Non-local Matching Condition and Scale-invariant Spectrum in Bouncing Cosmology

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In cosmological scenarios such as the pre-big bang scenario or the ekpyrotic scenario, a matching condition between the metric perturbations in the pre-big bang phase and those in the post big-bang phase is often assumed. Various matching conditions have been considered in the literature. Nevertheless obtaining a scale invariant CMB spectrum via a concrete mechanism remains impossible. In this paper, we examine this problem from the point of view of local causality. We begin with introducing the notion of local causality and explain how it constrains the form of the matching condition. We then prove a no-go theorem: independent of the details of the matching condition, a scale invariant spectrum is impossible as long as the local causality condition is satisfied. In our framework, it is easy to show that a violation of local causality around the bounce is needed in order to give a scale invariant spectrum. We study a specific scenario of this possibility by considering a nonlocal effective theory inspired by noncommutative geometry around the bounce and show that a scale invariant spectrum is possible. Moreover we demonstrate that the magnitude of the spectrum is compatible with observations if the bounce is assumed to occur at an energy scale which is a few orders of magnitude below the Planckian energy scale.

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I. INTRODUCTION

The studies of big bang singularity is an important arena where string theory and cosmology meet. Recently based on stringy dualities or extra dimensions arguments, attempts to resolve the big bang singularity such as the pre-big-bang \cite{1} or the ekpyrotic/cyclic cosmology \cite{2,3} has been put forwarded (see for example \cite{4} for review on string cosmology). To be an ambitious model of string cosmology, one hopes to reproduce the scale-invariant spectrum of density perturbations without invoking inflation. In both of these scenarios, one can expect the physics around the bounce (say around \( \eta^- < \eta < \eta^+ \)) to be nonperturbative and highly nontrivial. Without committing oneself to any specific form of the dynamics involved, a useful approach to this problem is to replace the dynamical evolution around the bounce by a nontrivial phase transition \cite{2,3}. In this approach, one evolves the Einstein equation far from the bounce (for time \( \eta < \eta^- \) and \( \eta > \eta^+ \)) where the classical equation can be applied, and then try to connect the physics at \( \eta^-, \eta^+ \) using an appropriate matching condition. We call these matching conditions to distinguish from the usual junction conditions \cite{33}.

The form of the matching conditions has important consequences on the form of the cosmic microwave background (CMB) spectrum. In the framework of known physics, it is found not to be the case both for the pre-big bang \cite{3,5} and the ekpyrotic scenarios \cite{2,10,11,12}. It is natural to ask to what extent this conclusion depends on the details of the matching conditions. In \cite{12} it was argued that the predictions of density perturbations of the bouncing cosmology are independent of the details of the matching condition near the bounce, with the reason that there exists a dynamical attractor such that every observer will follow the same history and result in the continuity of the curvature perturbation. Studies on models with regular bounce has been carried out \cite{12} and gave negative result on rather general assumptions \cite{14}.

The main goal of this paper is to understand to what extent the known matching conditions may get modified due to the high energy corrections to Einstein gravity and to examine whether or not there is any special class of matching conditions that could lead to a scale invariant spectrum. We will show that if the matching condition respects a local causality condition near the bounce, there is indeed no way to generate a scale invariant power spectrum, agreeing with the general result of \cite{12}. By local
causality, we mean a local event is not allowed to affect infinitely separated points through the bounce. More specifically, using our local matching condition, we find that a new mixing term between the subdominant mode in the pre-bounce era and the subdominant mode in the late time universe is allowed. This could be attributed to the anisotropic stress during the bounce. This new mixing term, however, cannot help to produce a scale-invariant mode that is dominant in the late time universe. To do this job, a mixing between the dominant mode in the pre-bounce era and the dominant mode in the late time is needed. However, this mixing is absent in general if the matching condition respects the local causality condition.

The important conclusion of our general treatment of the matching condition is that there has to be new nonlocal effects beyond general relativity in order to obtain a scale invariant power spectrum for the CMB fluctuations. In string theory, nonlocal effects such as noncommutative geometry has been widely considered. Motivated by this, we consider a modified equation of motion for the cosmological fluctuations inspired by the noncommutative field theory. We find that if the degree of nonlocality is not too strong, a scale invariant spectrum may appear. Moreover the magnitude of the density spectrum is compatible with observation if the bounce occurred at an energy scale that is a few orders of magnitude below the Planck scale.

The paper is organized as follows. In the next section, we discuss the notion of local causality and explain how it leads to the condition that there exists no negative powers of derivative in the matching condition. In section III we begin with a brief review of the linear cosmological perturbation theory, applied to the pre-big bang and ekpyrotic scenarios. Next we impose the condition of local causality and express our local matching condition in terms of the gauge invariant variables. We then demonstrate a no-go theorem: a scale invariant spectrum is generally impossible if a local matching condition is assumed. In section IV we go beyond the condition of local causality and consider a toy model inspired by the noncommutative field theory of the ekpyrotic scenario. We find in this model that it is possible to obtain a scale-invariant spectrum. Section V is devoted to conclusions and discussions. The paper is ended with a couple of appendices which address some of the more technical issues.

