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INTERFERENCE PHENOMENA OF NEUTRAL K MESONS *)

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A B S T R A C T

A systematic theoretical treatment of various interference phenomena of the neutral K mesons is presented. The phenomena are treated for the extreme case that CPT, CP and T invariance are all violated. Consequences of these invariances are then derived. Section 2 gives the relationships imposed on the propagators by the invariance conditions and section 3 presents the general analysis of the K_1, K_2 problem in terms of the propagator. In section 4 the space-time development of a K^0 beam is treated and this is applied to the general problem of K^0 decay in section 5. Section 5 also contains a discussion of the 2π decay mode while the leptonic modes are treated in section 6. Finally, in section 7 the scattering and regeneration of K_1 and K_2 mesons is discussed with particular reference to the regenerative forward scattering suggested by M.L. Good as a method for measuring the K_1 - K_2 mass difference.

E R R A T U M

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The condition imposed on the mass matrix by CP invariance alone, that the diagonal elements (as well as the off diagonal) are equal to each other was omitted. This can be easily established by the method used for the off diagonal elements on page 6. Additional changes in the text that result from this correction are as follows :

page 13 : Eq. (32) should read " $r = 1$ and $s = 1$ (CP invariance)".

page 16 : 2nd paragraph, lines 7 and 8 should read "... a maximal failure of CPT and CP invariance ...".

last paragraph should read "Note that when either CPT or CP invariance is valid ...".

page 21 : line 10, delete "CPT and".

page 24 : lines 6, 7, 8 "... while the condition on sr is $sr = 1$ Eq. (32)." Delete the following sentence.

page 25 : line 6, delete "... both CPT and ...".

page 30 : line 5, delete "... both CPT and ...".

line 6, delete "... both CPT and ...".

page 36 : lines 2 and 3 should read "... hence it depends on the question of CP invariance only if CPT also fails ¹⁸⁾. Since the quantity ...".

1. INTRODUCTION

Ever since Gell-Mann and Pais ¹⁾ suggested that the decay properties of neutral K mesons are characterized by states of mixed strangeness, the study of the time dependence of these states has proved to be a very fruitful source of information concerning the weak interactions of strange particles. Because the states $|K^0\rangle$ and $|\bar{K}^0\rangle$ produced by strong interactions are degenerate, the weak interactions are able, in spite of their small magnitude, to produce large interference effects. These interference effects are of second order in the weak interactions and are therefore capable of giving information concerning relative properties (such as relative phases) of different terms in the weak interaction Hamiltonian. This makes it possible to test certain selection rules and basic invariance principles in a very sensitive fashion ²⁾. It is also possible to use the interference phenomena to obtain an accurate determination of the very small mass difference between the K_1 and K_2 mesons ³⁾.

It is the purpose of this paper to present a systematic treatment of the various interference effects which have been discussed in a piecemeal fashion throughout the literature. The usual method of treatment of these problems makes use of the Wigner-Weisskopf perturbation theory. Although this method is adequate, it often obscures some of the important features of the problem with irrelevant details. A more general method, to be used here, is that presented recently by R. Jacob and the present author ⁴⁾, which makes use of the well-known fact that the time dependent properties of unstable particles are determined by the location of the poles of the propagator. In the degenerate case, the location of the poles reduces to an eigenvalue problem when the coupling is weak, as it is here. However, this eigenvalue problem is distinctly different in character from the usual diagonalization of the energy because the matrix to be diagonalized in the present case is not Hermitian. Furthermore, the matrix involves virtual

intermediate states which have a profound effect on the structure of the problem when there is a failure of either CP invariance or CPT invariance. The possibility of such a failure will be included here both because it will clarify the principles involved and because it will demonstrate in detail the manner in which the invariance principle may be tested by means of K mesonic interference phenomena. The eigenvalue problem in question has been treated in detail by Lee, Oehme and Yang²⁾ for the case in which CP alone fails. They used the Wigner-Weisskopf method, but the results obtained here are, of course, equivalent to theirs in that special case.

A persistent problem in the interpretation of the interference experiments has had to do with the difference in mass between the K_1 and K_2 mesons. The question of whether it is the momenta or the energies of these particles that are to be taken equal in the treatment of interference phenomena has often been raised. The method of reference⁴⁾ provides a clear answer to this question since it is based on an analysis of the phenomena in terms of wave packets.

In the following the general invariance properties of the K meson propagators are considered first in order to provide a basis for the discussion of CPT and CP invariance. In section 3, the eigenvalue problems for the propagator is treated and, in section 4, the space-time development of a general K^0 meson system is considered in terms of the 2×2 density matrix for the system. In the next two sections, the decay modes are treated and these are followed by a discussion of scattering and regeneration phenomena.

2. INVARIANCE PROPERTIES OF THE PROPAGATORS

The Heisenberg field operator for the K^0 field at the space-time point x is denoted by $\phi(x)$. The positive frequency part $\phi^+(x)$ of $\phi(x)$ annihilates a K^0 meson and the negative frequency part $\phi^-(\bar{x})$ of $\phi(\bar{x})$, where $\bar{x} = (-x, t)$, creates a $\bar{K}^0$ meson, since we define $\bar{K}^0$ as the result of operating on K^0 with CP. It is assumed that the weak interactions are included in the Hamiltonian generating this field.

The consequence of CPT invariance for the field operator may be expressed in terms of the anti-unitary operator denoted by CPT with the property ⁵⁾

$$CPT \phi(x) CPT = \phi^*(\bar{x}). \quad (1)$$

If the weak interactions are not CPT invariant, this condition does not apply to the true Heisenberg field, it only applies to the field which is "bare" in the sense of weak interactions (but does include all strong-interaction effects).

The consequences of CP invariance may be expressed in terms of a unitary operator CP having the property

$$CP \phi(x) CP = \phi^*(\bar{x}), \quad (2)$$

where $\bar{x}$ is defined as above by

$$\bar{x} = (-x, t). \quad (3)$$

4.

The Feynman propagator is a 2×2 matrix connecting the K^0 and $\bar{K}^0$ states to themselves and to each other. The matrix elements are defined in terms of vacuum expectation values of time ordered products (denoted by $\{ \}_{T}$) of field operators as follows ⁶⁾:

$$\langle K | \Delta'_F(x) | K \rangle = i \langle 0 | \{ \phi^*(x/2) \phi(-x/2) \}_T | 0 \rangle, \quad (4a)$$

$$\langle \bar{K} | \Delta'_F(x) | \bar{K} \rangle = i \langle 0 | \{ \phi(\bar{x}/2) \phi^*(-\bar{x}/2) \}_T | 0 \rangle, \quad (4b)$$

$$\langle K | \Delta'_F(x) | \bar{K} \rangle = i \langle 0 | \{ \phi^*(x/2) \phi^*(-\bar{x}/2) \}_T | 0 \rangle, \quad (4c)$$

$$\langle \bar{K} | \Delta'_F(x) | K \rangle = i \langle 0 | \{ \phi(\bar{x}/2) \phi(-x/2) \}_T | 0 \rangle. \quad (4d)$$

Since $\Delta'_F(x)$ is invariant under proper Lorentz transformations, it is a function only of $\{\underline{x}\}^2$ in its dependence on the three-vector $\underline{x}$. Hence

$$\Delta'_F(x) = \Delta'_F(\bar{x}). \quad (5)$$

The consequences of CPT invariance for the diagonal elements of Δ_F' may now be obtained by assuming the condition Eq. (1). From the definition Eq. (4b) it follows that

$$\langle \bar{K} | \Delta_F'(x) | \bar{K} \rangle = i \langle 0 | \theta(x) \phi(\bar{x}/2) \phi^*(-\bar{x}/2) + \theta(-x) \phi^*(-\bar{x}/2) \phi(\bar{x}/2) | 0 \rangle,$$

where $\theta(x)$ is the usual step function

$$\theta(x) = 0 \quad x_0 < 0,$$

$$\theta(x) = 1 \quad x_0 > 0.$$

Then Eq. (1) yields

$$\langle \bar{K} | \Delta_F'(x) | \bar{K} \rangle = i \langle 0 | \text{CPT} \{ \theta(x) \phi^*(-\bar{x}/2) \phi(\bar{x}/2) + \theta(-x) \phi(\bar{x}/2) \phi^*(-\bar{x}/2) \} \text{CPT} | 0 \rangle$$

From the anti-unitary character of CPT and the fact that the vacuum is invariant it follows that

$$\begin{aligned} \langle \bar{K} | \Delta_F'(x) | \bar{K} \rangle &= i \langle 0 | \theta(x) \phi^*(-\bar{x}/2) \phi(\bar{x}/2) + \theta(-x) \phi(\bar{x}/2) \phi^*(-\bar{x}/2) | 0 \rangle^* \\ &= i \langle 0 | \theta(x) \phi^*(\bar{x}/2) \phi(-\bar{x}/2) + \theta(-x) \phi(-\bar{x}/2) \phi^*(\bar{x}/2) | 0 \rangle \\ &= \langle K | \Delta_F'(\bar{x}) | K \rangle \end{aligned}$$

according to the definition Eq. (4a). In view of Eq. (5), the consequence of CPT invariance is

$$\langle \bar{K} | \Delta'_F(x) | \bar{K} \rangle = \langle K | \Delta'_F(x) | K \rangle, \quad (6)$$

the diagonal elements of the propagator matrix are equal if CPT is valid. There is no corresponding simple result for the off diagonal elements.

