ELASTIC UNITARITY OF THE ASYMPTOTIC AMPLITUDE:
CUTS IN THE ANGULAR MOMENTUM VARIABLE

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Much effort has been devoted lastly to the high-energy behaviour of the scattering amplitude under the assumption that the asymptotic properties are dominated by the Regge poles. The starting point of those speculations has been the work of Regge \(^1\) in which he showed, in the potential theory, the relation between the existence of bound states and resonances on one side, and the asymptotic properties of the scattering amplitudes in the transfer momentum. Many authors \(^2\) have suggested that the result of Regge might be extended to S matrix theory, i.e., that the exchange of bound states or resonances between high-energy scattering particles determines the asymptotic properties of the correspondent elastic amplitudes.

Recently \(^3\), we have solved a specific field theoretical model showing that this was indeed the case and that it allowed us to calculate properties of both elastic and inelastic scattering. This model also shows the limitation of the simple applications of Regge poles and indicates that the corrections might be so important as to obscure the pole description based on Regge trajectories.

In this letter we want to show — without using any specific dynamical model — that large corrections to the simple pole description have to be expected in any scattering field theory in which particles can be created and destroyed. Let us fix our attention on the true elastic scattering amplitude (no spin and isospin flip) for any strong interacting particle scattering. Let us denote as \(T_0(s, t)\) the Regge pole asymptotic contribution (exchange of one "bound state" with the quantum number of the vacuum):

\[
T_0(s, t) \sim C(t) \mathcal{A}(t) \left( - \cot \frac{\pi \mathcal{A}(t)}{\mathcal{A}} + i \right)
\]

(1)

where \(\sqrt{s}\) is the (high) c.m.s. energy in question, and \(t\) the square transfer momentum (negative in the physical region). The functions \(C(t)\) and \(\mathcal{A}(t)\) shall depend on the dynamical origin of the Regge pole: sufficiently general considerations, however, allow to state
\[ \chi(t) \leq 1 \quad \text{for} \quad t \leq 0 \]
\[ \text{Im} \chi(t) = 0 \]
\[ \frac{\text{Re} \chi(t)}{\text{Im} \chi(t)} > 0 \quad \text{for} \quad t \leq 4 M_n^2 . \]

We want to recall that a Regge pole interpolates a family of bound states or resonances with different angular momenta, but with the same dynamical origin. Eq. (1) represents the exchange of one Regge pole, i.e., the extrapolation to negative \( t \) of the effect of all one-resonance (or bound state) exchanges, the resonances belonging to the same family. However, in a relativistic \( S \) matrix theory, we should always have the possibility of exchanging more than one bound state or resonance of the Regge type between the scattering particles. Therefore we shall need a formalism which interpolates groups of two or more bound states or resonances. These corrections are indeed imposed by elastic unitarity in the \( s \) channel, which tells how to compose two amplitudes in a relativistic field theory. We shall investigate them by means of an iterative procedure based on the elastic unitarity and fixed momentum transfer dispersion relations.

Calling \( \mathcal{A}(s,t) \) the absorptive part of the amplitude, the iterative procedure shall be given by the following equation

\[ \mathcal{A}(s,t) = \frac{1}{(8 \pi)^2} \int \frac{T(s,t')}{T(s,t'')} \alpha \chi \text{Im} \mathcal{A} + \mathcal{A}_0(s,t) \]

where the notations are trivial: \( t' \) is the square transfer momentum between the initial and intermediate state, \( t'' \) between the intermediate and final state, the integration being over all the c.m. angles between the incident and intermediate state. \( \int \text{See Fig.1}\int \)
Besides, $\Lambda(s,t)$ and $T(s,t)$ are related by the fixed $t$ dispersion relation that reads asymptotically

$$T(s,t) = \frac{1}{\pi} \int_{0}^{\infty} A(s',t') \left[ \frac{1}{s'} + \frac{1}{s t - s'} \right] ds'$$