Here is our notation: \( \mu, \nu, \cdots \) run from 0 to 3, \( i, j, \cdots \) run from 1 to 3, \( x \) represents the spatial part of the coordinate \( x^\mu \).

II. LOCAL MATCHING CONDITION IN BOUNCING COSMOLOGIES

As we explained in the introduction, the situation is more complicated in the studies of bouncing cosmologies. Here one cannot evolve the classical general relativity in the region around the bounce since it is supposed to break down there. Since this region is supported generally on a nonzero measure set, one cannot apply the usual idea of junction condition which is imposed at a single hypersurface. One need a more general matching condition.

One approach which has been widely adopted in the literature is to assume the occurrence of a nontrivial phase transition around the bounce. Let \( \mathcal{O}(t, x) \) be the order parameter associated with the phase transition. Then the matching is carried out on the two hypersurfaces (\( \pm \) corresponds to initial and final time of the phase transition)

\[
\mathcal{O}(t^\pm, x) = \mathcal{O}_\pm^0 = \text{constant},
\]

by matching up the values of certain quantity which one may argue to be conserved during the phase transition. However there is weakness in this approach since the conservation law depends in some details of the dynamics during the phase transition.

In the following we will take a different approach which allow us to discuss all possible matching conditions in general. We will characterize a matching condition according to whether it respects a local causality condition of the physics involved. Our approach has the advantage that it is more robust without assuming the underlying dynamics. Also as we will see, it gives clear indication how new physics should appear in order to give a scale invariant density spectrum.

A. Requirement of local causality and local matching condition

Our guiding principle in constructing the matching condition is the condition of local causality. The statement is that no local event is allowed to affect infinitely separated points through the phase transition. Or equivalently the past causal region of any space-time point is of finite extent for a finite time duration. Although this assumption is intuitive and natural, a couple of remarks about its applicability is in order.

1. We note that if the spatial section of the universe during the bounce is compact and of the size comparable to the time scale of the bounce, local causality is not expected to hold. This would be the case if the size of the universe is the self dual radius of the string theory during the bounce. Several models with closed spatial section has been studied \[12\]. On the other hand, if the spatial section of the universe during the bounce remained of the size much larger than the time scale of the bounce, it is reasonable to assume the validity of local causality. This requires the flattness problem to be solved by some mechanism prior to the bounce \[10\]. We do not address the flatness problem in this paper and concentrate on the problem of the spectrum of the density fluctuation.

2. Due to the nonlocal nature of string, modification of the local causality condition occurs, typically at the interacting string level \[13\]. One can expect the violation of the local causality condition also occurs in some stringy
cosmologies. Another well known example of nonlocal physics is noncommutative geometry. It is possible that quantum gravity may lead to a description of noncommutative geometry during the bounce and a violation of the local causality condition. We will say more about this later.

Next we want to apply the requirement of the local causality to obtain our local matching condition. Let us begin with introducing two space-like hypersurfaces $S_-$ and $S_+$ separated by the bounce, where the former is before the bounce and the latter is after it. Before and after $S_\pm$, we assume that general relativity is valid and that linear perturbation provides a good approximation for the evolution of the fluctuation. The surface of the matching is specified by equations

$$O_\pm(t, x) = O_\pm^0 \text{ on } S_\pm$$

(2)

respectively. Here $O_\pm(t, x)$ are scalar quantities constructed out of metric and matter fields and $O_\pm^0$ are constants. Let us choose the synchronous gauge

$$g_{00} = -1, \quad g_{0i} = g_{0i} = 0,$$

(3)

We also choose the time coordinate to satisfy

$$t = t_\pm \text{ on } S_\pm$$

(4)

respectively. Note that the condition (3) and (4) are possible for any surface of the matching (2). This is due to the residual gauge degrees of freedom in the synchronous gauge.

We will be interested in the standard type of cosmological backgrounds which are homogeneous and isotropic,

$$ds^2 = -dt^2 + a^2(dx)^2 = a^2(-d\eta^2 + dx^2).$$

(5)

Here $a = a(\eta)$ and $\eta$ is the conformal time. Consider metric and scalar field perturbation of the form

$$\delta g_{\mu\nu} = \delta g_{\mu\nu}(k, \eta)e^{ikx} \quad \text{and} \quad \delta \varphi = \delta \varphi(k, \eta)e^{ikx},$$

(6)

where $k$ is the comoving wave number. We will be interested in perturbations with wavelength longer than the horizon size since these are the fluctuations that are relevant to the CMB observational data. The long wavelength limit means that the physical wavelength $a/k$ is much larger than the Hubble scale, i.e.

$$\frac{k}{aH} \bigg|_{\eta = \eta_\pm} \approx \frac{k}{\mathcal{H}} \bigg|_{\eta = \eta_\pm} \ll 1.$$  

(7)

Here $H = \dot{a}/a$ is the Hubble constant and $\mathcal{H} = a'/a$; dot represents the derivative with respect to cosmological time $t$ and prime represents the derivative with respect the conformal time $\eta$. One can convince oneself that this bound for long wavelength limit is very easy to be satisfied by the currently observed CMB data.