In the case of CP invariance, the useful result is found for the off diagonal elements. From the definition Eq. (4c) and the condition Eq. (2) it is found that

$$\begin{aligned} \langle K | \Delta'_F(x) | \bar{K} \rangle &= i \langle 0 | CP \{ \phi(\bar{x}/2) \phi(-x/2) \}_T CP | 0 \rangle \\ &= \langle \bar{K} | \Delta'_F(x) | K \rangle \end{aligned} \quad (7)$$

according to Eq. (4d), since the vacuum is invariant. Thus the off diagonal elements of the propagator matrix are equal if the interactions are all CP invariant.

3. REDUCTION OF PROPAGATOR TO NORMAL FORM

In principle the propagator may be calculated in terms of the contribution $\Pi^*(k^2)$ of the proper self-energy diagrams due to weak interactions. Then it has the form (the metric is such that $k^2 > 0$ is timelike)

$$\Delta'_F(k^2) = \lim_{z \rightarrow k^2 + i\epsilon} [z - m_0^2 - \Pi^*(z)]^{-1},$$

where m_0 is the mass of the K^0 in the absence of weak interactions. In this case Π^* is a 2×2 matrix connecting K and $\bar{K}$ states to themselves and to each other and z and m_0^2 are to be interpreted as multiples of the unit 2×2 matrix.

Since the coupling is very weak, it will be a good approximation to replace $\Pi^*(z)$ by $\Pi^*(m_0^2 + i\epsilon)$ in the neighbourhood of the poles of the propagator, as noted in Ref. ⁴⁾. This substitution includes the effects of wave function renormalization on the propagator to the given approximation. It is convenient to define the matrix

$$W = m_0^2 + \Pi^*(m_0^2 + i\epsilon), \quad (8)$$

and the matrix function

$$F(z) = (z - W)^{-1}. \quad (9)$$

Then the propagator is defined by

$$\Delta'_F(k^2) = \lim_{z \rightarrow k^2 + i\epsilon} F(z), \quad (10)$$

8.

and the poles z_1 and z_2 of the propagator (i.e., of $F(z)$), are the two solutions of the secular equation

$$\det(z_1 - W) = 0. \quad (11)$$

It is shown in Ref. 4) that

$$F(z) = \sum_{\alpha} \phi_{\alpha} \times \phi_{\alpha}^{\dagger} (z - z_{\alpha})^{-1}, \quad (12)$$

where ϕ_{α} is the two-component vector which is a solution of the characteristic equation

$$W \phi_{\alpha} = z_{\alpha} \phi_{\alpha}, \quad (13)$$

and the dual vector $\phi_{\alpha}^{\dagger}$ is the solution of the equation

$$\phi_{\alpha}^{\dagger} W = z_{\alpha} \phi_{\alpha}^{\dagger}. \quad (14)$$

From the eigenvalue equations (13) and (14) it follows that

$$\phi_{\alpha}^{\dagger} \phi_{\beta} = 0, \quad \beta \neq \alpha. \quad (15)$$

The relative normalization of these vectors may be determined from the condition

$$(z - W) F(z) = 1, \quad (16)$$

which follows from the definition, Eq. (9). Eq. (16) may be multiplied from the left by ϕ_β^+ to obtain

$$\phi_\beta^+ (z - W) \sum_\alpha \phi_\alpha \times \phi_\alpha^+ (z - z_\alpha)^{-1} = \phi_\beta^+, \quad (17)$$

after Eq. (12) has been used to substitute for $F(z)$. From Eq. (14), it follows that

$$(z - z_\beta) \sum_\alpha (\phi_\beta^+ \phi_\alpha) \phi_\alpha^+ (z - z_\alpha)^{-1} = \phi_\beta^+, \quad (18)$$

and the orthogonality condition Eq. (15) leads to the result

$$(\phi_\beta^+ \phi_\beta) \phi_\beta^+ = \phi_\beta^+, \quad (19)$$

from which the normalization

$$\phi_\beta^+ \phi_\beta = 1$$

follows. Therefore

$$\phi_\alpha^+ \phi_\beta = \delta_{\alpha\beta}. \quad (20)$$

Although equations (11), (13) and (14) do not correspond to a typical eigenvalue problem because W is not Hermitian, their solution in terms of the matrix elements of W is straightforward. The matrix elements of W are written as

$$\begin{aligned}\langle K|W|K\rangle &= m_0^2 + \omega, \\ \langle \bar{K}|W|\bar{K}\rangle &= m_0^2 + \omega', \\ \langle K|W|\bar{K}\rangle &= p^2, \\ \langle \bar{K}|W|K\rangle &= q^2,\end{aligned}\tag{21}$$

where p and q are to be interpreted as the positive branches of the square roots. The roots of Eq. (11) are then

$$E_\alpha - m_0^2 = \frac{1}{2}(\omega + \omega') - (-1)^\alpha \frac{1}{2}[(\omega - \omega')^2 + 4p^2q^2]^{\frac{1}{2}},\tag{22}$$

where $\alpha = 1$ or 2 .

If the components of the vectors $\phi_\alpha, \phi_\alpha^\dagger$ are denoted by $\phi_{\alpha K}, \phi_{\alpha \bar{K}}, \phi_{\alpha K}^\dagger, \phi_{\alpha \bar{K}}^\dagger$, they are found to satisfy, by virtue of equations (13) and (14), the conditions

$$\phi_{\alpha \bar{K}} = r[\gamma - (-1)^\alpha(1 + \gamma^2)^{\frac{1}{2}}]\phi_{\alpha K},\tag{23a}$$

$$\phi_{\alpha \bar{K}}^\dagger = r^{-1}[\gamma - (-1)^\alpha(1 + \gamma^2)^{\frac{1}{2}}]\phi_{\alpha K}^\dagger,\tag{23b}$$

where

$$r = \ell/p, \quad (24a)$$

$$\gamma = (\omega' - \omega)/2pq. \quad (24b)$$

From the normalization Eq. (20) follows the condition

$$\{1 + [\gamma - (-1)^\alpha (1 + \gamma^2)^{\frac{1}{2}}]^2\} \phi_{\alpha K}^\dagger \phi_{\alpha K} = 1. \quad (25)$$

Either one of the factors $\phi_{\alpha K}^\dagger$ or $\phi_{\alpha K}$ may be chosen arbitrarily and it is convenient to set

$$\phi_{\alpha K}^\dagger = \{1 + [\gamma - (1 + \gamma^2)^{\frac{1}{2}}]^2\}^{-\frac{1}{2}} \quad (26)$$

for the cases to be treated here. Then the definition

$$s = (1 + \gamma^2)^{\frac{1}{2}} - \gamma \quad (27)$$

serves to simplify the notation and the complete solution of the eigenvalue problem is

$$\begin{aligned} z_1 - m_0^2 &= \omega + pq s^{-1}, \\ z_2 - m_0^2 &= \omega - pq s, \end{aligned} \quad (28)$$

from Eq. (22),

$$\phi_1 = (1 + \Delta^2)^{-\frac{1}{2}} [\Delta^2 |K\rangle + \Delta r |\bar{K}\rangle], \quad (29a)$$

$$\phi_2 = (1 + \Delta^2)^{-\frac{1}{2}} [|K\rangle - \Delta r |\bar{K}\rangle], \quad (29b)$$

from Eq. (25) and Eq. (23a), and

$$\phi_1^T = (1 + \Delta^2)^{-\frac{1}{2}} [\langle K| + (\Delta r)^{-1} \langle \bar{K}|], \quad (29c)$$

$$\phi_2^T = (1 + \Delta^2)^{-\frac{1}{2}} [\langle K| - \Delta r^{-1} \langle \bar{K}|], \quad (29d)$$

from Eq. (26) and Eq. (23b). It is to be noted here that ϕ_1 and ϕ_2 are not in general orthogonal to one another while ϕ_1^T and ϕ_2^T are. It is for this reason that ϕ_1 and ϕ_2 are not to be interpreted as "states" in the usual quantum mechanical sense.