(4)

where the number of subtractions that could be necessary depends on the asymptotic behaviour of $A(s,t)$. Let us perform the first iteration by replacing $T(s,t)$ by $T_{0}(s,t)$ of Eq. (1) in (3). After some algebra, and making use of (4), this first iteration gives

$$T(s,t) = T_{0}(s,t) + T_{1}(s,t)$$

(5)

where

$$T_{1}(s,t) = \int_{0}^{s} \int_{0}^{s} P(t') \left( -\cot \frac{\pi t}{2} + i \right)$$

(6)

being

$$P(t') = \frac{2}{(8\pi)^{2}} \int_{-\infty}^{\infty} dt'' \int_{-\infty}^{\infty} dt''' \int_{-\infty}^{\infty} dt'''' \tau(t',t'') \tau(t''',t''') \left[ \cot \frac{\tau(t'',t''')}{2} \cot \frac{\tau(t',t''')}{2} + 1 \right]$$

(7)

$\tau(t,t')$ the triangle function.
\[
T(t, t') = \frac{\Theta(2tt' + 2tt'' + 2tt''' - t^2 - t'^2 - t''^2)}{\sqrt{2tt' + 2tt'' + 2tt''' - t^2 - t'^2 - t''^2}}
\] (8)

\(\Theta\) the usual step function; \(\mathcal{J}_M(t)\) and \(\mathcal{J}_m(t)\) are respectively the upper and lower values of \(\mathcal{J}\) for which the \(\mathcal{J}\) function given by (7) is different from zero.

We see from (6) that while \(T_0\) has an asymptotic power behaviour, \(T_1\) shows a superposition of powers up to a maximum value \(\mathcal{J}_M(t)\). Or, in a different language, the unitarity iteration of the Regge pole \(T_0\) generates a superposition of poles (cut) in the angular momentum variable. Using the very general properties (2) for \(\alpha(t)\) it is possible to show that

\[
\mathcal{J}_M(t) \geq \alpha'(t) + \alpha(0) - 1
\] (9)

where the equal sign holds only for \(t = 0\). In Eq. (9) we have an important result. Let us begin by discussing the case \(\alpha(0) = 1\), i.e., constant high-energy cross-sections. In this case (9) reads

\[
\mathcal{J}_M(t) \geq \alpha(t) \quad .
\] (10)

This means that the maximum power of \(s\) appearing in \(T_1\) shall always be bigger than \(\alpha(t)\) except for \(t = 0\) when both coincide; i.e., that except for the forward direction, \(T_1\) shall prevail over \(T_0\) in (5). In fact, the asymptotic behaviour of \(T_1(s, t)\) shall generally be given by
\[ T_1 \propto \frac{\int_M f(t)}{\log s} \]  \hspace{1cm} (11)

The situation would be different if \( \alpha(0) < 1 \). If, in fact,

\[ \alpha(0) = 1 - \varepsilon \]  \hspace{1cm} (12)

then \( \alpha(t) \) and \( \int_M(t) \) would show the behaviour indicated in fig.2. This means that there shall be a region of transfer momentum for which \( T_0 \) (the pole) dominates over \( T_1 \) (the cut), while for sufficiently big values (in magnitude) of the transfer momentum, again \( T_1 \) will prevail. The region \( t < 0 \) for which the pole prevails is smaller for smaller values of \( \varepsilon \) and shrinks to the point \( t = 0 \) when \( \varepsilon = 0 \). Before trying to discuss the physical meaning of the angular momentum cuts we have found, let us make two lateral remarks. The first one is to point out that the same method used to study the first iteration of (3) can be easily extended to any order of the iteration. We could easily understand that any successive iteration gives rise to a new superposition of powers of \( s \) going up to \( \int_M^{(n)}(t) \), for which

\[ \int_M^{(n)}(t) \geq \int_M^{(n-1)} + \alpha(0) - 1 \]

Therefore, for the case \( \alpha(0) = 1 \), any successive iteration should dominate over the preceding one, which makes quite possible that — similarly to (11) — the over-all asymptotic behaviour of \( T \) shall be

\[ T \propto s g(t \log s) \]
which is fundamentally different from the 0th iteration. For the previously discussed case \( \alpha (0) = 1 - E \) it is clear that

\[
\sum_{\ell} \langle \ell | \mathcal{M} | \ell \rangle = 1 - nE
\]

so that, as discussed for the first iteration, there shall be a region of small transfer momenta for which \( T_0 \) would dominate over \( T_1 \), \( T_2 \) over \( T_2 \), and so on.