Now we impose the condition of local causality. By local causality of the matching condition we means that there exists some finite region $\Omega$ on $S_-$ such that all the data $\{g_{ij}, g'_{ij}, \varphi, \varphi'\}$ at $P^0$ on $S_+$ is determined only by data $\{g_{ij}, g'_{ij}, \varphi, \varphi'\}$ on $\Omega$, modulo local spatial coordinate transformation (see FIG. 4). A subtle point in the problem of the matching condition is that it is not guaranteed that the Cauchy problem is solvable throughout the bounce. In this paper, we assume the solvability for long wavelength modes. In the long wavelength limit, the scalar field and metric on $\Omega$ can be expanded in terms of $\varphi, g_{ij}$ and derivatives of them at a point $P$ on $\Omega$. Given a one to one map $P \to P'$ from $S_-$ to $S_+$, the metric and scalar field at $P'$ is given by a function of the scalar field, metric and derivatives of them at $P$. For example, once the coordinate system on $S_+$ is given, the matching condition for the metric formally takes the form

$$g_{ij}\big|_{\eta = \eta_+, P'} = H_{ij}(g_{ij}, g'_{ij}, \varphi, \varphi', \nabla_k)\big|_{\eta = \eta_+, P'},$$

(8)

The function $H_{ij}$ is a rank 2 symmetric tensor with respect to the spatial coordinate transformation. Because of the local causality, no negative power of $\nabla_k$ can appear. Due to the tensor structure, it is clear that only even powers of $\nabla_k$ appear. The general form of the metric on $S_+$ is given by possible covariant combinations of metric and scalar fields with spatial derivatives on $S_-$. Choosing a map $P \to P'$ properly, no $x$-dependent quantity enters into the matching condition. Expanded in powers of $\nabla_k$, the matching condition (8) takes the form

$$g_{ij}\big|_{\eta = \eta_+, P'} = H_{ij}^{(0)}(g_{ij}, g'_{ij}, \varphi, \varphi')\big|_{\eta = \eta_+, P'} + \cdots,$$

(9)

where $\cdots$ denotes terms of higher order in $\nabla_k$. Now note that the only nontrivial tensors available for constructing the tensor $H_{ij}^{(0)}$ are $g_{ij}, g'_{ij}$ and $g^{ij}, g'^{ij}$. Employing a matrix notation, it is easy to see that the most general form of $H^{(0)}$ is:

$$H^{(0)} = \sum_{\chi} c_{\chi}(\varphi, \varphi') P_{\chi}(g, g', g^{-1}, g'^{-1}),$$

(10)

where for $n \geq 0$ integer,

$$P_{\chi} := f(AB)A(BA)^n$$

with $A = g$ or $g'$, $B = g^{-1}$ or $g'^{-1}$. (11)

$f(AB)$ represents all possible functions constructed out of $\text{Tr}(AB)^l$'s. The sum $\chi$ in (10) is over all possible
inequivalent forms of such factors. Similarly, we can generalize the above discussion to the matching condition of \( y_{ij} \) and for other generic tensors.

We remark that one may try to generalize the above to the gauge invariant quantities, but the requirement of local causality is less clear and less easy to formulate since spatial integrals of the metric fluctuations is included in the definition of the gauge invariant quantities. Our local matching condition is in the same spirit as the method of spatial gradient expansion (see, e.g., [18]). However, validity of a four dimensional effective theory during the bounce is not assumed in our approach. In the next section we will apply our local matching condition to the bouncing cosmologies. We will discuss the effect coming from fluctuation in the other scalar fields contribute to the entropy perturbation. We will discuss the effect of the entropy perturbation on the matching condition in the appendix [6].

In addition to the local causality condition, we will also assume the perfect fluid condition before and after the bounce. Perfect fluid condition requires the vanishing of the anisotropic stress. This is a reasonable assumption since in the single scalar field model, there is no anisotropic stress in the linear order of perturbation theory. In the radiation dominated era after the bounce, there is also no anisotropic stress if the mean free path is small, as is often assumed. We will discuss the effect of the anisotropic stress in the appendix [8].

In section III.C, we will show that one cannot obtain a scale invariant spectrum if the local matching condition is employed. This conclusion is unaffected by the effects discussed in appendix [6] and [8].

**III. DENSITY PERTURBATION IN BOUNCING COSMOLOGIES**

In this section, we will first briefly review the theory of cosmological perturbation [19]. We then apply the local matching condition to the bouncing cosmologies. Before the bounce, the model for the pre-big bang or ekpyrotic scenario is given by the four-dimensional effective theory with a scalar field coupled to gravity, the action is

\[
S = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} R + \frac{1}{2} (\partial \varphi)^2 - V(\varphi) \right). \tag{12}
\]

The scalar field \( \varphi \) is related to dilaton or the size of the extra-dimension. The evolution of fluctuation depends on the nontrivial potential \( V(\varphi) \) for the scalar field. The universe starts from a Minkowski spacetime with \( V \approx 0 \). In this paper we will consider the case of having a single scalar \( \varphi \) only. The generalization to many scalars is straightforward.

