Acceleration from M theory and Fine-tuning

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Abstract. The compactification of M theory with time dependent hyperbolic internal space gives an effective scalar field with exponential potential which provides a transient acceleration in Einstein frame in four dimensions. Ordinary matter and radiation are present in addition to the scalar field coming from compactification. We find that we have to fine-tune the initial conditions of the scalar field so that our Universe experiences acceleration now. During the evolution of our Universe, the volume of the internal space increases about 12 times. The time variation of the internal space results in a large time variation of the fine structure constant which violates the observational constraint on the variation of the fine structure constant. The large variation of the fine structure constant is a generic feature of transient acceleration models.

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1. Introduction

In recent years, some attempts have been made to derive a late-time accelerating universe from a fundamental theory of particle physics that incorporates gravity, like M theory or superstring theory. This process involves a compactification of eleven or ten-dimensional Einstein gravity to four dimensions. The no-go theorem states that we could not obtain accelerating universes in the Einstein frame in four dimensions by compactification of pure Einstein gravity in higher dimensions when the six or seven-dimensional internal space is a time independent nonsingular compact manifold without boundary [1, 2].

So any attempt to derive an accelerating universe must circumvent the no-go theorem. Townsend and Wohlfarth invoked a time dependent compactification of pure gravity in higher dimensions with hyperbolic internal space to circumvent the no-go theorem [3]. Their exact solution to the vacuum Einstein gravity in higher dimensions exhibits a short period of acceleration or transient acceleration which can’t explain the early inflation but may be used to explain the late time acceleration of our universe. The solution is the zero flux limit of spacelike branes as shown in [4]. If non-zero flux or forms are turned on, then it is shown that a transient acceleration exists for both compact internal hyperbolic and flat spaces [4, 5]. Other accelerating solutions were found by compactifying more complicated time dependent internal spaces [6-15]. When higher order corrections were added to the higher dimensional supergravity theory, inflationary solution was also found [16].

The application of time dependent compactification sheds some lights on the resolution of the late time acceleration within a fundamental theoretical framework. However, there is no stable point in the size of the internal space when the internal space is time dependent. This seems to be the common feature. From our four dimensional point of view, the negatively curved internal space provides a positive exponential potential. The generic feature of the potential is that it is too steep to sustain an eternal acceleration. In a realistic cosmological set up, matter and radiation are present in addition to the scalar field coming from the dimensional reduction, then the evolution of the scalar field will be different. It is worthy of studying whether the scalar field is dominant now and it provides the late time acceleration when matter and radiation are present. Therefore, we will study the model with matter and scalar field which comes from the time dependent compactification of M theory. We also discuss the evolution of the fine structure constant due to the time dependence of the internal space. The paper is organized as follows. In the next section, we will review the basic results of dimensional reduction. In section 3, we will discuss the evolution of the scalar field when matter and radiation are included. In section 4, we conclude the paper with some discussions.
2. Dimensional Reduction

We start with the the bosonic part of the low energy effective action for M theory \((D = 11)\) or superstrings \((D = 10)\) in \(D\) dimensions:

\[
S = \frac{1}{2\kappa_D^2} \int dx^D \sqrt{-G} \, (D)R,
\]

where we have dropped contributions from forms and dilations which are assumed to be stabilized when they exist, \(\kappa_D^2\) is the \(D\) dimensional gravitational constant. When we decompose the \(D\) dimensional space-time into \(4 + d\) dimensional space-time with the metric \(ds^2 = \gamma_{\mu\nu}(x)dx^\mu dx^\nu + \Phi^2(x)\hat{g}_{mn}(y)dy^m dy^n\), here the second half of the lower case Latin letters run from 1 to \(d\) and the Greek letters run from 0 to 3, we get

\[
S = \frac{1}{2\kappa_D^2} \int dx^d \sqrt{g} \int dx^4 \sqrt{-\Phi^4} [R(\gamma) + \Phi^{-2} \hat{R}(\hat{g}) - d(\Phi - d) \Phi^{-2} \gamma^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - 2d \Phi^{-1} \partial_\mu \Phi \partial^\mu \Phi].
\]

