rho = Tr[\Sigma^{\alpha\beta} E_\alpha^\rho E_\beta^\lambda * \xi_{\rho\sigma} * \xi_{\lambda T} * A^T] \).

The nature of the object \( \Sigma^{\alpha\beta} \) is explained in section VII. \( E_\alpha^\rho \) are tetrad vectors that transform from a locally inertial coordinate system, into a general curvilinear coordinate system. Greek indices \( \alpha, \beta, \), \( \delta, \epsilon, \gamma, \) and \( \kappa, \) are reserved for locally inertial coordinate systems. There is a particular explicit choice that we can make for these tetrads \( E_\alpha^\rho \). We can choose the tetrad vectors we already know from [1], for electromagnetic fields in curved space-times. Following the same notation in [1], we call \( E_\alpha^\rho \) the tetrad transformation properties of the Abelian environment.

Along the lines established in [1] we can study the \( SU(2) \) gauge transformation properties of these two vector fields. We observe that under local \( SU(2) \) gauge transformations \( S, *S, \) introduced in section VI, \( A_\mu \rightarrow \frac{1}{g} S^{-1} A_\mu \) \( S \) and \( *A_\mu \rightarrow \frac{1}{g} S^{-1} *A_\mu \) \( S * \). (34)

\( X^\rho \) and \( Y^\rho \) transform as,

\[
Tr[\Sigma^{\alpha\beta} E_\alpha^\rho E_\beta^\lambda * \xi_{\rho\sigma} * \xi_{\lambda T} A^T] \rightarrow Tr[\Sigma^{\alpha\beta} E_\alpha^\rho E_\beta^\lambda * \xi_{\rho\sigma} * \xi_{\lambda T} + 1 S^{-1} A^T S] + \frac{1}{g} Tr[\Sigma^{\alpha\beta} E_\alpha^\rho E_\beta^\lambda * \xi_{\rho\sigma} * \xi_{\lambda T} \partial^\rho (S)] (36)
\]

\[
Tr[\Sigma^{\alpha\beta} E_\alpha^\rho E_\beta^\lambda * \xi_{\rho\sigma} * \xi_{\lambda T} * A^T] \rightarrow Tr[\Sigma^{\alpha\beta} E_\alpha^\rho E_\beta^\lambda * \xi_{\rho\sigma} * \xi_{\lambda T} + 1 S^{-1} A^T * S] + \frac{1}{g} Tr[\Sigma^{\alpha\beta} E_\alpha^\rho E_\beta^\lambda * \xi_{\rho\sigma} * \xi_{\lambda T} \partial^\rho (S)] . (37)
\]

The field strength transforms as usual, \( f_{\mu\nu} \rightarrow S^{-1} f_{\mu\nu} S, \) and similar for the extremal \( \xi_{\mu\nu}, \) and their duals. Then, we can follow exactly the same guidelines laid out in [1], to study the gauge transformations of the tetrad vectors on blades one, and two. We can carry over the present work, all the analysis done in the gauge geometry section in [1]. It is clear that the vectors \( S^{(1)}_\mu \) and \( S^{(2)}_\mu, \) by virtue of their own construction, remain on blade one after the (34) transformation. It is also evident that after the transformation they are orthogonal. These two facts mean that the metric tensor is invariant under the transformations (34) when the two vectors are normalized. We are assuming for simplicity that \( S^{(1)}_\mu \) is timelike and \( S^{(2)}_\mu \) spacelike, both vectors non-trivial. Let us study then, the transformation of vectors (30-31) under the transformations (34). We can write,

\[
S^{(1)}_\mu = e^{\mu\nu} e^\nu_{\mu} Tr[\Sigma^{\alpha\beta} E_\alpha^\rho E_\beta^\lambda * \xi_{\rho\sigma} * \xi_{\lambda T} S^{-1} A^T S] + \frac{1}{g} e^{\mu\nu} e^\nu_{\mu} Tr[\Sigma^{\alpha\beta} E_\alpha^\rho E_\beta^\lambda * \xi_{\rho\sigma} * \xi_{\lambda T} \partial^\rho (S)] (38)
\]

