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SOME RIGOROUS INEQUALITIES SATISFIED BY DOUBLE-SPECTRAL FUNCTIONS

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A B S T R A C T

Combining the elastic unitarity condition in the elastic strips of the $\pi\pi$ scattering amplitude and the positiveness of the partial wave projections of the absorptive parts in the crossed channels we derive some rigorous inequalities on the spectral functions. Some applications are considered.

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I. INTRODUCTION

It is well known ¹⁾ that the elastic unitarity condition in a given channel provides explicit expressions of the double-spectral functions, as integral transforms of the absorptive parts in crossed channels. In this paper, it is shown that these explicit expressions, and the unitarity condition in crossed channels, enable us to find large regions where the double-spectral functions are strictly positive.

In Section II we investigate the case of neutral pseudoscalar particles with equal masses. In Section III we study the more realistic case of isotopic spin 1 pseudoscalar particles. It is found that the statement on the signs of the double-spectral functions is not general and holds only for certain values of total isospin. However, it is possible to get inequalities between double-spectral functions corresponding to different total isospins. In Section IV we discuss some applications of the previous results. The first one concerns the singularity of the amplitude in the complex ℓ plane at negative integral values of ℓ . It is shown that the singularity at $\ell = -1$ cannot disappear for certain values of the isospin. The second one concerns the possibility of resonances in the $\pi\pi$ system. It is shown that in the isospin 0 state and for energy below the inelastic threshold resonance in an angular momentum state $\ell \neq 0$ is unlikely.

II. NEUTRAL PSEUDOSCALAR PARTICLES (EQUAL MASSES μ)

The elastic unitarity condition in t channel can be written as follows

$$\rho_u(s, t) = \sqrt{\frac{t-4\mu^2}{t}} \frac{2}{\pi} \iint \frac{dz_1 dz_2}{\sqrt{z^2 + z_1^2 + z_2^2 - 2zz_1z_2 - 1}} A_s(s_1, t) A_s^*(s_2, t) \quad (1)$$

where

$$s_i = -\frac{t-4\mu^2}{2} (1-z_i) \quad (i=1, 2)$$

$$s = -\frac{t-4\mu^2}{2} (1-z) \quad (2)$$

$A_s(s, t)$ is the absorptive part in the s channel and $\rho_u(s, t)$ is one of the three double-spectral functions of the Mandelstam representation of the amplitude. The r.h.s. of Eq. (1) contains a single term because of the symmetry of the problem.

The domain of integration in the $z_1 z_2$ plane is contained in the square defined by

$$z_0 \leq z_i \leq z z_0 - \sqrt{(z^2-1)(z_0^2-1)} \quad (i=1, 2) \quad (3)$$

where

$$z_0 = \frac{t+4\mu^2}{t-4\mu^2} \quad (4)$$

The limits (3) can be written in a simpler way by using the new variables θ

$$\begin{aligned} z &= \operatorname{ch} \theta \\ z_i &= \operatorname{ch} \theta_i \quad (i = 0, 1, 2) \end{aligned} \quad (5)$$

Inequalities (3) then become

$$\theta_0 \leq \theta_i \leq \theta - \theta_0 \quad (i = 1, 2) \quad (3')$$

These inequalities can be fulfilled only if $\theta \geq 2\theta_0$. It is well known that this gives the boundary curve of the spectral region

$$s \geq \frac{16 \mu^2 t}{t - 4 \mu^2} \quad (6)$$

Now we have some information about the absorptive part $A_s(s_i, t)$. The unitarity condition in the s channel tells us that, outside the spectral region and for $t > 0$, $A_s(s_i, t)$ is real and positive. This can be shown as follows: let us expand $A_s(s_i, t)$ in partial waves in the s channel

$$A_s(s_i, t) = \sum_{\ell} (\ell + 1) A_{\ell}(s_i) P_{\ell} \left(1 + \frac{2t}{s_i - 4\mu^2} \right) \quad (7)$$

For $t > 4\mu^2$ and $s_i > 4\mu^2$, which is the region of interest, the argument of the Legendre polynomial is greater than 1, thus

$$P_{\ell} \left(1 + \frac{2t}{s_i - 4\mu^2} \right) \geq 1$$

4. .