After the conformal transformation

\[
\gamma_{\mu\nu} = \Phi^{-d} g_{\mu\nu},
\]

we get the four dimensional low energy effective action in Einstein frame

\[
S = \frac{V_d}{2\kappa_D^2} \int dx^d \sqrt{g} \left[ R(g) + \Phi^{-2-d} \hat{R}(\hat{g}) - \frac{d(d + 2)}{2} \Phi^{-2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \right],
\]

where \(V_d = \int dx^d \sqrt{g}\) is the volume of the \(d\) dimensional internal space with the metric \(\hat{g}_{mn}\) and \(\hat{R}(\hat{g})\) is assumed to be a constant. It is evident that we can identify \(\phi = \sqrt{d(d + 2)/2\kappa_D^2 \ln(\Phi)}\) as the canonical scalar field in 4 dimensions, where \(\kappa^2 = 8\pi G = \kappa_D^2/V_d\) is the four dimensional gravitational constant. Since we are interested in cosmological solutions, we take the metric of our \(D\) dimensional space-time as \(ds^2 = b^{-d}(t)(-dt^2 + a^2(t)\hat{g}_{ij}dx^i dx^j) + b^2(t)\hat{g}_{mn}dy^m dy^n\). The time \(t\) and the scale factor \(a(t)\) are the four dimensional quantities in Einstein frame. With this metric, we have

\[
^{(D)}G_{00} = 3 \frac{\dot{a}^2}{a^2} + \frac{3k}{a^2} - \frac{d(d + 2)}{4} \frac{\dot{b}^2}{b^2} + \frac{1}{2} b^{-2-d} \ddot{R},
\]

\[
^{(D)}G_{ij} = -a^2 \ddot{g}_{ij} \left[ \frac{2}{a^2} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} + \frac{d(d + 2)}{4} \frac{\dot{b}^2}{b^2} + \frac{1}{2} b^{-2-d} \ddot{R} \right],
\]

\[
^{(D)}G_{mn} = \hat{G}_{mn}(\hat{g}) - 3b^{d+2} \ddot{g}_{mn} \left[ \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{1}{6} (d + 2) \frac{\ddot{b}}{b} + \frac{(d + 2)^2 \dot{b}^2}{12 b^2} - \frac{1}{2} (d + 2) \frac{\dot{a} \dot{b}}{a b} \right],
\]

where \(\hat{G}_{mn}(\hat{g}) = \hat{R}_{mn} - \hat{g}_{mn} \hat{R}/2\) and \(\hat{R}_{mn} = -(d - 1) \delta^2 \hat{g}_{mn}\). The vacuum Einstein equations are

\[
3 \frac{\dddot{a}}{a^2} + \frac{3k}{a^2} = \frac{d(d + 2)}{4} \frac{\ddot{b}^2}{b^2} + \frac{1}{2} d(d - 1) \delta^2 b^{-2-d},
\]

\[
\frac{\ddot{b}}{b} - \frac{\dot{b}^2}{b^2} + 3 \frac{\dot{a} \dot{b}}{a b} = (d - 1) \delta^2 b^{-2-d},
\]

\[
\frac{\ddot{a}}{a} = -\frac{d(d + 2)}{6} \frac{\dot{b}^2}{b^2} + \frac{d(d - 1)}{6} \delta^2 b^{-d-2}.
\]
Equations (8-10) are not totally independent. For example, equation (10) can be derived from equations (8) and (9). In the following, we will assume our four dimensional universe is flat so that \( k = 0 \). To solve the above equations, we change the time coordinate \( t \) to \( \eta \) so that \( dt = a^3 d\eta \) and we use \( \dot{} \) to denote the derivative with respect to \( \eta \) in this section. By using \( \eta \), Equations (8-10) become

\[
3\frac{a'^2}{a^2} = \frac{d(d+2)}{4} \frac{b'^2}{b^2} + \frac{1}{2} d(d-1) \delta^2 a^6 b^{-2-d},
\]

\[
\frac{b''}{b} - \frac{b'^2}{b^2} = (d-1) \delta^2 a^6 b^{-2-d},
\]

\[
\frac{a''}{a} - \frac{a'^2}{a^2} = \frac{d(d-1)}{2} \delta^2 a^6 b^{-2-d},
\]

