THEORY OF HIGH-ENERGY SCATTERING AND MULTIPLE PRODUCTION

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We wish to report about the different consequences of a model for high-energy interactions. This model has been suggested to us by the structure of the strip approximation to the Mandelstam representation and can be simply understood as a generalization to very high energy of the peripheral model. The basic idea is that the main contributions to multiple production are given by a combination of a large number of low-energy processes. The graphs we are considering are shown in Fig. 1:

![Graphs showing low-energy two-body processes](image)

**Fig. 1**

Each bubble represents a low-energy two-body process. The number of multiperipheral graphs does, of course, increase with increasing energy. We wish to show that the sum of all multiperipheral effects exhibits, in the high-energy limit, particularly simple features, both for elastic scattering and for multiple production.

It is clear that the knowledge of the production amplitude allows to compute not only the production cross-section, but also the imaginary part of the elastic scattering amplitude through the unitarity relation (see Fig. 2):

\[
A(p, p', p_1, p_2') = \langle p', p_2' | T | p, p_2 \rangle = \sum_{\mu} \langle p', p_2' | T^+ | \mu \rangle \langle \mu | T | p, p_2 \rangle
\]

(1)
Let us consider the sum of all multiperipheral effects giving rise to $A$. This sum can be performed by making use of a recurrence relation which allows to compute the $(n+1)$ contribution, once the $n^{th}$ is known. This recurrence relation is visualized in Fig. 3:
\[ A_{n+1}(\rho, \rho', \rho'_2, \rho'_3) = \int dq dq' A_n(\rho, \rho', \rho'_2, \rho'_3) K(p, p', \rho'_2, \rho'_3) \]

(2)

Eq. (2) shows that \( A_{n+1} \) can be computed only when \( A_n \) is known not only on the mass shell, but in correspondence to all space-like values of the four momenta \( q, q' \). The recurrence formula does indeed allow to calculate all multi-peripheral contributions in terms of the low-energy one \( A_0 \). This procedure can be summarized by means of the integral equation

\[ A(s, u, v, t) = A_0(s, u, v, t) + \frac{i}{s} \int ds' du' dv' Q_e(s' s u v v') A(s', u', v', t) \]

(3)

where

\[ s = (p + p_2)^2, \quad u = p_2^2, \quad v = p_3^2 \]

\[ t = (p + p_2)^2 - (q - q')^2 = (p_3 - p_3')^2 \]

\[ s' = (q + p_1)^2 \quad u' = q^2 \quad v' = q'^2 \]

The integral equation

The knowledge of the solution \( A(s, u, v, t) \) of the integral equation is the fundamental problem of our work. Indeed, the on-mass-shell amplitude \( A(s, \mu^2, \mu^2, t) \) leads to the elastic diffraction cross-section, whereas it will be shown that the forward off-mass shell amplitude \( A(s, u, u, 0) \) leads to predictions concerning the average asymptotic properties of energy multiple production. In the asymptotic limit (high multiplicities), the integral equation reduces considerably. First,
the term \( A_0 \) can be dropped and the kernel turns out to depend only on the ratio \( s'/s \), so that the equation is invariant under the transformation \( s \to cs \), \( s' \to cs' \). This allows us to factorize the \( s \) dependence of the amplitude in the simple form

\[
A(suv, t) = s^{\alpha(t)} \mathcal{F}(uv, t)
\]

(4)

The problem is then reduced to the solution of a homogeneous integral equation for \( \mathcal{F}(u_1, u_2, t) \), whose solution determines both the exponent \( \alpha(t) \) and the eigenfunction \( \mathcal{F}(u_1, u_2, t) \). Both eigenvalues and eigenfunctions have a physical meaning: the eigenvalue gives the well-known shrinking of the diffraction peak, whereas the eigenfunction - as already pointed out - is connected with the average properties of multiple production. A form of the scattering amplitude analogous to Eq. (4) has been obtained by many people by adapting to high-energy scattering the results of Regge in potential theory. This analogy can be understood by considering that our multiperipheral graphs, observed in the crossed channel, are the relativistic analogous of the different iterations of the potential model used by Regge.

The predictions obtained by means of the model can be divided into two categories:

a) many general trends of the high-energy collisions do only depend on the transformation property of the integral equation, which is a consequence only of the topology of the multiperipheral graphs.
b) the specific numerical answers (like, for example, the value of the total cross-sections) do depend, of course, on the choice of \( A_0 \) and on the manner in which \( A_0 \) is continued off the mass shell. As is generally known, it has not yet been possible to find a completely satisfactory way of performing such a continuation, especially in the case of higher waves.

We have therefore concentrated our attention on the general model independent predictions, which we shall try to summarize now.

1) Elastic amplitude

The high-energy behaviour of the scattering amplitude \( T(s,t) \) is given by

\[
T(s,t) = -C(t) \left( \frac{2}{T \partial T/\partial t} \right) \pm 1
\]

where the first sign stands for amplitudes symmetric under crossing (\( s \leftrightarrow \bar{s} \)) as, for instance, absolute elastic scattering, while the second sign stands for antisymmetric amplitudes under the crossing. We obtain \( \frac{d \alpha}{dt} > 0 \). The exponent for the charge exchange amplitude is always smaller than the one for the purely elastic one. Eq. (5) turns out to be independent of the scattering particles, apart from the value of \( C(t) \). The \( C(t) \) can be factorized in such a manner that the relation between different amplitudes (dominated by the same pole) is the following:

\[
\frac{T_{xy}(s,t)}{T_{xy}(s,t)} = \frac{T_{xw}(s,t)}{T_{xw}(s,t)}
\]

where \( x, y, z \) and \( w \) represent any kind of particles.
2) Inelastic scattering

The average properties of multiple production are also easy to obtain. To find an average value, for instance, we must perform the sum over all multiperipheral processes with suitable weighing factors. The multiplicity, for instance, shall be given by

\[ \langle N \rangle = \frac{\sum_{n+1} A_n(s,0)}{\leq A_m(s,0)} \]

due to the fact that the \( n \)th peripherism gives just \( n+1 \) final states.