\[
S^{(2)}_\mu = \sqrt{-\frac{Q_{ym}}{2}} \left( e^{\mu\nu} e^\nu_{\mu} Tr[\Sigma^{\alpha\beta} E_\alpha^\rho E_\beta^\lambda * \xi_{\rho\sigma} * \xi_{\lambda T} S^{-1} A^T S] + \frac{1}{g} e^{\mu\nu} e^\nu_{\mu} Tr[\Sigma^{\alpha\beta} E_\alpha^\rho E_\beta^\lambda * \xi_{\rho\sigma} * \xi_{\lambda T} \partial^\rho (S)] \right) . (39)
\]

It is possible to rewrite equations (38–39) as,

\[
S^{(1)}_\mu = e^{\mu\nu} e^\nu_{\mu} Tr[S \Sigma^{\alpha\beta} S^{-1} E_\alpha^\rho E_\beta^\lambda * \xi_{\rho\sigma} * \xi_{\lambda T} A^T] + \frac{1}{g} e^{\mu\nu} e^\nu_{\mu} Tr[S \Sigma^{\alpha\beta} S^{-1} E_\alpha^\rho E_\beta^\lambda * \xi_{\rho\sigma} * \xi_{\lambda T} \partial^\rho (S) S^{-1}] (40)
\]

\[
S^{(2)}_\mu = \sqrt{-\frac{Q_{ym}}{2}} \left( e^{\mu\nu} e^\nu_{\mu} Tr[S \Sigma^{\alpha\beta} S^{-1} E_\alpha^\rho E_\beta^\lambda * \xi_{\rho\sigma} * \xi_{\lambda T} A^T] + \frac{1}{g} e^{\mu\nu} e^\nu_{\mu} Tr[S \Sigma^{\alpha\beta} S^{-1} E_\alpha^\rho E_\beta^\lambda * \xi_{\rho\sigma} * \xi_{\lambda T} \partial^\rho (S) S^{-1}] \right) . (41)
\]

For the sake of simplicity we are using the notation, \( \Lambda^{(-1)}_{\alpha\beta} = \Lambda^\alpha_{\beta}, \) and no confusion should arise with the transformed vectors \( \tilde{S}^{(1)}_\mu, \tilde{S}^{(2)}_\mu, \) for instance. Now, we can make use of the local transformation properties of the objects \( \Sigma^{\alpha\beta}, \) see section VII, and write,
\[
\hat{S}_n^\mu = \epsilon^{\mu\nu} \varepsilon^\sigma \varepsilon^\tau \theta^\rho \theta^\sigma \theta^\tau \theta^\rho \theta^\sigma \theta^\tau (S) S^{-1} + \nabla^\mu \nabla^\nu \nabla^\sigma \nabla^\tau \nabla^\rho \nabla^\sigma \nabla^\tau \nabla^\rho (S) S^{-1}
\]

(42)

\[
\hat{S}_n^\mu = \sqrt{-Q_{ym}}/2 \left( \epsilon^{\mu\nu} \varepsilon^\sigma \varepsilon^\tau \theta^\rho \theta^\sigma \theta^\tau \theta^\rho \theta^\sigma \theta^\tau (S) S^{-1} \right) + \frac{1}{g} \epsilon^{\mu\nu} \varepsilon^\sigma \varepsilon^\tau \theta^\rho \theta^\sigma \theta^\tau \theta^\rho \theta^\sigma \theta^\tau (S) S^{-1} \left( S \partial^\tau (S) S^{-1} \right)
\]

(43)

We would like to simplify the notation by calling,

\[
S_n^\mu = \epsilon^{\mu\nu} \varepsilon^\sigma \varepsilon^\tau \theta^\rho \theta^\sigma \theta^\tau \theta^\rho \theta^\sigma \theta^\tau (S) S^{-1}
\]