The consequences of the considerations of section 2 concerning the invariance of the propagator may now be applied here. The matrix elements of $\Delta_F'(k^2)$ are just the four-dimensional Fourier transforms of the matrix elements given by Eq. (4). Thus the condition following from CPT invariance is that expressed by Eq. (6)

$$\langle \bar{K} | \Delta_F'(k^2) | \bar{K} \rangle = \langle K | \Delta_F'(k^2) | K \rangle.$$

But this implies a corresponding relationship for W

$$\langle \bar{K} | W | \bar{K} \rangle = \langle K | W | K \rangle \quad (30)$$

or, from Eq. (21) and Eq. (24b)

$$\gamma = 0$$

For the parameters of Eq. (28) and Eq. (29), the consequence of CPT invariance is therefore

$$s = 1 \quad (\text{CPT invariance}). \quad (31)$$

The consequence of CP invariance is Eq. (7), the equality of the off diagonal elements or, from Eq. (21) and Eq. (24a)

$$r = 1 \quad (\text{CP invariance}). \quad (32)$$

In the event that both CPT and CP invariance are valid, the amplitudes Eq. (29) become

$$\begin{aligned} \phi_1 &= 2^{-\frac{1}{2}} (|K\rangle + |\bar{K}\rangle), \\ \phi_2 &= 2^{-\frac{1}{2}} (|K\rangle - |\bar{K}\rangle), \\ \phi_1^* &= 2^{-\frac{1}{2}} (\langle K| + \langle \bar{K}|), \\ \phi_2^* &= 2^{-\frac{1}{2}} (\langle K| - \langle \bar{K}|). \end{aligned} \quad (33)$$

The calculation of the parameters appearing in Eq. (28) and Eq. (29) from first principles cannot be carried out in a very satisfactory manner. For any given model of the weak interactions, the lowest order diagrams lead to divergences and for most models the degree of divergence is quadratic or worse. Furthermore, for most models, the reliable calculation of the modification of the weak interaction by strong coupling effects seems to be beyond the scope of presently

available techniques. Therefore, these parameters are best considered as phenomenological parameters to be obtained from experiment. In spite of the theoretical limitations, information of a very general character is to be obtained from such measurements, since basic laws such as the CPT theorem and CP invariance may be tested by comparison with relations like Eq. (31) and Eq. (32). It must also be kept in mind that even an estimate of the cut-off that must be applied in weak coupling theory will lead to some insight into the nature of weak coupling at high momentum transfers.

It is useful to give explicit expression to the real and imaginary parts of z_α by writing

$$z_1 = \left(m + \frac{1}{2}\Delta - \frac{1}{2}i\Gamma_1 \right)^2 \approx m^2 + m\Delta - i m \Gamma_1,$$

(34)

and

$$z_2 = \left(m - \frac{1}{2}\Delta - \frac{1}{2}i\Gamma_2 \right)^2 \approx m^2 - m\Delta - i m \Gamma_2,$$

where

$$m^2 = \frac{1}{2} \operatorname{Re} (z_1 + z_2). \quad (35)$$

Then Δ is the K_1 - K_2 mass difference and Γ_1 and Γ_2 are their total decay rates (at rest) as will be shown in section 4.

In general, one expects the self-energy part of a weak coupling diagram to be of second order in the weak coupling constant. Therefore, the roots $z_\alpha \sim m^2$ are also expected to be of second order. Thus, for example, the decay rates Γ_1 and Γ_2 are of order g^2 in the weak coupling constant g . The mass difference Δ is expected to be of the same order and this is the basis

of the order of magnitude estimate . . . . .

$$\Delta/\Gamma \approx 1, \quad (36)$$

which seems to be borne out by experiment ^{7), 8)}. An interesting comment concerning the mass difference has been made by Okun' and Pontecorvo ⁹⁾. If the weak interaction Hamiltonian may be written in terms of a universal weak current J_μ in the form $gJ_\mu^* J_\mu$, as suggested by Feynman and Gell-Mann ¹⁰⁾ and if the strangeness changing part of the current contains both $\Delta S = \Delta Q$ and $\Delta S = -\Delta Q$ terms, which is contrary to the assumptions of Feynman and Gell-Mann, then there would be a very large term of order g (rather than g^2) in the K_1-K_2 mass difference Δ . This is produced by that term, H_W^1 , of order g in the weak Hamiltonian which connects the states $|K\rangle$ and $|\bar{K}\rangle$, and therefore yields directly a contribution of order g to the off diagonal matrix elements of W . There is no corresponding term in the imaginary part Γ , of the mass since H_W^1 is Hermitian so that W has real eigenvalues to order g . This is a direct consequence of the fact that the K meson mass is below the threshold for any decay channel (such as an intermediate boson channel) with coupling constant $g^{\frac{1}{2}}$, therefore there is no absorptive term of this order in the matrix W .

The experimental value of the mass difference is of order g^2 , not g , but it seems to be well established ¹¹⁾ that both the $\Delta S = +\Delta Q$ and $\Delta S = -\Delta Q$ terms do occur in the leptonic decay of the K^0 . Hence, the notion of a weak Hamiltonian constructed directly from a single universal weak current seems to be ruled out.

A remark should be made here concerning a suggestion made by Glashow ¹²⁾. He noted that a term H_W^1 satisfying the $\Delta S = 2$ rule may be introduced without producing a large mass difference Δ if H_W^1 is odd under C and even under CP . Since H_W^1 is then odd under P , its matrix element between

two states of the same parity must vanish. But the off diagonal elements of W are equal to those of H_W^1 in lowest order, hence such a $\Delta S = 2$ interaction does not produce an off diagonal ($|K\rangle$ to $|\bar{K}\rangle$) matrix element of order g in the matrix W .

Note that, according to Eq. (28) and Eq. (34) the mass difference is given by

$$m\Delta = \frac{1}{2} \text{Re} \{ p q s^{-1} (1+s^2) \}. \quad (37a)$$

The contribution of a $\Delta S = 2$ term H_W^1 to pq is

$$pq = |\langle K | H_W^1 | \bar{K} \rangle|, \quad (38)$$

if H_W^1 is Hermitian. Hence the only way that a cancellation of terms of the order g in the mass difference can occur is if there is a maximal failure of CPT invariance so that $s^{-1}(1+s^2) \approx 0$ (compare Eq. (31)). Thus the only simple way to account for the experimental results is to assume that the weak Hamiltonian contains at least two terms, one with $\Delta S = \Delta Q$ the other with $\Delta S = -\Delta Q$ but that it contains no significant $\Delta S = 2$ term.

Note that when both CPT and CP invariance are valid, Eq. (37a) takes the simple form

$$\Delta = \text{Re} (p^2/m). \quad (37b)$$

4. SPACE-TIME DEVELOPMENT

The method of Ref. ⁴⁾ which is followed here makes use of a prototype experiment involving wave packets to derive the space-time development of the unstable system. Since the experimental limitations on the determination of both the momenta and positions of the particles are far outside the limits imposed by the uncertainty relations, the packets may be chosen so that both the momentum and position are well-determined relative to the experimental errors.

In the prototype experiment, the wave packets of the incident stable particles are assumed to meet (have their maximum overlap) in a small region centred at the point $\underline{x} = 0$, at time $t = 0$.