The other remark is that the properties we established for the pure inelastic amplitude (i.e., Regge pole with no spin, isospin and strangeness) are trivially extended to other amplitudes (we concentrate, as an example, on charge exchange). In fact, as \( \alpha'_{c, e}(t) > \alpha_{c, e}(t) \), then the largest contribution due to Regge pole iterations occurs when the charge exchange amplitude \( T_0 \) (Regge pole with isospin different from zero) is not iterated with itself, but with the absolute elastic amplitude.

Eq. (9) will now read *)

\[
\int_{\mathcal{M}_{c, e}} (t) \geq \alpha'_{c, e}(t) + \alpha'(0) - 1
\]

and therefore all the discussions of the relative importance of \( T_1 \) over \( T_0 \) (and so for other iterations) - as function of the value of \( \alpha'_{c, e}(0) \) - we did before, apply equally well for the case of charge exchange (or exchange of any quantum number).

Let us now discuss the physical reasons for the existence of the cut contribution. Let us first ask why they were not guessed to be present in the argument obtained by crossing the Regge results on potential scattering 2). From our point of view, the reason is clear and is due to the non-relativistic method in the \( t \) channel that the potential approach implies. If, in fact,

*) If the amplitude in question, instead of being even under crossing as the absolute elastic one, happens to be odd, then the cotangent must be replaced by the tangent in Eqs. (1) or (6).
one would have allowed particle creation, then the cuts would appear as soon as the threshold is overcome, so that two particles could give rise not only to one two-particle bound state, but to two of them. This is perhaps better understood in the light of the identity of a relativistic Bethe-Salpeter bound state problem with a model for high energy elastic and inelastic scattering recently proposed \(^3\). Seen in this way (presence of poles due to bound states, presence of cuts due to production threshold), the result we obtained seems indeed very natural.

The relation between these cuts and the angular momentum variable is less clear; to explore it we would need to be able to compose non-integer angular momenta \(^4\), our experience in this new field being still premature for a simple understanding. A special case for which the meaning of the composition is simpler is the case \( t = 0 \). In this case we must analyse the spin of a system of zero mass being \( \alpha'(0) \) the spin of one zero energy bound state. Having no kinetic energy at disposal (no orbital momentum), a system composed by two bound states would have a spin composed by two \( \alpha'(0) \) spins: this means a maximum of \( 2\alpha'(0) \). This is just the beginning of our continuum (a factor \( s^{-1} \) being purely kinematical).

Let us try to understand next the behaviour of Fig.2, i.e., the fact that for momentum transfers bigger than a certain value, the cut dominates over the pole. We already said that the cut is associated with more than a two-particle (bound or not bound) exchanged system. This means, in the dispersive language, that its singularities in \( t \), at fixed \( s \), lie outside the strip, i.e., further away from the physical negative region. This implies that their dependence in \( t \) (for \( t \leq 0 \)) is weaker than the strip (pole) contribution, \( \frac{\text{slope of } \sum H(t)}{\text{slope of } \alpha(t)} \) in Fig.2 smaller than the slope of \( \alpha(t) \), implying therefore that if for a certain value of \( t \) both contributions are comparable, then for even bigger transfer momentum \( T_1 \) (cut) would dominate over \( T_0 \) (pole).
This gives a new meaning to the speculation on Regge poles at high energy: they are in some way "polological" arguments similar to the "polological" arguments we are now familiar with, in the usual $s, t$ variables. This means that the contributions of the poles are expected to be dominant only in certain regions (as OPEC dominates NN scattering only for some waves), while for other regions, the further singularities can well compete. What we showed in this note is that for diffraction scattering, this region is indeed small and, actually, it shrinks only to the forward direction if the behaviour of high energies total cross-section is really a constant one.

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