(44)

\[
S_n^\mu = \sqrt{-Q_{ym}}/2 \epsilon^{\mu\nu} \varepsilon^\sigma \varepsilon^\tau \theta^\rho \theta^\sigma \theta^\tau \theta^\rho \theta^\sigma \theta^\tau (S) S^{-1} \left( S \partial^\tau (S) S^{-1} \right)
\]

(45)

Then, we can write equations (42-43) as,

\[
\hat{S}_n^\mu = S_n^\mu + C^\prime S_n^\mu + D^\prime S_n^\mu
\]

(46)

\[
\hat{S}_n^\mu = S_n^\mu + E^\prime S_n^\mu + F^\prime S_n^\mu
\]

(47)

Again, if we carefully follow the steps in the section gauge geometry in [1], we can conclude that,

\[
E^\prime = D^\prime
\]

(48)

\[
F^\prime = C^\prime
\]

(49)

where,

\[
C^\prime = \frac{1}{g} \left( -Q_{ym}/2 \right) \varepsilon^\sigma \varepsilon^\tau \theta^\rho \theta^\sigma \theta^\tau \theta^\rho \theta^\sigma \theta^\tau (S) S^{-1} \left( S \partial^\tau (S) S^{-1} \right)
\]

(50)

\[
D^\prime = \frac{1}{g} \left( -Q_{ym}/2 \right) \varepsilon^\sigma \varepsilon^\tau \theta^\rho \theta^\sigma \theta^\tau \theta^\rho \theta^\sigma \theta^\tau (S) S^{-1} \left( S \partial^\tau (S) S^{-1} \right)
\]

(51)

We would like as well, to calculate the norm of the transformed vectors \( \tilde{S}_n^\mu \) and \( \hat{S}_n^\mu \),

\[
\tilde{S}_n^\mu = \left[ (1 + C^\prime)^2 - D^\prime \right] \tilde{S}_n^\mu
\]

(52)

\[
\hat{S}_n^\mu = \left[ (1 + C^\prime)^2 - D^\prime \right] \hat{S}_n^\mu
\]

(53)

where the relation \( S_n^\mu \) \( S_n^\mu = -S_n^\mu \) \( S_n^\mu \) has been used.

It is possible at this point to repeat all the discussion about the different cases that arise according to the sign of \( (1 + C^\prime)^2 - D^\prime \), as it was done in [1]. It is also straightforward to understand that a similar analysis can be done on blade two, for the transformation of the vectors \( S_n^\mu \) and \( S_n^\mu \) that we are assuming to be spacelike. Our first conclusion from the results above, is that \( SU(2) \) local gauge transformations, generate the composition of several transformations. There is a local tetrad transformation, generated by a locally inertial coordinate transformation \( \Lambda^\alpha_\delta \), of the electromagnetic tetrads \( E^\rho_\alpha \). The normalized tetrad vectors \( S_n^\mu \) and \( S_n^\mu \), undergo a Lorentz transformation on the blade they generate. The two normalized vectors \( \tilde{S}_n^\mu \), \( \hat{S}_n^\mu \), end up on the same blade one, generated by the original normalized generators of the blade, \( \left( \frac{\tilde{S}_n^\mu}{\sqrt{-\sigma^{\mu\nu} \tilde{S}_n^\rho \tilde{S}_n^\rho}}, \frac{\hat{S}_n^\mu}{\sqrt{-\sigma^{\mu\nu} \hat{S}_n^\rho \hat{S}_n^\rho}} \right) \), as it was highlighted at the beginning of this section. Therefore, the gauge invariance of the metric tensor is assured. We can continue making several important remarks about these tetrad transformations. Within the set of LB1 tetrad transformations, there is an identity transformation that corresponds to the identity in \( SU(2) \). To every LB1 tetrad transformation, which in turn is generated by \( S \) in \( SU(2) \), there corresponds an inverse, generated by \( S^{-1} \). We observe also the following. Since locally inertial coordinate transformations in general do not commute, then the locally \( SU(2) \) generated transformations are non-Abelian. The non-Abelianity of \( SU(2) \) is mirrored by the non-commutativity of these locally inertial coordinate transformations. The key role in this non-commutativity is played by the object \( \Sigma^{\alpha\beta} \), that translates local \( SU(2) \) gauge
transformations, into locally inertial Lorentz transformations. Another issue of relevance is related to the analysis of the “memory” of these transformations. We would like to know explicitly, if a second LB1 tetrad transformation, generated by a local gauge transformation $S_2$, is going to “remember” the existence of the first one, generated by $S_1$.