The virtual K^0 meson is produced there and it goes on to either interact or decay into wave packets which are emitted from a region centred on a space-time point which is simply denoted by $x = (\underline{x}, t)$. Then it is shown in Ref. ⁴⁾ that the 2×2 S matrix between these wave packet states is, to a very good approximation

$$S(x) = \sum_{\alpha} \phi_{\alpha} \times \phi_{\alpha}^T \int d^3k \psi(k, (\underline{z}_{\alpha} + k^2)^{\frac{1}{2}}) e^{i\underline{k} \cdot \underline{x}} \exp[-i(\underline{z}_{\alpha} + k^2)^{\frac{1}{2}} t], \quad (39)$$

where $\psi(k, k_0)$ is an overlap integral of the incident and emerging wave packets. Since those wave packets have very well-defined momenta and energies ψ has a very strong maximum at that value of $\underline{k} = \underline{K}$ and $k_0 = K_0$ satisfying the kinematic conditions for a collision producing the particle of momentum $\underline{K}$ and energy K_0 . $\underline{K}$ also satisfies the kinematic conditions for the decay or interaction of such a particle. At the same time, the Fourier transform of $\psi(\underline{k}, (m^2 + \underline{k}^2)^{\frac{1}{2}})$ is a very strongly localized wave packet:

$$\int d^3k \psi(\underline{k}, (m^2 + \underline{k}^2)^{\frac{1}{2}}) e^{i\underline{k} \cdot \underline{x}} = A(\underline{x}) e^{i\underline{K} \cdot \underline{x}}, \quad (40)$$

where $A(\underline{x})$ has a strong maximum at $\underline{x} = 0$. The width of the packet $\psi(\underline{k}, (m^2 + \underline{k}^2)^{1/2})$ may be assumed to be large compared to the uncertainty in the energy associated with the finite lifetime. Since this uncertainty is of the order of $(z_\alpha - m^2)/m$, the substitution of z_α for m^2 in $\psi(\underline{k}, (m^2 + \underline{k}^2)^{1/2})$ will not alter the function appreciably. Hence, to a very good approximation

$$\int d^3k \psi(\underline{k}, (z_\alpha + \underline{k}^2)^{1/2}) e^{i\underline{k} \cdot \underline{z}} = A(\underline{z}) e^{i\underline{K} \cdot \underline{z}}, \quad (41)$$

where $A(\underline{x})$ is independent of α . Here, $\underline{K}$ is to be interpreted as the expectation value of the momentum determined from a number of events by means of the kinematical conditions.

The usual wave-packet approximation now gives for the integrals appearing in Eq. (39)

$$\int d^3k \psi(\underline{k}, (z_\alpha + \underline{k}^2)^{1/2}) e^{i\underline{k} \cdot \underline{z}} \exp[-i(z_\alpha + \underline{k}^2)^{1/2} t] = A(\underline{z} - \underline{v}_\alpha t) e^{i\underline{K} \cdot \underline{z}} e^{-iK_{0\alpha} t}, \quad (42)$$

where $\underline{v}_\alpha$ is the group velocity given by

$$\underline{v}_\alpha = \underline{K} / K_{0\alpha}, \quad (43)$$

with

$$K_{0\alpha} = (z_\alpha + \underline{K}^2)^{1/2}. \quad (44)$$

The order of magnitude of the difference $|\underline{v}_1|t - |\underline{v}_2|t$ is easily found from the measured lifetime and mass difference. It turns out to be

$$|\underline{v}_1|t - |\underline{v}_2|t \approx |\underline{v}|t(\Delta/m), \quad (45)$$

where

$$\underline{v} = \underline{K}/K_0 \quad (46)$$

is the physical group velocity. Since $\Delta/m \approx 10^{-14}$ and vt is of the order of 10 cm, the difference Eq. (45) is very small compared to any reasonable choice for the width of the packet $A(x)$, hence the difference may be neglected and Eq. (39) may be put in the form

$$S(x) = A(\underline{x} - \underline{v}t) e^{i\underline{K} \cdot \underline{x}} \sum_{\alpha} \phi_{\alpha} \times \phi_{\alpha}^{\dagger} e^{-iK_{0\alpha}t}. \quad (47)$$

Finally, from the definition Eq. (44) of $K_{0\alpha}$ and Eq. (34) for z_{α} , it follows that

$$K_{0\alpha} \approx K_0 - [(-1)^{\alpha} \Delta + i\Gamma_{\alpha}](m/2K_0) \quad (48)$$

with

$$K_0 = (m^2 + \underline{K}^2)^{\frac{1}{2}}. \quad (49)$$

Thus

$$S(x) = A(x - vt) e^{iK \cdot x} e^{-iK_0 t} \sum_{\alpha} \phi_{\alpha} \times \phi_{\alpha}^{\dagger} e^{-i\omega_{\alpha} t}, \quad (50)$$

where

$$\omega_{\alpha} = -[(-1)^{\alpha} \Delta + i\Gamma_{\alpha}^{\dagger}](m/2K_0). \quad (51)$$

By means of Eq. (50), the space-time development of the 2×2 density matrix $\rho(x)$ relating states of strangeness ± 1 may now be obtained. If the density matrix before the initial collision is ρ_0 then

$$\rho(x) = S(x) \rho_0 S^{\dagger}(x).$$

Therefore

$$\rho(x) = |1 + \Delta^2|^{-1} |A(x - vt)|^2 \sum_{\alpha, \beta} \rho^{\alpha\beta} \exp[-i(\omega_{\alpha} - \omega_{\beta}^*)t], \quad (52)$$

where

$$\rho^{\alpha\beta} = |1 + \Delta^2| (\phi_{\alpha}^{\dagger} \rho_0 \phi_{\beta}^{\dagger*}) \phi_{\alpha} \times \phi_{\beta}^*. \quad (53)$$

Usually an initial pure state of definite strangeness is of interest and, in particular, the state with $S = 1$ will be of interest here. Then

$$\rho_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (54)$$

and

$$|1 + \Delta^2| (\phi_{\alpha}^{\dagger} \rho_0 \phi_{\beta}^{\dagger*}) = 1 \quad (55)$$

as a result of the choice of normalization, Eq. (26). Thus

$$\rho^{\alpha\beta} = \phi_{\alpha} \times \phi_{\beta}^* \quad (56)$$

when the initial state has $S = 1$. This form of $\rho^{\alpha\beta}$ corresponds to a density matrix for mixtures of the "pure" states ϕ_1 and ϕ_2 . The meaning of these states is clear from Eq. (52), they are the states of well-defined mass and lifetime, hence they are to be interpreted as the K_1 and K_2 states. However, these are not pure states in the usual sense because in general they are not connected to the states $|K\rangle$ and $|\bar{K}\rangle$ by a unitary transformation, but instead, by the transformation Eq. (29a) and Eq. (29b). Only in the case of CPT and CP invariance does this transformation reduce to a unitary transformation. Then ϕ_{α} and $\phi_{\alpha}^{\dagger}$ are given by Eq. (33), which is in agreement with the usual definition of the K_1 and K_2 .

In the general case, care must be exercised in the interpretation of ϕ_1 and ϕ_2 since they only give the proper relative weights of the K_1 and K_2 meson states when the initial state is a K^0 , in accordance with Eq. (54). For any other initial state, the weight factor $(\phi_{\alpha}^{\dagger} \rho_0 \phi_{\beta}^{\dagger*})$ in Eq. (53) must be taken into account.

5. DECAY PROCESSES

Each decay mode, including the 2π mode, 3π mode and leptonic modes, is denoted here by a latin index j and the corresponding decay amplitudes of the K and $\bar{K}$ mesons by D_{jK} and $D_{j\bar{K}}$. The "decay matrix", D , is then the rectangular, two-column matrix with elements D_{jK} and $D_{j\bar{K}}$. The density matrix for the decay modes is given as a function of x by $D\rho(x)D^*$. According to Eq. (52), the different time dependent expressions $\exp[-i(\omega_\alpha - \omega_\beta^*)t]$ for the decay modes occur with the factors

$$D\rho^{\alpha\beta}D^* = |1 + \lambda^2|(\phi_\alpha^\dagger \rho_0 \phi_\beta^{\dagger*})(D\phi_\alpha) \times (D\phi_\beta)^*. \quad (57)$$

Therefore one may again speak of the "pure" state of the decay mode with the amplitude

$$D^\alpha = D\phi_\alpha, \quad (58)$$

which has the component

$$D_j^\alpha = D_{jK}\phi_{\alpha K} + D_{j\bar{K}}\phi_{\alpha\bar{K}} \quad (59)$$

in the j^{th} decay mode. There is, of course, the connection

$$\Gamma_\alpha = \sum_j |D_j^\alpha|^2 \quad (60)$$

between the total decay rate and the lifetime parameter Γ_α . This can be established by considering the "pure" initial state $\rho_0 = \phi_\alpha \times \phi_\alpha^*$ for a given α , and comparing the rate of decay of the probability of this state, given by the time derivative of Eq. (52), with the total rate at which the decay modes are produced.