Moreover, according to the optical theorem, $A_e(s_i)$ is positive. It follows at once that $A_s(s_i, t)$ is also positive. However, the expansion (8) converges only outside the spectral region, that is to say for

$$\theta_i \leq 2\theta_0 \quad (8)$$

Let us return to the limits $(\theta_i)^{3'}$. We see that when $\theta \leq 3\theta_0$ the inequality (8) is a consequence of $(\theta_i)^{3'}$. This means that in order to know the spectral function $\rho_u(s, t)$ in the region $2\theta_0 \leq \theta \leq 3\theta_0$ we need only know the absorptive part $A_s(s_i, t)$ in the region $\theta_0 \leq \theta_i \leq 2\theta_0$, where it is positive. From this and the fact that the kernel in (1) is positive it follows that $\rho_u(s, t)$ is positive when

$$\theta \leq 3\theta_0 \quad (9)$$

This result is, of course, valid only in the elastic region of the t channel

$$4\mu^2 \leq t \leq 16\mu^2 \quad (10)$$

Using (2), (4) and (5), condition (9) becomes ^{*)}

$$s \leq 4\mu^2 \left(\frac{3t + 4\mu^2}{t - 4\mu^2} \right)^2 \quad (9')$$

^{*)} The curve $s = 4\mu^2 \left(\frac{3t + 4\mu^2}{t - 4\mu^2} \right)^2$ is the Landau curve of the following graph

Figure 1 shows the region defined by (6), (9') and (10). The function $\rho_u(s,t)$, being symmetrical in its arguments, is also positive for

$$4\mu^2 \leq s \leq 16\mu^2$$

$$\frac{16\mu^2 s}{s-4\mu^2} \leq t \leq 4\mu^2 \left(\frac{3s+4\mu^2}{s-4\mu^2} \right)^2 \quad (11)$$

III. INTRODUCTION OF ISOSPIN 1

Let us define at first our notations. Labelling the pion isospin states by the conventional isotopic vector index that takes on the values 1, 2, 3 the elastic amplitude reads as follows

$$\langle \gamma \delta | T | \alpha \beta \rangle = A^1(s, t, u) \delta_{\alpha\beta} \delta_{\gamma\delta} + A^2(s, t, u) \delta_{\alpha\gamma} \delta_{\beta\delta} + A^3(s, t, u) \delta_{\alpha\delta} \delta_{\beta\gamma}$$

Taking into account crossing symmetries, we can write the Mandelstam representations of A^1

$$\begin{aligned} A^1(s, t, u) = & \frac{1}{\pi^2} \iint \frac{\sigma(t', u') dt' du'}{(t'-t)(u'-u)} + \frac{1}{\pi^2} \iint \frac{\rho(u', s') du' ds'}{(u'-u)(s'-s)} + \\ & + \frac{1}{\pi^2} \iint \frac{\rho(t', s') dt' ds'}{(t'-t)(s'-s)} \end{aligned} \quad (12)$$

and $A^2(s, t, u) = A^1(t, u, s)$

$$A^3(s, t, u) = A^1(u, s, t)$$

For simplicity we have written neither terms due to single spectral functions nor subtraction terms. This defines two double-spectral functions $\sigma(x, y)$ and $\rho(x, y)$, the first one being symmetrical: $\sigma(x, y) = \sigma(y, x)$.

In the following it will be convenient to write alternatively

$$\begin{aligned} A^J(s, t, u) = & \frac{1}{\pi^2} \iint \frac{\rho_s^J(t', u') dt' du'}{(t'-t)(u'-u)} + \frac{1}{\pi^2} \iint \frac{\rho_t^J(u', s') du' ds'}{(u'-u)(s'-s)} + \\ & + \frac{1}{\pi^2} \iint \frac{\rho_u^J(s', t') ds' dt'}{(s'-s)(t'-t)} \quad (J=1, 2, 3) \end{aligned} \quad (13)$$

We define also the amplitude $M^{sI}(s,t,u)$ (respectively M^{tI}, M^{uI}) of total isospin I in the s (respectively t, u) channel. These amplitudes are connected to the A^J by well-known matrices

$$\begin{aligned} M^{sI} &= \sum_J \alpha_J^{sI} A^J \\ M^{tI} &= \sum_J \alpha_J^{tI} A^J \\ M^{uI} &= \sum_J \alpha_J^{uI} A^J \end{aligned} \quad (14)$$

where

$$\alpha^s = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \quad \alpha^t = \begin{pmatrix} 1 & 3 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \alpha^u = \begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad (15)$$