To this end, let us just write for instance the vector $\tilde{h}$ that they are going to transform under SU(2) local functions and the corresponding three LB1 scalar functions $(2)$ into the three LB1 groups, associated to our already chosen three sets of tetrads. Let us then make three choices for the tensor $\tilde{\omega}_{\nu}$, out of all the possible ones, we can study the mutual relation between the SU(2) group, and the three LB1 (or LB2) groups, associated to our already chosen three sets of tetrads. Let us then study the LB1 transformations for one of these sets of tetrads. For all the other tetrads and also for the LB2 rotations the analysis is just analogous. For each SU(2) element $S$, there exist three local scalar functions $\theta^\alpha, i = 1, 2, 3$, see section VI. Borrowing the notation and line of thinking from the section group isomorphism in [1], we can write a set of three equations relating these three SU(2) local functions and the corresponding three LB1 scalar functions $\phi^\alpha(h)$, $\theta^\alpha, i = 1, 2, 3$.

$$
\frac{1}{g}(\sim)_{\text{sym}}/2) Tr[\Sigma^\alpha\beta E_\alpha^\rho E_\beta^\lambda * \xi_{\rho\sigma} * \xi_{\lambda\tau} S^{-1} \partial^\tau(S)] = (\sim)_{\text{sym}}/2) Tr[\Lambda^\alpha_\beta \Lambda_\alpha^\gamma E_\alpha^\rho E_\beta^\lambda * \xi_{\rho\sigma} * \xi_{\lambda\tau} \partial^\tau(S) S^{-1}] \\
= -C'(h) S'(1)_{(1)} + D'(h) S'(2)_{(2)} + M'(h) S'(3)_{(3)} + N'(h) S'(4)_{(4)} \\
(57)
$$

such that,

$$
D'(h) = (1 + C'(h)) \tanh(\phi'(h)) \quad \text{for proper transformations} \tag{58}
$$

$$
D'(h) = (1 + C'(h)) / \tanh(\phi'(h)) \quad \text{for improper transformations} \tag{59}
$$

The index $h$ runs from one to three, representing the three tetrads, and the summation convention is not applied on $(h)$. $M'(h)$ and $N'(h)$ arise for blade two in a similar fashion as $C'(h)$ and $D'(h)$ arise for blade one [1]. Therefore, (57) are three sets of equations that relate the three local $\theta^i$, implicitly included in $S$, and the three local $\phi^\alpha(h)$. We would like to analyze one more issue. Let us suppose that we map the local gauge group SU(2) into the three LB1 groups. We are interested in the injectivity of such a group mapping. Let us suppose then, that $S_1$ and $S_2$ generate the same $\phi'(h)$ for $h = 1 \ldots 3$. The product $S_1 S_2^{-1}$ should generate the identity, meaning that $S_1 = S_2$. Therefore, the injectivity of the mapping remains proved. In section VI it is also proved that the image of this group mapping is not a subgroup of the three LB1 groups. Thus, we are able now to formulate the following results,
Theorem 1 The mapping between the local gauge group SU(2) of transformations is isomorphic, to the tensor product of the three groups of LB1 tetrad transformations.

Following analogously the reasoning laid out in [1], in addition to the ideas above, we can also state,

Theorem 2 The mapping between the local gauge group SU(2) of transformations is isomorphic, to the tensor product of the three groups of LB2 tetrad transformations.