Let \( g = a^2 b^{-d} \), then equations (12) and (13) give \((\ln g)'' = 0\), so \( a^2 b^{-d} = C_3 \exp(C_4 \eta)\). Take \( f = a^6 b^{-d-2} \), equations (12) and (13) give \((\ln f)'' = 2(d-1) \delta^2 f\), so \( a^{-6} b^{d+2} = 4(d-1)^2 \delta^2 C_1^2 \sinh^2[(\eta + C_2)/2C_1]\). Substitute the above solutions to equation (11), we get the relationship \( C_1^2 C_2^2 = d/3(d+2) \). Therefore, after re-scaling of \( \eta \) with the choice \( C_2 = 0 \) and \( C_4 = 2 \), the general solution is

\[
a(\eta) = A \exp \left[ -\frac{d+2}{2(d-1)} \eta \right] \sinh^{-d/2(d+1)} \left[ \frac{3(d+2)}{d} |\eta| \right],
\]

\[
b(\eta) = B \exp \left( -\frac{3}{d-1} \eta \right) \sinh^{-1/(d-1)} \left[ \frac{3(d+2)}{d} |\eta| \right],
\]

where \( B = [d(d-1)^2 \delta^2 C_3^3/3(d+2)]^{-1/2(d-1)} \) and \( A = C_3^{1/2} B^{d/2} \). So \( a(\eta) \to 0 \) and \( b(\eta) \to \infty \) when \( \eta \to -\infty \) if \( d > 1 \). In terms of \( \eta \), the Universe starts from \( -\infty \) and the infinite future time is \( \eta = 0 \). Because

\[
\frac{a'}{a} = -\frac{d+2}{2(d-1)} + \sqrt{\frac{3d(d+2)}{2(d-1)}} \coth \left[ \frac{3(d+2)}{d} |\eta| \right],
\]

so we have \( \dot{a} > 0 \) if \( d > 1 \), i.e., the Universe expands. To get acceleration \( \ddot{a} > 0 \), we need to require

\[
\frac{a'^2}{a^2} < \frac{3(d+2)}{4(d-1)} \sinh^{-2} \left[ \sqrt{\frac{3(d+2)}{d}} |\eta| \right] = m(\eta).
\]

The evolution of \( m(\eta) - \frac{a'^2}{a^2} \) is shown in Figure 1. We see that there exists a period of acceleration. However, the duration of acceleration is not long enough to explain the flatness and horizon problems, but it can be used to explain the late time acceleration. Note that

\[
\frac{b'}{b} = -\frac{3}{d-1} + \sqrt{\frac{3(d+2)}{d(d-1)^2}} \coth \left[ \frac{3(d+2)}{d} |\eta| \right],
\]

it is possible that \( \dot{b} > 0 \) when \( \eta \to 0 \), i.e., at late times or in the far future, the volume of extra dimensions will increase. In the far future, \( \eta \to 0 \), we have

\[
\frac{b'/b}{a'/a} \to \frac{2}{d}.
\]

Therefore, extra dimensions will manifest themselves in the future.
Figure 1. The evolution of acceleration is plotted in the solid line by using the left $y$ axis. The dotted line plots the evolution of the ratio of $\rho_\phi/\rho_m$ by using the right $y$ axis. In this plot, some constants of proportionality are not included.

3. Four Dimensional Cosmology

From four dimensional point of view, equations (8-10) describe the standard cosmology plus a scalar field $\phi = \sqrt{d(d+2)/2\kappa^2\ln(b(t))}$ with the exponential potential $V(\phi) = [d(d-1)\delta^2/2\kappa^2]\exp(-\sqrt{2(d+2)/d}\phi/m_{pl})$ as the matter source. Since we are interested in the late time acceleration instead of the early time inflation, we should also include radiation and dust in the matter contents. With all these matter sources, the standard cosmology is

$$H^2 = \frac{\ddot{a}}{a^2} = \frac{1}{3m_{pl}^2} \left[ \rho_m + \rho_r + \frac{1}{2}\dot{\phi}^2 + V(\phi) \right],$$  