The invariances CPT, CP and T impose certain conditions on the matrix D. If the decay state j' is that obtained from j by the operation CP, then the phases may be chosen (for all decay modes other than the 3π modes if we neglect final state interactions for the leptonic modes) so that the following conditions result from the indicated invariances

$$\text{CPT: } D_{j'\bar{K}} = D_{jK}^*, \quad (61a)$$

$$\text{CP: } D_{j'\bar{K}} = D_{jK}, \quad (61b)$$

$$\text{T: } D_{jK}^* = D_{j\bar{K}}. \quad (61c)$$

For the 2π mode, which is the dominant decay mode, only one state j need be considered since CP leaves a 2π state unchanged. Thus

$$j'(2\pi) = j(2\pi) \quad (62)$$

in general. The K_1 and K_2 amplitudes for this decay mode are found from Eq. (59) to be

$$D^\alpha(2\pi) = D_K(2\pi) \phi_{\alpha K} + D_{\bar{K}}(2\pi) \phi_{\alpha \bar{K}}. \quad (63)$$

Then, from Eq. (29a) and Eq. (29b)

$$D''(2\pi) = (1+a^2)^{-\frac{1}{2}} [a^2 D_K(2\pi) + a D_{\bar{K}}(2\pi)] \quad (64a)$$

and

$$D^{(2)}_{(2\pi)} = (1 + \alpha^2)^{-\frac{1}{2}} [D_K(2\pi) - \alpha r D_{\bar{K}}(2\pi)] \quad (64b)$$

are the decay amplitudes of the K_1 and K_2 into the 2π mode. The condition on $D(2\pi)$ from CP invariance is from Eq. (61b) and Eq. (62)

$$D_K(2\pi) = D_{\bar{K}}(2\pi), \quad (65)$$

while the condition on r is $r = 1$ [Eq. (32)]. It is interesting to note that the 2π amplitude of the K_2 does not vanish from CP invariance alone; the condition of CPT invariance, $s = 1$ [Eq. (31)] must also be valid.

In terms of the ratio ξ of the $\bar{K}$ to K amplitudes,

$$D_{\bar{K}}(2\pi) = \xi D_K(2\pi), \quad (66)$$

the condition Eq. (65) for CP invariance reads

$$\xi = 1, \quad (67)$$

and

$$\xi = \frac{D^{(2)}_{(2\pi)}}{D^{(u)}_{(2\pi)}} = \frac{1}{\alpha} \frac{1 - \alpha r \xi}{\alpha + r \xi} \quad (68)$$

provides a measure of the degree of violation of CPT and CP invariance. The quantity $|\xi|^2$ is just the rate of 2π decay of the K_2 relative to the K_1 meson if the initial state is a K_0 meson, in accordance with Eq. (54). Experimentally ¹³⁾ $|\xi|^2 < 10^{-5}$ so that $rs\xi$ is evidently close to one. The simplest conclusion that can be drawn from this is that $r \approx \xi \approx s \approx 1$ which would imply both CPT and CP invariance for the non-leptonic interactions.

In this connection it is important to keep in mind the fact that the conditions determining r and s involve much more than the dominant 2π amplitude. Through the quantity W given by Eq. (8), intermediate 3π states, and possibly 4π states, etc., may contribute appreciably to the determination of r and s . Virtual leptonic states also contribute but it is assumed here that their coupling is weaker than the non-leptonic coupling. Since the experimental limit on $|\xi|$ is comparable to the ratio of leptonic to non-leptonic rates, no conclusion concerning CPT or CP invariance of the leptonic interaction can be obtained at this time from this experiment. It can only be concluded that, if there is a failure of CPT or CP invariance, it probably is limited to leptonic interactions.

6. LEPTONIC DECAYS

The amplitude for leptonic decay may be expressed in terms of form factors for the various possible types of weak interaction. A comprehensive discussion of the way in which the leptonic decay spectrum may be used to determine the nature of the interaction has been given by Pais and Treiman¹⁴⁾. Experimental results obtained to date seem to exclude tensor interactions and to be consistent with the existence of just V and A interactions¹⁵⁾. For the sake of simplicity, the present considerations will be limited to the V, A theory. In that case the matrix element of the weak Hamiltonian, H_W , between a K meson state and a state of pion plus lepton (ℓ) and neutrino takes the form

$$\langle \pi \ell \nu | H_W | K \rangle = (F_1 K_\lambda + F_2 q_\lambda) (\bar{u}_\ell \gamma_\lambda (1 + \gamma_5) \nu_\nu), \quad (69)$$

where u and v are the usual Dirac spinors for the positive and negative energy states, K_λ is a component of the K meson four-momentum and q_λ is a component of the four-momentum transfer between K meson and pion, and the form factors F_1 and F_2 are functions of the invariant q^2 . The term in q_λ may be reduced to the form of a scalar term by making use of the Dirac equation for the leptons

$$\langle \pi \ell \nu | H_W | K \rangle = F_1(q^2) \bar{u}_\ell \gamma_\lambda K_\lambda (1 + \gamma_5) \nu_\nu + i m_\ell F_2(q^2) \bar{u}_\ell (1 + \gamma_5) \nu_\nu, \quad (70)$$

where m_ℓ is the mass of the lepton.

From the fact that muon and electron decay rates are comparable it can be concluded that $F_2 \gtrsim F_1$, if the same form factors may be used for both muon and electron. Hence the term in F_2 is very small for the electron mode and it will be neglected. Thus only one form factor need be considered in the case of the electron and analysis of the electron mode is thereby greatly simplified. For this reason, attention will be limited to the electron mode here, but it should be kept in mind that a number of interesting experiments arise out of the interference between the terms F_1 and F_2 and these are only feasible for the muon mode.

It is also consistent with the data ¹⁵⁾ to neglect the q^2 dependence of the form factor F_1 and thereby reduce the problem to the determination of a constant. However, there are several such constants corresponding to the different elements of the decay matrix. These may be written as

$$\begin{aligned} D_K(\pi^- e^+ \nu) &= f, \\ D_{\bar{K}}(\pi^- e^+ \nu) &= g, \\ D_K(\pi^+ e^- \bar{\nu}) &= g', \\ D_{\bar{K}}(\pi^+ e^- \bar{\nu}) &= f', \end{aligned} \tag{71}$$

where the common spin-dependent factor $\bar{u}_e \gamma_\lambda K_\lambda (1 + \gamma_5) v_\nu$ has now been dropped to simplify the writing.

The amplitudes may be further analysed in terms of $\Delta I = \frac{1}{2}$ and $\Delta I = \frac{3}{2}$ terms to clarify the relationship with the corresponding decay modes of the K^+ meson. If, for example, $f_{1/2}$ and $f_{3/2}$ are the amplitudes corresponding to the indicated change in isotopic spin, then

$$f = f_{\frac{1}{2}} + f_{\frac{3}{2}} \quad (72a)$$

and

$$D_+(\pi^0 e^+ \nu) = 2^{-\frac{1}{2}} (2f_{\frac{3}{2}} - f_{\frac{1}{2}}) \quad (73a)$$

is the amplitude for K_{e3}^+ decay. Similarly

$$f' = f'_{\frac{1}{2}} + f'_{\frac{3}{2}} \quad (72b)$$

and

$$D_-(\pi^0 e^- \bar{\nu}) = 2^{-\frac{1}{2}} (2f'_{\frac{3}{2}} - f'_{\frac{1}{2}}). \quad (73b)$$

The amplitudes g , which necessarily correspond to $\Delta I = 3/2$, do not occur for the decay of the charged K meson.