V. CONCLUSIONS

The SU(2) local gauge group of transformations associated with Yang-Mills fields, finds its counterpart in geometrical structures. To find this relation between gauge, and geometrical structures we build in a succession, different sets of tetrad vector fields. First, the three extremal fields and complexions that arise from the diagonalization of each component of the Yang-Mills stress-energy tensor allow for the construction of three sets of tetrad vectors that have a similar structure than their Abelian counterparts [1], but lack the key property they have. This property is related to the fact that in the Abelian environment associated with electromagnetic fields, the local gauge transformation of the tetrad vectors induces a Lorentz transformation on blade one, such that the two vectors that generate this blade, remain on the blade after the transformation. Similar for rotations on blade two. This property is essential as far as we ask for the metric tensor to remain invariant under U(1) transformations in the Abelian case. We demand a similar property for the metric tensor in spacetimes where SU(2) Yang-Mills fields are present. That is the reason why we take on the task of finding tetrads that have transformation properties analogous to the Abelian, in this non-Abelian environment. Once we build these new tetrads in section III, we study their transformation properties. They have an inherent freedom in the choice of two vector fields. These vectors chosen for this particular example in SU(2) × U(1) Yang-Mills geometrodynamics, clearly show in a few steps, that there is a group morphism between SU(2), and LB1. Analogous for LB2. In fact there is a morphism for each tetrad, and we learnt in section II that it is possible to consider as many tetrads as generators has the gauge group. These group mappings clearly show the relation between the local gauge structures and their geometrical counterparts. We must not overlook that in the architecture of our construction, there is the presence of objects related to the Weyl irreducible representation of the Lorentz group. There is also the presence of electromagnetic fields. It is possible then to wonder if there could be a relation to the spacetime associated with a nuclear weak interaction. The question stands about the possibility of extending these constructions to other field structures that involve other irreducible representations of the Lorentz group. The procedure to build the tetrads, has been laid out in a way that automatically allows for its extension, for instance, to SU(3) gauge theories in a curved spacetime. There is an underlying program in these ideas. We quote from [7], "at one time it was even hoped that the rest of physics could be brought into a geometric formulation, but this hope has met with dissapointment, and the geometric interpretation of the theory of gravitation has dwindled to a mere analogy, which lingers in our language in terms like “metric”, “affine connection”, and “curvature”, but is not otherwise very useful". By establishing a link between the local gauge groups of transformations and local geometrical groups of transformations, like in [1], or in the present manuscript, we are trying to bring the gauge theories into a geometric formulation. The geometrization of the gauge theories is where we are aiming at.

VI. APPENDIX I

Following the notation in [6] we write the elements in SU(2) as,

\[ S = \sigma_o \cos(\theta/2) + i \sigma_j \hat{\theta}^j \sin(\theta/2) = \sum \frac{1}{n!} \left( \sum_{i=1}^{3} \frac{i}{2} \sigma_i \theta^i \right)^n, \]  

where \( \sigma_o \) is the identity, \( \sigma_j \) for \( j = 1 \ldots 3 \) are the usual Pauli matrices, and the summation convention is applied for \( j = 1 \ldots 3 \). We can then define the function \( \theta \) as,

\[ \left( \sum_{i=1}^{3} \frac{i}{2} \sigma_i \theta^i \right)^2 = -\sigma_o \left( \frac{\theta}{2} \right)^2, \]

\[ \theta^2 = \sum_{i=1}^{3} (\theta^i)^2, \quad \text{where } \hat{\theta}^i \text{ is given by, } \hat{\theta}^i = \theta^i / | \theta |. \]
Let us consider the $SU(2)$ $2\pi$ parameter sphere in $\theta$ [6], and let us evaluate $S$, and $\partial_\lambda S$ at $\theta = 0$. $S$ is just $\sigma_\theta$. We can write the derivative $\partial_\lambda S$ as,

$$\partial_\lambda S|_{\theta=0} = \left( (-1/2) \sigma_\theta \sin(\theta/2) \partial_\lambda \theta + (1/2) \sigma_\theta \hat{\theta} \sin(\theta/2) \partial_\lambda \theta + i \sigma_\theta \partial_\lambda \hat{\theta} \sin(\theta/2) \right)|_{\theta=0}. \quad (63)$$