(20)

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0,$$  

(21)

$$\frac{\ddot{a}}{a} = -\frac{1}{3m_{pl}^2} \left[ \frac{1}{2}\rho_m + \rho_r + \dot{\phi}^2 - V(\phi) \right],$$  

(22)

where $\rho_m = \rho_{m0}a^3_0/a^3$ and $\rho_r = \rho_{r0}a^4_0/a^4$, the subscript 0 means the present value of the variable. Without dust and radiation, the Universe with the scalar field experiences a transient acceleration as discussed in the previous section. Here we are concerned with the question if the acceleration persists when dust and radiation are present. For a cosmology with a barotropic fluid which has an equation of state $p_b = w_b\rho_b$ and a scalar field which has the exponential potential $V(\phi) = V_0\exp(-\lambda\phi/m_{pl})$, it was found that [17, 18]:

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The equation of state parameter of the scalar field is the late time attractor solution is the scalar field dominated power-law solution and the equation with $\Omega_{\phi} = 1$ and $w_\phi = -1 + \lambda^2/3$.

For the scalar field coming from compactification, we have $\lambda^2 = 2(d+2)/d < 3$. So the late time attractor solution is the scalar field dominated power-law solution and the equation of state parameter of the scalar field is

$$w_\phi = -1 + \frac{\lambda^2}{3} = -\frac{d-4}{3d}. \quad (23)$$

If we take $d = 7$, we get $w_\phi = -1/7 > -1/3$. If we take $d = 6$, we get $w_\phi = -1/9 > -1/3$. The late time attractor solution gives a decelerating universe. Whatever the initial conditions of the scalar field are, the scalar field can always start from subdominant at early times and become dominant at late times. In order for the scalar field to contribute about 70% to the total energy density and provide an accelerating universe at present time, it must have not reached the fixed point yet. Therefore it is necessary to fine-tune the initial conditions of the scalar field [19].

Before we study the fine-tuning problem, we would like to further examine the exact solutions obtained in the previous section first. By using the exact solutions [14] and [15], we get the energy density for the scalar field $\phi$

$$\rho_\phi = \frac{d(d+2)b^2}{4\kappa^2} + \frac{1}{2\kappa^2}d(d-1)\delta^2b^{d-2}$$

$$= a^{-6\kappa^2} \left[ \frac{d(d+2)b^2}{4} + \frac{3(d+2)}{2(d-1)} \sinh^{-2} \left( \frac{3(d+2)}{d} |\eta| \right) \right]. \quad (24)$$

Since $\rho_m \propto a^{-3}$, the evolution of $\rho_\phi/\rho_m$ is shown in Figure 1. During the acceleration phase, we see that the ratio of $\rho_\phi/\rho_m$ increases with time. Therefore, we expect the scalar field contributes more and more to the total energy density.

Now we are ready to solve equations (20)-(22) numerically. It is more convenient to use the variable $u = \ln(a/a_0)$ instead of the cosmic time $t$. Let $y = \phi/m_{pl}$, equations (20) and (21) become

$$\left( \frac{H}{H_0} \right)^2 = \frac{\Omega_{m0} \exp(-3u) + \Omega_{r0} \exp(-4u) + \Omega_{V0} \exp(-\lambda y)}{1 - y^2/6}. \quad (25)$$

$$y'' = 3\lambda\Omega_V \exp(-\lambda y) - \left[ \frac{3}{2} \Omega_m + \Omega_r + 3\Omega_V \exp(-\lambda y) \right] y', \quad (26)$$

where $\Omega_m = \rho_m/3H^2m_{pl}^2 = \Omega_{m0}(H_0/H)^2 \exp(-3u)$, $\Omega_V = V_1/3H^2m_{pl}^2 = \Omega_{V0}(H_0/H)^2$, $V_1 = d(d-1)\delta^2/2\kappa^2$, $\Omega_\phi = y^2/6 + \Omega_V \exp(-\lambda y)$, $\Omega_r = \rho_r/3H^2m_{pl}^2 = \Omega_{r0}(H_0/H)^2 \exp(-4u)$, and we use $'$ to denote differentiation with respect to $u$ in this section. By using equation (22), we get