We shall write the Mandelstam representation of M^{sI} as follows

$$\begin{aligned} M^{sI}(s,t,u) &= \frac{1}{\pi^2} \iint \frac{\sigma_s^{sI}(t',u') dt' du'}{(t'-t)(u'-u)} + \frac{1}{\pi^2} \iint \frac{\sigma_t^{sI}(u',s') du' ds'}{(u'-u)(s'-s)} + \\ &+ \frac{1}{\pi^2} \iint \frac{\sigma_u^{sI}(s',t') ds' dt'}{(s'-s)(t'-t)} \end{aligned} \quad (16)$$

and similar expressions for M^{tI} and M^{uI} . Obviously, the double spectral functions σ^I are connected to the ρ^J by the α matrices

$$\sigma_{s,t,u}^{sI} = \sum_J \alpha_J^{sI} \rho_{s,t,u}^J \quad (17)$$

We need the absorptive parts of these different amplitudes in the s , t and u channels. We denote them by a subscript. For example, A_s^J and M_s^{tI} are the absorptive parts in the s channel of A^J and M^{tI} .

Now, we are able to write down the unitarity condition. In t channel ($4\mu^2 \leq t \leq 16\mu^2$) and for isospin I it reads as follows

$$\sigma_u^{tI}(s, t) = \sqrt{\frac{t-4\mu^2}{t}} \frac{1}{\pi} \iint \frac{dz_1 dz_2}{\sqrt{z_1^2 + z_2^2 + z_1^2 z_2^2 - 2z_1 z_2 - 1}} \left\{ M_s^{tI}(s_1, t) M_u^{tI*}(u_2, t) + M_u^{tI}(u_1, t) M_s^{tI*}(s_2, t) \right\} \quad (18)$$

where

$$\begin{aligned} s_i &= u_i = -\frac{t-4\mu^2}{2} (1-z_i) \quad (i=1, 2) \\ s &= -\frac{t-4\mu^2}{2} (1-z) \end{aligned} \quad (19)$$

and the domain of integration is the same as in (1). Using the symmetry properties of the amplitudes, one easily sees that the two terms between brackets in (18) give the same contribution. Moreover, one has

$$M_u^{tI}(u_2, t) = (-)^I M_s^{tI}(s_2, t) \quad (I=0, 1, 2)$$

Finally (18) becomes

$$\sigma_u^{tI}(s, t) = (-)^I \sqrt{\frac{t-4\mu^2}{t}} \frac{2}{\pi} \iint \frac{dz_1 dz_2}{\sqrt{z_1^2 + z_2^2 + z_1^2 z_2^2 - 2z_1 z_2 - 1}} M_s^{tI}(s_1, t) M_s^{tI*}(s_2, t) \quad (20)$$

which is quite similar to Eq. (1). In the following we shall write Eq. (20) in a more concise way

$$\sigma_u^{tI}(s,t) = (-)^I K_t \left\{ M_s^{tI} M_s^{tI*} \right\} \quad (20')$$

K_t is the unitarity operator in t channel.

At this point, it is not possible to follow the same reasoning as in the previous Section because $M_s^{tI}(s_i, t)$ has no definite sign in the region defined by (8). However, the absorptive part $M_s^{sI}(s_i, t)$ corresponding to isospin I in the s (and not t) channel has the property of being positive for $\theta_i \leq 2\theta_0$. This can be seen by expanding M_s^{sI} in partial waves in s channel, as in Eq. (7). Thus, we must express M_s^{tI} in terms of M_s^{sI} and this can be done by using the α matrices

$$M_s^{tI} = \sum_{J I'} \alpha_{J I'}^{t I} (\alpha^s)^{-1 J}_{I'} M_s^{s I'} \quad (21)$$

Hence

$$\sigma_u^{tI}(s,t) = \sum_{I' I''} \sum_{J I''}^I K_t \left\{ M_s^{s I'} M_s^{s I''*} \right\} \quad (22)$$

where

$$\sum_{I' I''}^I = (-)^I \sum_{J' J''} \alpha_{J' I'}^{t I} \alpha_{J'' I''}^{t I} (\alpha^s)^{-1 J'}_{I'} (\alpha^s)^{-1 J''}_{I''}$$