We remark that: 1. In general there could be other fields in the model. One can divide the scalar fields into the background one (adiabatic field) and the others (entropy fields) [20]. The scalar field in [20] is the adiabatic field and entropy fields are zero by definition. The effect coming from fluctuation in the other scalar fields contribute to the entropy perturbation. We will discuss the effect of the entropy perturbation on the matching condition in the appendix [6].

2. In addition to the local causality condition, we will also assume the perfect fluid condition before and after the bounce. Perfect fluid condition requires the vanishing of the anisotropic stress. This is a reasonable assumption since in the single scalar field model, there is no anisotropic stress in the linear order of perturbation theory. In the radiation dominated era after the bounce, there is also no anisotropic stress if the mean free path is small, as is often assumed. We will discuss the effect of the anisotropic stress in the appendix [8].

We will apply our local matching condition (9) to the bouncing cosmologies. Before the bounce is not assumed in our approach. In the next section we will apply our local matching condition (9) to the bouncing cosmologies. We then apply the local matching condition to the bouncing cosmologies. Before the bounce is not assumed in our approach. In the next section we will apply our local matching condition (9) to the bouncing cosmologies. Before the bounce is not assumed in our approach. In the next section we will apply our local matching condition (9) to the bouncing cosmologies.

**A. Review of cosmological perturbation theory**

Metric fluctuations are classified by their properties under the spatial rotations into scalar, vector and tensor ones. In the linear perturbation theory, time evolution of these perturbations are decoupled with each other. In this paper, we will focus on the scalar ones (the consideration of the vector and tensor modes can be proceeded similarly) which are defined by

\[
\delta g_{\mu\nu} = a^2 \left( \frac{2\phi}{-B_{ij}} - \frac{B_{ij}}{2(\psi \delta_{ij} - E_{ij})} \right), \tag{13}
\]

where \( \phi, B, \psi \) and \( E \) are scalar functions on the constant-time hypersurface. Indices \( i \) means taking three dimensional covariant derivative in spatial direction. In our case it is the ordinary derivative. Indices \( i, j \cdots \) are raised and lowered by the Kronecker delta.

There is a gauge symmetry for the fluctuations of metric due to the residual diffeomorphism on the background metric, and the gauge transformations affecting scalar fluctuations are of the form

\[
\eta \rightarrow \tilde{\eta} = \eta + \xi^0(\eta, x) \quad \text{and} \quad x^i \rightarrow \tilde{x}^i = x^i + \delta^i j \partial_j \xi(\eta, x). \tag{14}
\]

In the linear order, functions \( \phi, B, \psi \) and \( E \) in the metric fluctuation are changed to \( \tilde{\phi}, \tilde{B}, \tilde{\psi} \) and \( \tilde{E} \) as follows.

\[
\tilde{\phi} = \phi - (a'/a)\xi^0 - \xi^0' , \quad \tilde{\psi} = \psi + (a'/a)\xi^0 , \quad \tilde{B} = B + \xi^0 - \xi' , \quad \tilde{E} = E - \xi, \tag{15}
\]

Gauge invariant quantities are

\[
\Phi = \phi + (1/a)(B - E')a', \quad \Psi = \psi - (a'/a)(B - E'), \tag{16}
\]

The energy momentum tensor of the action [12] describes a perfect fluid. It follows that \( \Phi = \Psi \). We will assume that the perfect fluid condition is satisfied for \( \eta < \eta_- \) and \( \eta > \eta_+ \).

For the fluctuation of the scalar field we have the constraint equation coming from the equation of motion,

\[
\delta \varphi^{(gi)} = \left( \frac{3}{2} \frac{2}{\varphi'} \right)^{-1} (\Psi' + \mathcal{H} \Phi), \tag{17}
\]

where

\[
\delta \varphi^{(gi)} := \delta \varphi + \varphi'(B - E') \tag{18}
\]

is the gauge invariant fluctuation of the scalar field and \( l \) is the Planck length. Thus the scalar fluctuation \( \Phi \) is the only independent quantity to consider. By expanding the Einstein equation around the background metric, one gets the time-dependent equation for \( \Phi \), which can be put into a compact form in terms of the new variable

\[
u''(k, \eta) + \left( k^2 - \frac{(z^{-1})''}{z^{-1}} \right) u(k, \eta) = 0, \tag{19}
\]

where

\[
\nu''(k, \eta) = \nu''(k, \eta) + \left( k^2 - \frac{(z^{-1})''}{z^{-1}} \right) u(k, \eta) = 0, \tag{19}
\]
where $z = a \varphi' / \mathcal{H}$. This is a Schrödinger type equation and can be solved once the initial condition is specified. The initial condition is given by assuming that the universe starts from the Minkowski space where $k^2 \gg (z^{-1})''/z^{-1}$. In this limit, we get a plane wave solution with the normalization fixed by the canonical quantization, the result is

$$u(k, \eta) = -\frac{3}{2} l^2 k^{-3/2} e^{-i k \eta}. \quad (20)$$

Note that only the positive energy mode ($e^{-i k \eta}$) is chosen as the initial condition.