The invariance conditions, Eq. (61), may be expressed in terms of these amplitudes as follows

$$CPT: f' = f^*, g' = g^*, \quad (74a)$$

$$CP: f' = f, g' = g, \quad (74b)$$

$$T: f^* = f, f'^* = f', g^* = g, g'^* = g'. \quad (74c)$$

The time dependence of the leptonic decay of the K^0 is characterized by the appearance of three terms corresponding to the factors

$$\exp(-\Gamma_1 m t / K_0), \quad (75a)$$

$$\exp(-\Gamma_2 m t / K_0), \quad (75b)$$

and

$$\exp[-i(\omega_\alpha - \omega_\beta^*)t], \quad \alpha \neq \beta, \quad (75c)$$

appearing in Eq. (52). The third term (and its conjugate complex) is the interference term which is such a rich source of information concerning the leptonic interactions. The coefficients of these terms are given by Eq. (57) in terms of K_1 and K_2 decay amplitudes, Eq. (59), which are now expressed in terms of the amplitudes, f , g , etc., by

$$\begin{aligned} D^{(1)}(\pi^- e^+ \nu) &= (1+A^2)^{-\frac{1}{2}}(A^2 f + A r g), \\ D^{(2)}(\pi^- e^+ \nu) &= (1+A^2)^{-\frac{1}{2}}(f - A r g), \\ D^{(1)}(\pi^+ e^- \bar{\nu}) &= (1+A^2)^{-\frac{1}{2}}(A^2 g' + A r f'), \\ D^{(2)}(\pi^+ e^- \bar{\nu}) &= (1+A^2)^{-\frac{1}{2}}(g' - A r f'), \end{aligned} \quad (76)$$

according to Eq. (29).

The relative rate of the K_1 decay into the $\pi^- e^+ \nu$ mode is proportional to $|D^{(1)}(\pi^- e^+ \nu)|^2$, the contribution of the interference term is proportional to $D^{(1)}(\pi^- e^+ \nu)^* D^{(2)}(\pi^- e^+ \nu)$, and so on, as indicated by Eq. (57). These terms appear with the coefficient $|1+s^2(\phi_\alpha^\dagger \rho_\phi \phi_\beta^{r*})|$ which is just equal to one when the initial state is the K^0 state. Relative intensities, charge

asymmetries and other such quantities may therefore be expressed directly in terms of the parameters r, s, f, g , etc., by means of Eq. (76). For example, the charge asymmetry in K_2 decay is

$$|D^{(2)}(\pi^- e^+ \nu)|^2 - |D^{(2)}(\pi^+ e^- \bar{\nu})|^2 = |1 + \alpha^2| (|f - \alpha r g|^2 - |g' - \alpha r f'|^2),$$

which vanishes when both CPT and CP invariance are valid.

The well-known results for the amplitudes, Eq. (76), when both CPT and CP hold are found from Eqs. (31), (32) and (74) to be

$$\begin{aligned} D^{(1)}(\pi^- e^+ \nu) &= 2^{-\frac{1}{2}}(f+g), \\ D^{(2)}(\pi^- e^+ \nu) &= 2^{-\frac{1}{2}}(f-g), \\ D^{(1)}(\pi^+ e^- \bar{\nu}) &= 2^{-\frac{1}{2}}(f+g), \\ D^{(2)}(\pi^+ e^- \bar{\nu}) &= -2^{-\frac{1}{2}}(f-g). \end{aligned} \quad (77)$$

The time dependence of the decay into $(\pi^- e^+ \nu)$ then contains the four terms

$$\frac{1}{2}(f+g)^2 \exp(-\Gamma_1 m t / K_0), \quad (78a)$$

$$\frac{1}{2}(f-g)^2 \exp(-\Gamma_2 m t / K_0), \quad (78b)$$

$$\frac{1}{2}(f^2 - g^2) \exp(-i m t \Delta / K_0) \exp[-\frac{1}{2}(\Gamma_1 + \Gamma_2) m t / K_2], \quad (78c)$$

$$\frac{1}{2}(f^2 - g^2) \exp(i m t \Delta / K_0) \exp[-\frac{1}{2}(\Gamma_1 + \Gamma_2) m t / K_0], \quad (78d)$$

while the decay into $(\pi^+ e^- \bar{\nu})$ contains the terms (78a) and (78b) but has (78c) and (78d) replaced by

$$-\frac{1}{2}(f^{\nu}-g^{\nu}) \exp(-im\Delta t/\kappa_0) \exp\left[-\frac{1}{2}(\Gamma_1+\Gamma_2)mt/\kappa_0\right], \quad (78e)$$

$$-\frac{1}{2}(f^{\nu}+g^{\nu}) \exp(im\Delta t/\kappa_0) \exp\left[-\frac{1}{2}(\Gamma_1+\Gamma_2)mt/\kappa_0\right]. \quad (78f)$$

The coefficients of these terms depend on the initial state through ρ_0 as indicated by Eq. (57) but, for a pure initial state of strangeness +1, these coefficients are all equal.

7. SCATTERING AND REGENERATION PHENOMENA

The importance of regeneration phenomena for the K^0 mesons was first pointed out by Pais and Piccioni¹⁶⁾. The effect in question results from the fact that the scattering and interactions of the K^0 and $\bar{K}^0$ are due to strong interactions which conserve strangeness. However, the interactions of K^0 and $\bar{K}^0$ are different so that, when a particle in a state of mixed strangeness is scattered, there is a change in the degree of mixing. For example, a beam of K_2^0 mesons is in the mixed state ϕ_2 . If it is scattered from a target, the $|K^0\rangle$ and $|\bar{K}^0\rangle$ components in ϕ_2 are affected differently so that the resultant state is a different mixture of $|K^0\rangle$ and $|\bar{K}^0\rangle$ which can be written as a linear combination of ϕ_1 and ϕ_2 . Hence the K_1^0 meson is regenerated from the original pure beam of K_2^0 mesons.

The regeneration process is to be analyzed in terms of the scattering amplitude $\mathcal{T}(\underline{k}', \underline{p}'; \underline{k}, \underline{p})$ of the target nuclei for K mesons of momentum $\underline{k}$ being scattered to states of momentum $\underline{k}'$ by a target nucleus of initial momentum $\underline{p}$ and final momentum $\underline{p}' = \underline{p} + \underline{k} - \underline{k}'$. $\mathcal{T}(\underline{k}', \underline{p}'; \underline{k}, \underline{p})$ is a 2×2 matrix with elements labelled by the K^0 and $\bar{K}^0$ states. Because of strangeness conservation, only the diagonal elements of $\mathcal{T}(\underline{k}', \underline{p}'; \underline{k}, \underline{p})$ are different from zero in this representation. The initial and final states of the target nucleus will be taken to be wave packets $\Psi_i(\underline{p})$ and $\Psi_f(\underline{p}')$. Then only the 2×2 matrix

$$\mathcal{T}(\underline{k}', \underline{k}) = \int d^3 p \int d^3 p' \delta(\underline{p}' + \underline{k} - \underline{p} - \underline{k}) \Psi_f^*(\underline{p}') \mathcal{T}(\underline{p}', \underline{k}'; \underline{p}, \underline{k}) \Psi_i(\underline{p}) \quad (79)$$

enters into the discussion. Again, the off diagonal elements of $\mathcal{T}(\underline{k}', \underline{k})$ vanish and the diagonal elements are denoted by

$$\langle K^0 | T(\underline{k}', \underline{k}) | K^0 \rangle = T(\underline{k}', \underline{k})$$

and

(80)

$$\langle \bar{K}^0 | T(\underline{k}', \underline{k}) | \bar{K}^0 \rangle = T'(\underline{k}', \underline{k}).$$

Consider a K^0 beam produced at the space-time point $x = 0$, scattered by a target nucleus at the point x' (in the sense that the wave packets $\psi_i(p)$ and $\psi_f(p')$ describe packets entering and leaving the point x' , respectively) and being detected at point x . The S matrix for this process may be found by the same methods used to obtain Eq. (39). Note that the propagator enters this problem twice, once between production and scattering and once between scattering and detection. The result is

$$\begin{aligned} S(x, x') = & \sum_{\alpha} \int d^3 k \phi_{\alpha} \times \phi_{\alpha}^{\dagger} \psi_d^*(\underline{k}', (z_{\alpha} + \underline{k}^2)^{\frac{1}{2}}) \psi_p(\underline{k}, (z_{\alpha} + \underline{k}^2)^{\frac{1}{2}}) \\ & \times e^{i \underline{k} \cdot \underline{x}} \exp[-i(z_{\alpha} + \underline{k}^2)^{\frac{1}{2}} t] \\ & + \sum_{\alpha, \beta} \int d^3 k' \int d^3 k \phi_{\beta} \times \phi_{\beta}^{\dagger} T(\underline{k}', \underline{k}) \phi_{\alpha} \times \phi_{\alpha}^{\dagger} \psi_d^*(\underline{k}', (z_{\beta} + \underline{k}'^2)^{\frac{1}{2}}) \\ & \times \psi_p(\underline{k}, (z_{\alpha} + \underline{k}^2)^{\frac{1}{2}}) |z_{\alpha} - z_{\beta}|^{-1} \exp[i \underline{k}' |z_{\alpha} - z_{\beta}|] \\ & \times \exp[-i(z_{\beta} + \underline{k}'^2)^{\frac{1}{2}} (t - t')] e^{i \underline{k}' \cdot \underline{x}'} \exp[-i(z_{\alpha} + \underline{k}^2)^{\frac{1}{2}} t'], \end{aligned} \quad (81)$$

where $\psi_p(\underline{k}, k_0)$ and $\psi_d(\underline{k}', k'_0)$ are the overlap integrals of the wave packets forming the production and detection vertices respectively. The first term describes the unscattered wave and the second term describes the scattered wave in the asymptotic region.