The expression for \( \langle N \rangle \) is indeed extremely simple to obtain:

\[ \langle N \rangle = \frac{\int \sigma A(s,0) A(s,0) ds}{\sigma A(s,0)} = \sqrt{\alpha} \log s + c \]

If we wish to obtain the average spectra of secondaries, i.e., which is the number \( dN_s(k) \) of secondaries with 4 momentum \( k \) : what must be done is to compute how many secondaries in any multiperipherism can have such a momentum and then sum over all multiperipherism. The procedure is indeed simple and can be visualized in Eq. (6) and Fig. 4:

\[ N_s(k) = \sqrt{\int A(s, s_1) A(s_2, u_2) I(s, s_1, s_2, u_1, u_2)} \]  \hspace{1cm} (6)
The actual evaluation leads to

\[ dN_s(k) = \frac{F(m^2, k^2)}{d^3k} \frac{dE_{lab}}{E_{lab}} \]  

(7)

where \( m^2 \) represents the mass, \( k^2 \) the transverse momentum, and \( E_{lab} \) the lab. energy of the secondary in question.

Eq. (7) has the feature that the spectrum of mass and transverse momentum is independent of both the incident and secondary energies (as suggested by experiments). The energy spectrum (for not too high or too low energies) is given by \( \frac{1}{E_{lab}} \). The transverse spectrum is strongly peaked in the forward direction, and its actual shape \( F(k_{T}^2) \) is model dependent.

3) **Relation between asymptotic properties and bound states**

All our equations, established for \( t \leq 0 \), can be continued to positive \( t \). At \( t = 4m^2 \), \( \chi \) develops an imaginary part. Eq. (7) shows already that, for the value \( t = t_B \) for which \( \chi(t_B) \) is an integer, \( \text{Re}T \) develops a pole.
This means that the particles exchange a bound state (or resonance if \( t_B > \sqrt{4 \mu^2} \)) of mass \( t_B \) and of spin \( \alpha'(t_B) \). This relation between asymptotic properties of scattering and bound states can be understood without making appeal to polology. It is possible to show, in fact, that our fundamental integral equation for \( \alpha \) integer, written for \( t > 0 \), coincides with the Bethe-Salpeter ladder equation for a bound state of angular momentum \( \alpha' \).

4) **Cuts in the angular momentum variable**

The weakest point of the whole approach is that unitarity in the \( s \) channel is not taken into account. A similar difficulty appears in all approaches based on the Regge theory, since in the potential model \( s \) means only momentum transfer, and therefore unitarity in this channel has no meaning.

Now, Froissart has shown that unitarity in the \( s \) channel, together with the Mandelstam representation, do not allow cross-sections increasing more rapidly than \((\log s)^2\). So this means that Eq. (4) is only acceptable for \( \alpha(t) \geq 1 \), for \( t \leq 0 \). Now both our equation and the potential model do not give any such limitation for \( \alpha(t) \).

In order to have an idea of the effect of unitarity, we have tried to take into account the corrections due to elastic unitarity corresponding to effects of multiple scattering. These new terms show a new interesting feature: they give rise to continuous distributions of powers (cuts in the angular momentum):

\[
A_{\text{cutf.}}(st) = \int_{s_{\text{max}}}^{s} \int_{\frac{1}{2}}^{1} \mathcal{P}(st) \, dt \, \, d\xi
\]

The position of the upper limit of integration for \( t \to 0 \) is simply given by:

\[
\xi_{\text{max}}(t) = 2 \alpha(t) - 1
\]
So the dominance of the pole on the cut depends on whether $\alpha(0)$ is larger or smaller than 1. For $\alpha(0) > 1$ the cut dominates on the pole, so the whole theory breaks down, as predicted by the Froissart theorem. In the $\alpha(0) = 1$ case, which seems to be suggested by experiment, the cut and the pole coincide.

We wish to emphasize that we have not proved the existence of cuts in the angular momentum, but have just given arguments in favour of their existence. It is, however, interesting to note that these cuts enter in the theory in a non-trivial manner, and that they are a possible mechanism by which unitarity leads to the Froissart limitation in the cross-section.

In conclusion, we have seen that in the framework of our work, many characteristics of high-energy physics show simple regular features. One of such features concerns the Regge pole behaviour of elastic scattering. We want to stress, however, that many other characteristics as, for instance, multiplicities and spectra of production processes, show analogous regularities which - in our opinion - have exactly the same physical basis.

A very interesting confirmation of the results concerning relativistic elastic scattering has been obtained independently by B. Lee and R. Sawyer. These authors have succeeded in extending to the Bethe-Salpeter equation the methods of Regge and Blankenbecler-Goldberger for Yukawa scattering and have proven that the amplitude is meromorphic in the complex angular momentum half-plane $\text{Re } \ell > -\frac{3}{2}$.

An expression for $\alpha(t)$, in the case of weak coupling, is obtained, and coincides - in that limit - with the one obtained by Bertocchi, Fubini and Tonin.