On the other hand, we are interested in the long wavelength limit, $k^2 \ll (z^{-1})''/z^{-1}$ near the bounce, therefore

$$u''(k, \eta) - \frac{(z^{-1})''}{z^{-1}} u(k, \eta) = 0. \quad (21)$$

The solution at the leading order of $k$-expansion is

$$l^{-2} u = S(k) \left( \frac{1}{z} + O(k^2) \right) + \frac{3}{2} D(k) \left( \frac{1}{z} \int d \eta (z^2) + O(k^2) \right). \quad (22)$$

Note that the correction starts from $k^2$ order. From (22) we have

$$\Phi = S(k) \left( l^3 \frac{\mathcal{H}}{a^2} + O(k^2) \right) + D(k) \left( \frac{1}{a} \left( \frac{1}{a} \int d \eta a^2 \right) + O(k^2) \right). \quad (23)$$

The coefficients $S(k)$ and $D(k)$ are time independent. Their form are fixed by extrapolating (22) to the short wavelength limit and compare with (20) at the regime $k^2 \sim (z^{-1})''/z^{-1}$. The details depend on the time evolution of $\Phi$ at this intermediate regime and thus depends on the potential $V(\varphi)$. A scale invariant spectrum

$$P_\Phi(k) = \frac{k^3}{2 \pi^2} |\Phi(k)|^2 \quad (24)$$

is obtained if $\Phi(k) \propto k^{-3/2}$.

In bouncing cosmologies, the term proportional to $S$ grows before the bounce whereas the term proportional to $D$ is constant with respect to time. After the bounce, assuming the universe is filled with radiation and that there is no entropy perturbation, then the evolution of fluctuation is governed by the same equation (19) as before. In the long wavelength limit, the solution for the fluctuation is in the same form as (23). We note that the $S$-mode is now decaying in time and thus the $D$-mode becomes dominant, particularly at the time of decoupling.

The important task is to determine the $k$-dependence of $S(+) (k)$ and $D(+) (k)$ after the bounce. Using the matching condition, one can relate them to the $S(-) (k)$ and $D(-) (k)$ before the bounce.

In general a mixing between the modes $S$ and $D$ may occur,

$$\begin{pmatrix} S(+) \\ D(+) \end{pmatrix} = M \begin{pmatrix} S(-) \\ D(-) \end{pmatrix}. \quad (25)$$

Here, matching matrix $(M)_{ij}$ are functions of $k$. If the mixing is right, the desired form of $D(+) \sim k^{-3/2}$ may be generated and hence result in a scale invariant spectrum in CMB.

Below we derive the form of the matching between $(S(-), D(-))$ and $(S(+), D(+))$. In particular we will find that local causality implies that

$$M_{21} = 0, \quad M_{22} = 1, \quad \text{and thus} \quad D(+) = D(-). \quad (26)$$

### B. Consequences of the local matching condition: mixing matrix

Our discussion so far is general and we can consider the matching condition on any surface. In the analysis below, we will take the matching surface $S_\text{m}$ to be the one given by $\varphi = \text{constant}$. We can choose a different matching surface with $\mathcal{O}_- (\eta, x) = \text{constant}$ for a different quantity $\mathcal{O}_-$. However as explained in the appendix $\mathbb{E}$ the metric and scalar field fluctuations turns out to be the same as (27) and (28) up to the order we consider and leads to the same matching condition (47) below.

To justify the linear matching condition, one needs to examine the magnitude of the fluctuation of physical quantities on the surface of the matching. In $\mathbb{R}$, validity of the linear perturbation theory is demonstrated using the attractor property of the background solution. In particular it has been shown that in the synchronous gauge $\delta g_{00} = \delta g_{0i} = 0$, linear perturbation theory remains valid near the bounce. Another advantage of using the synchronous gauge is that we can choose the matching surface such that the scalar field has no fluctuation (up to the order of $k$ we consider) over it. In fact as we show in the appendix $\mathbb{A}$ we can exploit the residual gauge symmetry of this gauge and choose the hypersurfaces $S_\text{m}$ such that $\varphi$ is constant over $S_\text{m}$, i.e. $\delta \varphi = 0$. Therefore we will use the synchronous gauge. The form of the metric and scalar field fluctuation in this gauge is also computed in the appendix $\mathbb{A}$ with the result that, around $\eta = \eta_\text{m}$.
Here we have used the fact that $c = O(k^{n+2})$. And such that $\delta \varphi = 0$ at $\eta = \eta_-$. Here $a_\Phi$ is a constant specifying the order in the wave number $k$ of the gauge invariant observable $\Phi$, i.e., $\Phi = O(k^{n+2})$. In our following analysis, we will be interested in the leading order terms only, and so any terms with $O(k^{n+2})$ are effectively zero in the long wavelength limit. Note also that

$$c_X = c_X^{(B)} + O(k^{n+2}) \sim c_X^{(B)}$$

for the coefficient $c_X^{(B)}$ appearing in the general matching condition $\Psi$. Here the superscript $B$ denotes the background.