$$\frac{\ddot{a}}{aH^2} = -q = \Omega_\phi - \frac{1}{2} y^2 - \frac{1}{2} \Omega_m - \Omega_r. \quad (27)$$
From equation (27), we can see that $y'_0 < 2 \Omega_\phi$. On the other hand, $\Omega_\phi$ is small at early times, so $y'$ must also be small at early times. Therefore, $y'$ is small from the past to the present, which means that the scalar field changes very slowly. In other words, the scalar field behaves more like a cosmological constant so that $\Omega_V \exp(-\lambda y_0) \sim 0.7$ and $\Omega_V \exp(-\lambda y) \sim 10^{-36}$ at nucleosynthesis when $a/a_0 \sim 10^{-10}$. From equation (26), we see that at very early time when radiation was dominated, $y'' \sim -y'$. So for small positive $y'$, $y'$ will decrease first until it reaches below $10^{-36}$, then it will start to increase again. That is another way to see why the scalar field changes very slowly. In the recent past, we have $y'' \sim 3\lambda \Omega_\phi$, so $y'$ became relatively large and the scalar field would change relatively fast in the recent past. For the numerical calculation, we start from the epoch of nucleosynthesis around $a/a_0 \sim 10^{-10}$ or $u = -23$. The initial conditions are chosen as $\Omega_{V,i} \exp(-\lambda y_i) \sim 10^{-36}$ and $y'_i = 0.01$. The evolution of $\Omega_\phi$, $\Omega_m$ and $\Omega_r$ are shown in Figure 2. With the choice of the initial conditions of the scalar field, the behaviors of the energy densities are consistent with the current observations. We also plot the evolutions of the dimensionless acceleration $-q = \dot{a}/aH^2$ and the scalar field in Figure 3. From Figure 3, we see that the scalar field increases very slowly for most of the time and it starts to increase relatively fast in the recent past when it became to dominate the energy component. This behavior confirmed the above discussions. $\phi/m_{pl}$ increases about 2 during the whole period of evolution, so the volume increases about $\exp(2\sqrt{2d/(d+2)}) = 12.1$ times.
4. Discussions

Upon compactification of pure gravity in higher dimensions with time dependent hyperbolic internal space, we get a scalar field with exponential potential in four dimensions. The potential of the scalar field is too steep to give an accelerated attractor solution. With suitable fine-tuning of the initial conditions of the scalar field, our Universe experiences a transient acceleration now. The scalar field changed very slowly until recently and it behaves like a cosmological constant. Due to the scalar field dynamics, the volume of the internal space increases about $12^{1.1}$ times. In the far future, the internal space will become infinite and $(\dot{b}/b)/(\dot{a}/a)$ goes to $2/d$ asymptotically. The change of the volume of the internal space results in the variation of the fundamental constants. Cline and Vinet found that the time variation of the gravitational constant violated the observational constraint in the string frame for time-varying extra dimensions [20]. It is known that the effective four dimensional fine structure constant is inversely proportional to the volume of the internal space, so

$$\left| \frac{\dot{\alpha}}{\alpha} \right| = d \left| \frac{\dot{b}}{b} \right| = \sqrt{\frac{2d}{d+2}} H |y'|.$$  \hfill (28)

The experimental data gives $|\dot{\alpha}/\alpha| \lesssim 10^{-14}$ yr$^{-1}$ [21]. Substitute this constraint and $H_0 \sim 10^{-10}$ to equation (28), we get

$$|y'| \lesssim 10^{-5}.$$  \hfill (29)
As we discussed in the last section, the scalar field changes relatively fast during the acceleration phase and the bound (29) is violated at present. Although the time dependent compactification of M theory or superstring theory can give a transient acceleration which is consistent with the current astronomical observations, the time variation of the internal space will result in large variation of the fine structure constant. This result is quite general and can apply to other models with transient acceleration. The change of fine structure constant due to time dependent internal space was also discussed in [14]. They discussed the scalar field dominated region only. In our discussion, we include radiation and cold dark matter, so our result is more general.

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