Again, as in section 4, narrow packets centred around momenta $\underline{\underline{K}}$ and $\underline{\underline{K}}'$ may be assumed and the result takes the form

$$S(x',x) = A_p(\underline{\underline{x}} - \underline{\underline{x}}' - \underline{\underline{v}}(t-t')) \left\{ A_d^*(\underline{\underline{x}} - \underline{\underline{x}}' - \underline{\underline{v}}(t-t')) e^{i\underline{\underline{K}} \cdot \underline{\underline{x}}} e^{-iK_0 t} \right. \\ \times \sum_{\alpha} \phi_{\alpha} \times \phi_{\alpha}^{\dagger} \exp(-i\omega_{\alpha} t) \\ \left. + A_d^*(\underline{\underline{x}} - \underline{\underline{x}}' - \underline{\underline{v}}(t-t')) \sum_{\alpha, \beta} \bar{\Phi}_{\beta\alpha}(x',x) \right\}, \quad (82)$$

where $A_p(\underline{\underline{x}})$ and $A_d(\underline{\underline{x}})$ are determined from ψ_p and ψ_d respectively by equations of the form of Eq. (41). The prime on $\underline{\underline{x}}'$ (and ω_{β}' , below) signifies that $\underline{\underline{K}}$ has been replaced by $\underline{\underline{K}}'$ in $\underline{\underline{v}}$ (and ω_{β}). The functions

$$\bar{\Phi}_{\beta\alpha}(x',x) = e^{i\underline{\underline{K}}' \cdot \underline{\underline{x}}'} \exp[-i(K_0 + \omega_{\alpha})t] |\underline{\underline{x}} - \underline{\underline{x}}'|^{-1} \exp(i\underline{\underline{K}}' \cdot (\underline{\underline{x}} - \underline{\underline{x}}')) \\ \times \exp[-i(K_0' + \omega_{\beta}')(t-t')] \phi_{\beta} \times \phi_{\alpha}^{\dagger} \langle \phi_{\beta}^{\dagger} \tau(\underline{\underline{K}}', \underline{\underline{K}}) \phi_{\alpha} \rangle \quad (83)$$

describes the scattered wave of type ϕ_{β} which is produced by an incident wave ϕ_{α} .

It is important to recognize that the momenta $\underline{\underline{K}}$ and $\underline{\underline{K}}'$ appearing in $\bar{\Phi}_{\beta\alpha}$ are independent of the indices α and β , that is, independent of whether a K_1 or K_2 state is being considered. These momenta are determined by kinematics from the momenta of the particles entering or leaving the production, scattering or detection vertices. The K meson appears only in virtual states, described here in terms of the propagator. The appearance of distinct complex masses for the K_1 and K_2 in the final expression for the amplitude is a mathematical result following from the existence of the two

distinct poles in the propagator when it is expressed as a function of the energy variable k_0 . Although these poles lie in the unphysical part of the k_0 plane, the integral over physical values of k_0 is very well approximated by taking the Cauchy integral around the pole. Thus the effective energies of the K_1 and K_2 mesons are different (and complex) but the momenta are unaffected by the choice of mass¹⁷⁾.

The regenerative aspects of the scattering are governed by the matrix element of $\mathcal{T}(\underline{K}', \underline{K})$ appearing in Eq. (83). This matrix element will be denoted by $\mathcal{T}_{\beta\alpha}(\underline{K}', \underline{K})$:

$$\mathcal{T}_{\beta\alpha}(\underline{K}', \underline{K}) = \langle \phi_\beta^\dagger \mathcal{T}(\underline{K}', \underline{K}) \phi_\alpha \rangle. \quad (84)$$

From Eq. (80) it follows that

$$\mathcal{T}_{\beta\alpha}(\underline{K}', \underline{K}) = \phi_{\beta K}^\dagger \mathcal{T}(\underline{K}', \underline{K}) \phi_{\alpha K} + \phi_{\beta \bar{K}}^\dagger \mathcal{T}'(\underline{K}', \underline{K}) \phi_{\alpha \bar{K}}. \quad (85)$$

Then Eq. (29) yields the results

$$\begin{aligned} \mathcal{T}_{11}(\underline{K}', \underline{K}) &= (1 + \Delta^2)^{-1} (\Delta^2 \mathcal{T} + \mathcal{T}'), \\ \mathcal{T}_{22}(\underline{K}', \underline{K}) &= (1 + \Delta^2)^{-1} (\mathcal{T} + \Delta^2 \mathcal{T}'), \\ \mathcal{T}_{12}(\underline{K}', \underline{K}) &= (1 + \Delta^2)^{-1} (\mathcal{T} - \mathcal{T}'), \\ \mathcal{T}_{21}(\underline{K}', \underline{K}) &= (1 + \Delta^2)^{-1} (\mathcal{T} - \mathcal{T}') \Delta^2. \end{aligned} \quad (86)$$

One of the interesting aspects of Eq. (36) is that it does not contain the parameter r , hence it does not depend on the question of CP invariance¹⁸⁾. However, the quantity s is directly related to the question of CPT invariance, hence a measurement of the scattering amplitude could, in principle, be used as a test of the CPT theorem.

If the weak interactions are assumed to satisfy the CPT theorem, then, by Eq. (31), $s = 1$ and

$$\begin{aligned} T_{11} = T_{22} &= \frac{1}{2}(\tau + \tau'), \\ T_{12} = T_{21} &= \frac{1}{2}(\tau - \tau'), \end{aligned} \quad (87)$$

which are the usual expressions for the scattering amplitudes.

A particularly interesting regeneration phenomenon is the one suggested by M.L. Good³⁾ and used by R.H. Good et al.⁷⁾ as a method for measuring the mass difference Δ . This depends on the constructive interference between the K_1 waves regenerated by the scattering of a plane wave of K_2 mesons by the nuclei in a thick plate. The interference occurs only in the forward direction and produces a narrow peak on top of the diffraction pattern.

An expression for the interference peak may be obtained from Eq. (82). If the point $\underline{x} = 0$ of production of the K meson is at a large distance from the point $\underline{x}'$ of scattering, then the $\alpha = 1$ terms will be damped out in Eq. (82) for any reasonable choice of the wave packet $A_p(\underline{x})$, since then the elapsed time t' will be large compared to the K_1 lifetime. In this way, the scattering of a pure K_2 wave is assured.

The condition on the packet $A_p(\underline{x})$ is not very restrictive, it is required to be very small compared to the spacing between the point of production and the point of scattering. To assure coherence of the scattered wave, the packet is taken to be very large compared to the experimental error involved in determining the position of a scattering or detection event. Then, for the following considerations, A_p may be assumed to be a constant. On the other hand, in order that the detection phenomena be coherent, the packet $A_d(\underline{x})$ must be highly localized. This guarantees that waves scattered from different parts of the plate arrive (and are detected) at the same place and at the same time. Thus

$$A_d(\underline{x}) \approx 0, \underline{x} \neq 0. \quad (88)$$

It follows that contributions to the scattered wave in Eq. (82) or Eq. (83) arise only from points $\underline{x}'$ in the plate for which

$$\underline{x} - \underline{x}' = \underline{v}'(t - t'), \quad (89)$$

that is, the distance travelled is related to the time elapsed by the group velocity, an obvious result.