As a first application of our local matching condition, we will show that the covariance of the matching condition $\Psi$ proves the continuity through the bounce of the term proportional to $D(-)\delta_{ij}$ in the metric fluctuation $\delta g_{\mu\nu}$. To see this, we substitute (27) into the general matching condition $\Psi$ and, for the moment, assume that the term $k_i k_j S^{(-)} \cdots$ is higher order in $k$ compared to the $D(-)$ term. Due to the structure of $P_X$, we get generally

$$P_X(g, g', g^{-1}, g'^{-1}) = d_X(a, a') \times (1 + 2D(-)) + \cdots$$

to leading order in $k$. Here $d_X(a, a')$ is a homogeneous monomial of $a, a', a^{-1}, a'^{-1}$ of degree 1. Therefore we obtain

$$g^{(+)\mu\nu}(\eta) = a^2(1 + 2D(-))\delta_{ij} + \cdots$$

where

$$a_+ = a_+(a(\eta), a'(\eta), \cdots)|_{\eta = \eta_-}.$$  

Here we have used the fact that $c_X^{(B)}$ depends only on $a$ and $a'$ and we have denoted

$$a^2 := \sum_X c_X^{(B)}d_X.$$  

As a result, we obtain

$$\delta g^{(+)\mu\nu}(\eta) = 2a^2D(-)\delta_{ij} + O(k^{n+2}).$$

Applying the same matching argument to the time derivative of the metric, we get

$$\delta g^{(+)\mu\nu}'(\eta) = 2(a^2)'D(-)\delta_{ij} + O(k^{n+2}).$$

where the constant $(a^2)'$ is defined in a similar fashion as $\Psi$. Having $\delta g^{(+)\mu\nu}$ and its first time derivative on $\eta = \eta_+$ surface, then $\delta g^{(+)\mu\nu}$ can be determined for $\eta > \eta_+$. In the synchronous gauge, we have

$$\delta g^{(+)\mu\nu}(\eta, k) = 2a^2_+ \left( \begin{array}{c} 0 \\ 0 \\ D(-)\delta_{ij} + O(k^{n+2}) \end{array} \right).$$

This concludes the continuity of the term proportional to $D(-)\delta_{ij}$ through the bounce. This result is consistent with the argument given in [12].

Next we include also the $k_i k_j$ terms in the local matching condition $\Psi$. The most general form of $\delta g^{(+)\mu\nu}$ after the bounce is

$$\delta g^{(+)\mu\nu}(\eta, k) = 2a^2_+ \left( \begin{array}{c} 0 \\ 0 \\ (D(-) + O(k^{n+2}))\delta_{ij} + O(k^{n+2}) \end{array} \right).$$

Here $M_S$ and $M_D$ are some functions of $\eta$ whose form can be fixed by requiring the perfect fluid condition. Indeed from the definitions of the gauge invariant quantities $\Phi$, $\Psi$ one has

$$\Phi = (1/a) \left[ (S^{(-)}M_S + D^{(-)}M_D)'a \right] + O(k^{n+2}),$$

$$\Psi = D(-) - (a'/a) (S^{(-)}M_S + D^{(-)}M_D)' + O(k^{n+2}).$$

Perfect fluid condition $\Phi = \Psi$ requires

$$(1/a)[M'_S a]' = -(a'/a)M'_S,$$

and

$$(1/a)[M'_D a]' = 1 -(a'/a)M'_D.$$  

Solving gives

$$M'_S = C_Sl^2/a^2,$$

and gives

$$M'_D = a^{-2} \left( \int d\eta a^2 + C_D l^2 \right).$$

where $C_S$ and $C_D$ are arbitrary constants. Putting this into (22),

$$\Phi = -C_S S^{(-)} \frac{H^2}{a^2} + \frac{1}{a} D^{(-)} \left( \frac{1}{a} \int d\eta a^2 + C_D l^2 \right)' + O(k^{n_s + 2})$$

$$= -C_S S^{(-)} + C_D D^{(-)} \frac{H^2}{a^2} + D^{(-)} \frac{1}{a} \left( \frac{1}{a} \int d\eta a^2 \right)'$$

$$+ O(k^{n_s + 2}).$$

(44)

The form of the fluctuation agrees with the form (22) in the lowest order and thus confirms the absence of the entropy perturbation after the bounce in the long wavelength limit up to the order $k^{n_s}$. An immediate consequence of (44) is that

$$D^{(+)} = D^{(-)}.$$

(45)

This condition expressing the continuity of the curvature perturbation was shown in [21] as the consequence of the conservation law of energy momentum. In this paper, we have established this as the result of the covariance of the matching condition.

Note that our above result (44) and (45) were obtained by assuming that in the leading order of the $k_i k_j S^{(-)} \ldots$ higher order in $k$ compared to the $D^{(-)}$ term. This is true for the ekpyrotic scenario (see [24] below), but not for the pre-big bang scenario [30]. For a general potential of the form (48), one can check that the term $k_i k_j S^{(-)} \ldots$ will be dominant over the $D^{(-)}$ term whenever $1/3 \leq p < 1$. Whenever the assumption is violated, one needs to take into account of the $k_i k_j S^{(-)}$ term in our above analysis and this leads to higher order corrections to $M_{21}$ of the form $M_{21} \sim k^2$. And we obtain that

$$D^{(+)} = D^{(-)} + A k^2 S^{(-)},$$

with $A$ a real number, (46)

in the leading order of $k$. This agrees with that of [11].