The calculation of the matrices Σ^I yields

$$\Sigma^0 = \begin{pmatrix} \frac{1}{9} & \frac{1}{3} & \frac{5}{9} \\ \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{5}{9} & \frac{5}{3} & \frac{25}{9} \end{pmatrix} \quad \Sigma^1 = \begin{pmatrix} -\frac{1}{9} & -\frac{1}{6} & \frac{5}{18} \\ -\frac{1}{6} & -\frac{1}{4} & \frac{5}{12} \\ \frac{5}{18} & \frac{5}{12} & -\frac{25}{36} \end{pmatrix} \quad \Sigma^2 = \begin{pmatrix} \frac{1}{9} & -\frac{1}{6} & \frac{1}{18} \\ -\frac{1}{6} & \frac{1}{4} & -\frac{1}{12} \\ \frac{1}{18} & -\frac{1}{12} & \frac{1}{36} \end{pmatrix} \quad (23)$$

Now, we can apply the reasoning of the previous Section to Eq. (22) because M_s^{sI} is positive and the operator K_t is defined by a positive kernel. But only the first of the three matrices Σ^I has all its elements of the same sign. Consequently, among the three double-spectral functions σ_u^{to} , σ_u^{t1} and σ_u^{t2} only the first one is positive in the region defined by (6), (9') and (10)*). However, one easily checks that the linear combinations $(\sigma_u^{to} + \sigma_u^{t1})$, $(\sigma_u^{to} - 2\sigma_u^{t1})$, $(\sigma_u^{to} + 2\sigma_u^{t2})$ and $(\sigma_u^{to} - \sigma_u^{t2})$ are also positive. Finally, one has

$$\begin{aligned} \sigma_u^{to} &> 0 \\ -\sigma_u^{to} &< \sigma_u^{t1} < \frac{1}{2} \sigma_u^{to} \\ -\frac{1}{2} \sigma_u^{to} &< \sigma_u^{t2} < \sigma_u^{to} \end{aligned} \quad (24)$$

*) The necessary and sufficient condition on $\Sigma_{I'I''}^I$ for the quantity $\sum_{I'I''} \Sigma_{I'I''}^I M_s^{sI'} M_s^{sI''}$ to be non-negative is $\Sigma_{I'I''}^I \geq 0$ for every I' and I'' . One could think that, taking into account the integration over z_1 and z_2 , σ_u^{tI} could be non-negative with less stringent conditions on Σ^I . However, it is possible to show that, if one matrix element $\Sigma_{I'I''}^I$ is negative, one can find particular absorptive parts M_s^{sI} , such that σ_u^{tI} is negative.

Now let us investigate the unitarity condition in terms of the double-spectral functions $\sigma(s,t)$ and $\rho(s,t)$ defined in (12). From (17) and (22) one gets

$$\rho_u^J(s,t) = \sum_{I'I''} \theta_{I'I''}^J K_t \left\{ M_s^{sI'} M_s^{sI''*} \right\} \quad (25)$$

where

$$\theta_{I'I''}^J = \sum_I (\alpha^t)^{-1} \sum_I^J \sum_{I'I''}^I$$

In terms of $\sigma(s,t)$ and $\rho(s,t)$ Eq. (25) can be written as

$$\begin{aligned} \rho(t,s) &= \sum_{I'I''} \theta_{I'I''}^1 K_t \left\{ M_s^{sI'} M_s^{sI''*} \right\} \\ \rho(s,t) &= \sum_{I'I''} \theta_{I'I''}^2 K_t \left\{ M_s^{sI'} M_s^{sI''*} \right\} \\ \sigma(s,t) &= \sum_{I'I''} \theta_{I'I''}^3 K_t \left\{ M_s^{sI'} M_s^{sI''*} \right\} \end{aligned} \quad (25')$$

The calculation of the matrices θ^J yields

$$\theta^1 = \begin{pmatrix} \frac{1}{9} & 0 & -\frac{1}{9} \\ 0 & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{9} & -\frac{1}{4} & \frac{13}{36} \end{pmatrix} \quad \theta^2 = \begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{4} & \frac{7}{12} \\ \frac{1}{6} & \frac{7}{12} & \frac{11}{12} \end{pmatrix} \quad \theta^3 = \begin{pmatrix} 0 & -\frac{1}{6} & \frac{1}{6} \\ -\frac{1}{6} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \end{pmatrix} \quad (26)$$