If we consider for instance, all possible geodesics through the $2\pi$ parameter sphere origin, then we must conclude that the four components of $\partial_\lambda S|_{\theta=0}$ can take on any value, ranging from $-\infty$ to $+\infty$. Then, accordingly, the vector components of $Tr[\bar{A}_\beta^\lambda \Lambda_\gamma^\beta \Sigma^{\delta \gamma} E^\rho_\gamma E^\lambda_\rho \xi_{\rho\sigma} \partial^\nu (S) S^{-1}]$, can take on any values ranging again, from $-\infty$ to $+\infty$. Borrowing once more the notation and line of thinking from [1], specially the section gauge geometry, we can see that in correspondence to the scalars we named $C$ and $D$ in [1] we get the scalars $C'$ and $D'$.

Since the vector components of $Tr[\bar{A}_\beta^\lambda \Lambda_\gamma^\beta \Sigma^{\delta \gamma} E^\rho_\gamma E^\lambda_\rho \xi_{\rho\sigma} \partial^\nu (S) S^{-1}]$, can take on any values, positive or negative, then we must conclude that $1 + C'$ and $D'$ can take on any possible real values. Borrowing again the ideas from [1], we can analyze as an example, the case where $1 + C' > D' > 0$, and $0 > C' > -1$. Let us suppose in addition that $\partial_\rho \theta$, $\partial_\rho \hat{\theta}$ and $\hat{\theta}$ have finite components at the origin. We can always consider the geodesic through the origin of the $2\pi$ sphere, such that $\theta_n = n \theta^i$, where $n$ is a natural number. Now, $\theta_n = n \theta$ and $\hat{\theta}_n = \hat{\theta}^i$, but $\partial_\rho \theta_n = n \partial_\rho \theta$. Then, at the origin of the parameter sphere, $\theta^i = 0$, $\hat{\theta}_n = 0$ and $\partial_\rho \theta_n|_{\theta=0} = n \partial_\rho \theta|_{\theta=0}$. Putting all this together we have that for the new geodesic, for $n$ sufficiently large, $D'_n > 0 > 1 + C'_n$. This implies one important conclusion. The $SU(2)$ group of local gauge transformations, generates proper and improper LB1 transformations. Therefore the image of $SU(2)$ is not associated to a subgroup of LB1.

VII. APPENDIX II

The object $\sigma^{\mu\nu}$ is defined as $\sigma^{\mu\nu} = \sigma_+^\mu \sigma_-^\nu - \sigma_-^\mu \sigma_+^\nu$, [8] [9]. The object $\sigma_\pm^\mu$ arises when building the Weyl representation for left handed and right handed spinors. According to [9], it is defined as $\sigma_\pm^\mu = (1, \pm \sigma^i)$, where $\sigma^i$ are the Pauli matrices for $i = 1 \cdots 3$. Under the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ spinor representations of the Lorentz group it transforms as,

$$S_{(1/2)}^{-1} \sigma_\pm^\mu S_{(1/2)} = \Lambda^\mu_{\ \nu} \sigma_\pm^\nu. \quad (64)$$

Equation (64) means that under the spinor representation of the Lorentz group, $\sigma_\pm^\mu$ transform as vectors. In (64), the matrices $S_{(1/2)}$ are local, as well as $\Lambda^\mu_{\ \nu}$ [9]. The $SU(2)$ elements can be considered to belong to the Weyl spinor representation of the Lorentz group. Since the group $SU(2)$ has a homomorphic relationship to $SO(3)$, they just represent local space rotations. It is also possible to define the object $\Sigma^{\mu\nu} = \sigma_+^\mu \sigma_-^\nu - \sigma_-^\mu \sigma_+^\nu$, analogously. Nonetheless, it is relevant to understand that in order for the two vector fields $X^\rho$ and $Y^\rho$ to be real valued, we have to choose the object $\Sigma^{\mu\nu}$ such that it is Hermitic. Then, a possible choice could be for instance, $\Sigma^{\mu\nu} = i \left( \sigma^{\mu\nu} + \sigma^{\nu\mu} \right)$. This a particularly suitable choice when we consider Lorentz space rotations of the electromagnetic tetrad vectors.

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