In conclusion, the matching condition in the leading order can be expressed in the form of a mixing matrix as follows

$$\begin{pmatrix} S^{(+)} \\ D^{(+)} \end{pmatrix} = M \begin{pmatrix} S^{(-)} \\ D^{(-)} \end{pmatrix} = \begin{pmatrix} C_S & C_D \\ A k^2 & 1 \end{pmatrix} \begin{pmatrix} S^{(-)} \\ D^{(-)} \end{pmatrix},$$

(47)

with undetermined coefficients $C_S$ and $C_D$. [11] is the most general matching condition that is consistent with the requirement of local causality. This is one of the main result of this paper.

Utilizing the explicit spectra [30] and [24] below, the condition [10] implies that it is impossible to generate a scale invariant spectrum for the pre-big bang and the ekpyrotic scenario. For the general potential [18], a possibility of obtaining a scale invariant spectrum was discussed in [11] where $p = 2/3$ and $D^{(-)} \sim k^{3/2}$, $S^{(-)} \sim k^{-7/2}$ were found. Unfortunately, the solution turned out not to be a stable attractor point [22]. It is still an open problem to obtain more general form of initial spectrums which could result in the scale invariant spectrum after the bounce.

Finally we remark that if one instead uses the Lichnerowicz junction condition on the constant energy surface, one can apply the result of [9] and gets $C_S = 1, C_D = 0$. Deviation of $C_S$ and $C_D$ from unity and zero respectively is possible in general. For example, in appendix D we discuss the effect of having anisotropic stress during the bounce and how the values of $C_S$ and $C_D$ may be affected. We also remark that here we have only considered a single scalar coupled to gravity. For the effect of multi-scalar scenarios on the matching condition, we refer the reader to appendix [10] for some discussions.

C. Bouncing cosmologies and density spectrum

In the studies of bouncing cosmologies, a potential is supposed to be generated by the nonperturbative effects in string theory and it takes the form

$$V(\phi) = -V_0 e^{-\sqrt{\pi} \phi}.$$

(48)

Different values of the parameter $p$ give different cosmological models. For example, $0 < p << 1$ in the ekpyrotic scenario, and $p = \frac{1}{2}$ in the pre-big bang scenario. In this case, an exact solution for the background metric [3] can be obtained. We have

$$a = a_0 |\eta|^{\frac{1}{1-p}}$$

(49)

and

$$z = \left( \frac{3t^2}{2} \right)^{-1/2} a_0 p^{-1/2} |\eta|^{\frac{1}{1-p}},$$

(50)

where $a_0$ is a parameter with the dimension of length. Moreover, the equation [13] for the metric fluctuation can be solved explicitly by [11]

$$u = \left( \frac{3t^2}{2} \right) \sqrt{\frac{\pi}{2} k^{-1} |\eta|^{1/2} H_{1/2}^{(-1)}(k|\eta|)},$$

(51)

where $H_{1/2}^{(-1)}(k|\eta|)$ is the Hankel function of the first kind and $\nu = \frac{1}{2} + \frac{1}{1-p}$. Expanding the solution in $k$, we get

$$S = \left( \frac{3t^2}{4} \right)^{1/2} \frac{2}{\sin \nu \pi} a_0^{-i} \Gamma(-\nu + 1) p^{-1/2} k^{-1-\nu},$$

$$D = -\left( \frac{3t^2}{4} \right)^{1/2} \frac{2}{\sin \nu \pi} a_0^{-1} e^{-i \nu \pi} \Gamma(\nu + 1) p^{1/2} \frac{1}{1-p} k^{1+\nu}.$$

(52)

These expressions give the $k$-dependence of the $S$ and $D$ for a given $p$. It is easy to obtain that before the bounce,
S and D scales as
\[ S^{(-)}(k) \sim k^{-2}, \quad D^{(-)}(k) \sim k^{0}, \]
for the pre-big bang scenario,
\[ S^{(-)}(k) \sim k^{-3/2}, \quad D^{(-)}(k) \sim k^{-1/2}, \]
for the ekpyrotic scenario. (54)

As has been argued above, scale invariant spectrum is not resulted in these scenarios.

In conclusion, we have shown a no-go theorem: as long as the local causal matching condition is respected, one cannot obtain a scale invariant spectrum in the post-bounce era for both the pre-big bang scenario and the ekpyrotic scenario. This no-go theorem can however be easily lifted. In the next section we discuss the possibility of having nonlocal causality during the bounce and study the possible effects on the resulting spectrum. In particular we demonstrate that by allowing nonlocal effects during the bounce, a scale invariant spectrum can be generated.