This gives, as previously, the following inequalities valid in the region defined by (6), (9') and (10)

$$\begin{aligned}\rho(s, t) &> 0 \\ \rho(t, s) &> -\frac{3}{7} \rho(s, t) \\ -\rho(s, t) &< \sigma(s, t) < \rho(s, t)\end{aligned}\tag{27}$$

Finally, we are interested in the implications of the unitarity condition upon the functions σ_u^{sI} and σ_u^{uI} . From (17) and (25) one gets

$$\begin{aligned}\sigma_u^{sI}(s, t) &= \sum_{I' I''} \xi_{I' I''}^I K_t \left\{ M_s^{sI'} M_s^{sI''*} \right\} \\ \sigma_u^{uI}(s, t) &= \sum_{I' I''} \eta_{I' I''}^I K_t \left\{ M_s^{sI'} M_s^{sI''*} \right\}\end{aligned}\tag{28}$$

where

$$\begin{aligned}\xi_{I' I''}^I &= \sum_J \alpha_J^{sI} \theta_{I' I''}^J \\ \eta_{I' I''}^I &= \sum_J \alpha_J^{uI} \theta_{I' I''}^J\end{aligned}$$

The calculation of the matrices ξ^I and η^I yields

$$\begin{aligned} \xi^0 &= \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{5}{3} \end{pmatrix} & \xi^1 &= \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{5}{12} \\ 0 & \frac{5}{12} & \frac{5}{4} \end{pmatrix} & \xi^2 &= \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{3} & \frac{3}{4} & \frac{7}{12} \end{pmatrix} \\ \eta^0 &= \begin{pmatrix} \frac{1}{9} & -\frac{1}{3} & \frac{5}{9} \\ -\frac{1}{3} & \frac{1}{2} & \frac{5}{6} \\ \frac{5}{9} & \frac{5}{6} & \frac{5}{18} \end{pmatrix} & \eta^1 &= \begin{pmatrix} \frac{1}{9} & -\frac{1}{6} & -\frac{5}{18} \\ -\frac{1}{6} & 0 & -\frac{5}{6} \\ -\frac{5}{18} & -\frac{5}{6} & -\frac{5}{9} \end{pmatrix} & \eta^2 &= \begin{pmatrix} \frac{1}{9} & \frac{1}{6} & \frac{1}{18} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{18} & \frac{1}{3} & \frac{23}{18} \end{pmatrix} \end{aligned} \quad (29)$$

From this follow the inequalities, again valid in the region defined by (6), (9') and (10)

$$\sigma_u^{sI}(s, t) > 0 \quad \text{for } I = 0, 1, 2 \quad (30)$$

and *)

$$\begin{aligned} \sigma_u^{u^2}(s, t) &> 0 \\ -2 \sigma_u^{u^2}(s, t) &< \sigma_u^{u^0}(s, t) < 10 \sigma_u^{u^2}(s, t) \\ -5 \sigma_u^{u^2}(s, t) &< \sigma_u^{u^1}(s, t) < \sigma_u^{u^2}(s, t) \end{aligned} \quad (31)$$

*) These inequalities are not the only ones that can be found. For example, one has also

$$\sigma_u^{u^0}(s, t) - \sigma_u^{u^1}(s, t) + \sigma_u^{u^2}(s, t) > 0$$

which is not contained in (31).

Noting the symmetry property of $\sigma_u^{uI}(s,t)$

$$\sigma_u^{uI}(s,t) = (t)^I \sigma_u^{uI}(t,s)$$

one has, in the region defined by (11), the following inequalities

$$\begin{aligned} \sigma_u^{u2}(s,t) &> 0 \\ -2 \sigma_u^{u2}(s,t) &< \sigma_u^{u0}(s,t) < 10 \sigma_u^{u2}(s,t) \\ -\sigma_u^{u2}(s,t) &< \sigma_u^{u1}(s,t) < 5 \sigma_u^{u2}(s,t) \end{aligned} \tag{32}$$

IV. SOME APPLICATIONS OF THE PREVIOUS RESULTS

First of all the inequalities obtained in the preceding Section are rigorous and any solution of the Mandelstam equation for the $\pi\pi$ system should satisfy these inequalities.