To obtain the relative intensity of the interference, it is now necessary to sum $\Phi_{\beta 2}$, given by Eq. (83), over all nuclei in the target plate, which is now assumed to be an infinite slab of thickness L normal to the direction $(\underline{k})$ of the incident K_2 beam. If the density N of nuclei is introduced, the sum becomes an integral over the points $\underline{x}'$. Eq. (89) may be used to express t' as a function of $\underline{x}'$. Furthermore, according to Eq. (89), $\underline{v}'$ and $\underline{k}' = \underline{v}'K'_0$ are functions of $\underline{x}'$ in the sense that they have the direction of $\underline{x}' - \underline{x}$. With this understanding, the relative regenerated amplitude at the

point x takes the form ¹⁹⁾

$$R_\beta(x) = 2\pi N \phi_\beta e^{i\vec{k} \cdot \vec{x}} \exp[-i(k_0 + \omega_2)t] \int_{y_1}^{y_2} dy \int_0^\infty \rho d\rho T_{\beta 2}(\vec{k}', \vec{k}) \\ \times R^{-1} e^{i\vec{k}' \cdot \vec{R}} e^{-i\vec{k} \cdot \vec{y}} \exp[-i(k'_0 - k_0 + \omega'_\beta - \omega_2)R/v], \quad (90)$$

where the variables

$$\vec{R} = \vec{x} - \vec{x}', \quad (91a)$$

$$y = \vec{k} \cdot \vec{R} / k, \quad (91b)$$

$$\rho = (R^2 - y^2)^{\frac{1}{2}} \quad (91c)$$

have been introduced and y_1 and y_2 are defined by

$$y_i = \vec{k} \cdot \vec{R}_i / k, \quad (91d)$$

where $\vec{R}_1$ and $\vec{R}_2$ are points on the front and back of the plate, respectively, so that

$$y_1 - y_2 = L. \quad (92)$$

Equation (91c) makes it possible to substitute R as an integration variable in place of ρ . Then an integration by parts leads to an expansion of the integral in powers of $(LK')^{-1}$, and only the first term in this expansion need be retained. In this approximation

$$R_\beta(x) = 2\pi N \phi_\beta e^{i\vec{K}_\beta \cdot \vec{x}} \exp[-i(\kappa_0 + \omega_2)t] \frac{v^2 \mathcal{T}_{\beta 2}(\vec{K}_\beta, \vec{K}_1)}{(vK + \omega_2 - \omega_\beta)} \\ \times \frac{\exp[i(\omega_2 - \omega_\beta)y/v]}{(\omega_2 - \omega_\beta)} \{ \exp[i(\omega_2 - \omega_\beta)L/v] - 1 \}, \quad (93)$$

where now only the forward scattering $\vec{K}' = \vec{K}$ appears because the entire contribution arises from $\rho = 0$ in this approximation. This is, of course, a generalization of the usual relationship between forward scattering amplitude and index of refraction. The absorption of the K mesons in the plate may easily be included in Eq. (93) by introducing a factor $e^{-\frac{1}{2}aL}$, where a is the absorption coefficient, which is nearly the same for K_1 and K_2 mesons ^{3), 7)}.

Although $R_1(x)$ gives the amplitude for regenerated K_1 mesons at x , this is not the quantity of interest since it is the 2π decay mode that is detected as an indicator of K_1 mesons. The total relative amplitude for 2π decay, including the effect of the unscattered wave, may be obtained from Eq. (63), Eq. (68), Eq. (82) and Eq. (93) in the form

$$A_{2\pi}(x) = e^{-\frac{1}{2}aL} e^{-i\omega_2 t} D^{(1)}(2\pi) \left\{ \xi + \frac{2\pi N v^2 \mathcal{T}_{12}(\vec{K}_1, \vec{K}_2)}{(vK + \omega_2 - \omega_1)(\omega_2 - \omega_1)} \exp[i(\omega_2 - \omega_1)y/v] [\exp[i(\omega_2 - \omega_1)L/v] - 1] \right\}, \quad (94)$$

where a term proportional to $\xi \mathcal{T}_{22}$ has been dropped since it corresponds to forward scattering of the K_2 and is therefore already included in the absorption of the K_2 beam by the introduction of the factor $e^{-\frac{1}{2}aL}$ as indicated above.

If the weak interactions are CPT and CP invariant, $\xi = 0$, $\mathcal{T}_{12}$ is given by Eq. (87), and the amplitude Eq. (94) becomes identical with that obtained in Ref. ⁷⁾. However, a failure of either invariance condition leads to $\xi \neq 0$ so that the intensity will include a term proportional to ξ . This point has been recognized as a possible test of CP invariance by Whately ²⁰⁾.

The relative intensity, as a function of the time $t_1 = y_1/v$, required for the K meson to travel from the front side of the plate to the point x outside the plate, is given to first order in ξ , by

$$I_{2\pi}(t_1) = \frac{4\pi^2 N^2 e^{-\alpha L}}{m^2} \frac{|\mathcal{T}_{12} \Lambda|^2}{\Delta^2 + \frac{1}{4}(\Gamma_1 - \Gamma_2)^2} \exp(-m\Gamma_1 t_1 / K_0)$$

$$- \frac{4\pi N e^{-\alpha L}}{m} \operatorname{Re} \left\{ \frac{\xi^* \mathcal{T}_{12} \Lambda \exp(-im\Delta t_1 / K_0)}{\Delta - \frac{1}{2}i(\Gamma_1 - \Gamma_2)} \right\} \exp\left[-\frac{1}{2}m(\Gamma_1 + \Gamma_2)t_1 / K_0\right], \quad (95)$$

where

$$\Lambda = \exp(-imL\Delta/vK_0) \exp\left[\frac{1}{2}mL(\Gamma_2 - \Gamma_1)/vK_0\right] - 1. \quad (96)$$

Use has been made of Eq. (51) to express ω_α in terms of its real and imaginary parts, and of Eq. (46). The excellent approximation of neglecting $|\omega_1 - \omega_2|$ compared to vK has also been used. $\mathcal{T}_{12}$ in this equation stands for the forward scattering amplitude.

The first term in Eq. (95) is just the same as that given by M.L. Good ³⁾, while the second term gives the above-mentioned interference effect which would arise if the weak interactions violated CPT or CP invariance. As noted in section 5, it is reasonably well established that $|\xi| < 3 \times 10^{-3}$. This is to be compared to $\pi N m^{-1} |\mathcal{T}_{12} \Lambda| \left| \Delta^2 + \frac{1}{4}\Gamma_1^2 \right|^{-\frac{1}{2}}$ which is of the same order of magnitude. Hence the interference term would be expected to be detectable if either CPT or CP invariance is violated.

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FOOTNOTES AND REFERENCES

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- 17) In this connection, R.H. Good et al., Ref. ⁷⁾ incorrectly assumed that, as a consequence of the conservation laws, the energies of the K_1 and K_2 are, in effect, equal after scattering but that the momenta are different because of the difference in mass. Actually conservation of momentum and energy is taken into account in obtaining Eq. (81) before the integration over k_0 is carried out. The extension of the k_0 integration into the complex plane then requires a slight analytic extension of the overlap integrals of the wave packets which originally involve the physical momenta and energies.

In spite of the error, the final result of Ref. ⁷⁾ is correct because only the phase $(\underline{K}_{\alpha} \cdot \underline{x} - K_{0\alpha} t)$ is relevant to their calculation and the relationship between $\underline{x}$ and t , $\underline{x} = \underline{v}t$ is the same for the K_1 and K_2 mesons. From the relationship $K_{0\alpha}^2 = z_{\alpha}^2 + \underline{K}_{\alpha}^2$, it follows that the small (complex) mass difference $z_1 - z_2$ enters the phase in the same way whether it is assumed that $K_1 = K_2$ or $K_{10} = K_{20}$. This remark is due to M.L. Good (private communication).

- 18) M.L. Good, Ref. ³⁾, has noted the fact that the scattering amplitudes are independent of the question of CP invariance in the case that the CPT theorem is valid.
- 19) The factor ϕ_2 from Eq. (83) is dropped here since it is a common factor expressing the fact that the beam incident on the target contains only K_2 mesons.
- 20) M. Whately, private communication.