Now we want to discuss two particular consequences of the previous results concerning the singularity of the amplitude $\phi_\ell(t)$ at $\ell = -1$ ²⁾ and the possibility of resonances of the $\pi\pi$ system in the elastic region.

There exists an argument by Gribov and Pomeranchuk³⁾ according to which the amplitude $\phi_\ell(t)$ where ℓ is the angular momentum in the t channel exhibits a pathological behaviour at negative integral values of ℓ (if analytic continuation can be performed this far). The argument rests on the fact that, when writing a dispersion relation for $\phi_\ell(t)$ in the t plane, the contribution due to the left-hand cut is a meromorphic function in the whole ℓ plane, with simple poles at negative integral values of ℓ . The residue at the point $\ell = -n-1$ is given by

$$\int_{-z_0}^{z_0} P_n(z) \rho_t(s, u) dz$$

(where the integral is taken over the spectral region), and is zero unless $t < -32\mu^2$ in case of pseudoscalar particles. It is important that, at least for one value of t , the residues should not vanish. The results of Section III enable us to find values of t , for which these residues are indeed different from zero. In fact, if we study, for example, the amplitude $\phi_\ell^{tI}(t)$ corresponding to total isospin I in the t channel the residues of interest are given by

$$\int P_n(z) \sigma_t^{tI}(u, s) dz$$

The double-spectral functions σ_t^{tI} satisfy the same inequalities (31) and (32) as σ_u^{uI} . We see at once that the residue $\int \sigma_t^{t2}(u,s) dz$ at $\ell = -1$, for isospin 2, is strictly positive when the integration is taken over the correct region that is to say (see Fig. 2) for

$$-(32 + \frac{4}{3}) \mu^2 \leq t < -32 \mu^2 \quad (33)$$

If we study the amplitude $\phi_\ell^{sI}(t)$ corresponding to total isospin I in the s channel (the poles of which yield the asymptotic behaviour in the s channel for fixed isospin I), the residues of interest are given by $\int P_n(z) \sigma_t^{sI}(u,s) dz$. Looking at the inequalities (24) and (30), we see that the residue $\int \sigma_t^{so}(u,s) dz$ at $\ell = -1$ for isospin 0 is strictly positive when t satisfies (33). This means that the singularity at $\ell = -1$ is indeed present for ϕ_ℓ^{t2} and ϕ_ℓ^{so} .

As another consequence of the results of the previous Section, let us consider the expression of the absorptive part M_t^{t0} . If there is no subtraction and no simple spectral function, one simply has

$$M_t^{t0} = \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{\sigma_u^{t0}(s',t) ds'}{s' - s} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{\sigma_s^{t0}(t,u') du'}{u' - u}$$

Taking into account the symmetry property

$$\sigma_s^{t0}(t,x) = \sigma_u^{t0}(x,t)$$

one can write

$$M_t^{t_0} = \frac{1}{\pi} \int_{4\mu^2}^{\infty} dx \, \sigma_u^{t_0}(x, t) \left\{ \frac{1}{x-s} + \frac{1}{x-u} \right\} \quad (34)$$

When t is fixed below the inelastic threshold of the t channel we see that a part of the integration in (34) is taken over the region defined by (6), (9') and (10). We can reasonably expect that this part of the integration is the most important one when both s and u are negative (physical region of the t channel). If it is so, looking at the first of the inequalities (24), we see at once that $M_t^{t_0}$ is positive. This means that, if there is a resonance in the $\pi\pi$ system, in the isospin zero state below the first inelastic threshold the resonant wave must be the s wave (if there were a resonance in the d wave, $M_t^{t_0}$ should change sign). This result has a limited validity because

- i) we have disregarded subtraction terms and
- ii) we have neglected the double-spectral function for large s .

R E F E R E N C E S

- 1) S. Mandelstam, Phys. Rev. 112, 1344 (1958).
- 2) We are indebted to Professor Charap for having pointed to us this possible application.
- 3) V.N. Gribov and I.Ya. Pomeranchuk, Proceedings of the 1962 CERN Conference, 522.

FIG. 1

FIG. 2