IV. NON-LOCAL BOUNCE AND SCALE-INVARIANT SPECTRUM

In string theory, noncommutative geometry is known to be the possible and easily realized in terms of D-brane. See for the review [24]. One possibility to violate the local causality condition is the emergence of noncommutative geometry. Non-local causal structure is a general consequence of noncommutative field theories [24]. In noncommutative space, it is known that the low-energy effective action typically includes nonlocal term. For example in the noncommutative \( \varphi^4 \) theory the low energy effective action provides a nonlocal term [25].

\[ \mathcal{L}_{\text{eff}} = \int d^4p \frac{1}{2} \left( p^2 + m^2 + \frac{g^2}{96\pi^2(p \circ p + \Lambda^2)} \right) \varphi(p)\varphi(-p) + \cdots, \] (55)

where \( p \circ p = |p_i(\theta^{ij}) p_j| \) and \( \theta^{ij} \) is the noncommutative parameter with the relation
\[ [x^i, x^j] = i\theta^{ij}. \] (56)

Here \( x^i \) and \( p_i \) are dimensionful quantities and correspond to \( ax^i \) and \( ki/a \) in our paper. It is clear from [65] that the time evolution of \( \varphi \) has an IR pole if the ultraviolet cut-off scale \( \Lambda \) is taken to infinity.

We now consider the possible modification on the time evolution of the metric fluctuation due to nonlocal effects. Since it is not known how the gravity couple to the noncommutative field theory, we will study a modified equation for the metric fluctuation as inspired by [65]. We assume that within a time interval \( \eta_1 < \eta < \eta_3 \) around the bounce, nonlocal effects become important and this is captured by the C-term in the following equation for the fluctuation
\[ u''(k, \eta) + \left( k^2 + (C(\eta)k^{-\alpha})^2 - \frac{(z-1)^{2\alpha}}{z^{2\alpha}} \right) u(k, \eta) = 0. \] (57)

Here \( C(\eta) > 0 \) are dimensionless quantities. The time function \( C(\eta) \) is taken to be non-zero in the time interval \( \eta_1 < \eta < \eta_3 \). Moreover, as a simplification of our analysis, we assume that within the bounce period \( C(\eta) \) is constant and the \((C(\eta)k^{-\alpha})^2\) term dominates over the other two terms. This requires gravitational effects to be suppressed during the bounce. We do not argue the mechanism of suppression and take the model as a toy model to demonstrate possible nonlocal effects in the matching condition.

We consider three regimes for the time evolution of the fluctuation. The dividing line is the size of the Hubble scale. We assume that nonlocal effects starts to play a prominent role as the size of the Hubble scale gets down and becomes \( H = 1/l_b \). The length size \( l_b \) characterizes the onset of new physics at sub-\( l_b \) scale. Therefore as the universe approaches the bounce, its size shrinks rapidly until \( H = 1/l_b \) at time \( \eta = \eta_- < 0 \). Below that one can no longer trust the semi-classical gravity picture, and the universe enters into the nonlocal regime dictated by the term \((C(\eta)k^{-\alpha})^2\) in (57) until \( \eta = \eta_+ \). At this moment, we again have \( H = 1/l_b \) and the semi-classical gravity picture is resumed. See FIG. [2].

Therefore, outside the bounce period, we set \( C(\eta) = 0 \), and (57) reduces to
\[ u''(k, \eta) - \frac{(z-1)''}{z-1} u(k, \eta) = 0 \] (58)
for the superhorizon modes. Moreover, from now on, we will focus on the ekpyrotic scenario with the type of scalar potential [48], i.e., \( V(\varphi) = -V_0 e^{-\sqrt{\frac{2}{3}} \varphi} \), and we assume
\[ p = \begin{cases} 
0 < q << 1 & \text{if } \eta < \eta_- , \\
1/2 & \text{if } \eta > \eta_+ .
\end{cases} \] (59)
The choices of \( p \) correspond to the following picture in the ekpyrotic scenario: before the bounce (i.e., \( \eta < \eta_- \)), the brane is slowly approaching the bounce, and after the bounce (i.e., \( \eta > \eta_+ \)) the brane universe enters the radiation dominated era. The time dependence of the
where $a_0, b_0$ are dimensionful length-scale constants. We should mention that for simplicity we have extrapolated the solution $\bar{a}$ so that $a(\eta = 0) = 0$. In general, one can relax this condition by introducing the shift so that $a(\eta) = b_0(\eta - \eta_0)$ for $\eta > \eta_+$. The analysis and the results are essentially the same. We will not consider this possibility. As mentioned, the Hubble size serves as a dimensionful length-scale constant. We also introduce the “period” of the bounce as the dividing line at $\eta = \eta_-$ and $\eta_+$, and these moments are characterized by

$$ H(\eta_-) = H(\eta_+) = l_b^{-1}. \quad (61) $$

Recall $H := \frac{p}{(1-p)\eta}$, then from the above condition we obtain

$$ \eta_+ = \sqrt{\frac{q}{b_0}}, \quad \text{and} \quad \eta_- = -\sqrt{\frac{q}{a_0}}. \quad (62) $$

It follows from (60) and (61) that

$$ a(\eta_-) / a(\eta_+) = q \eta_+ / |\eta_-|. \quad (63) $$

We also introduce the “period” of the bounce as

$$ \Delta := \eta_+ - \eta_- \approx \eta_+ \quad (64) $$

since $q \ll 1$.

The fluctuation in each regime can then be solved easily and we have

$$ u \approx \begin{cases} S^{(-)} \frac{\sqrt{q}}{A_{0}^{1/2}} + D^{(-)} \frac{3\Lambda b_0^{2}|\eta_-|^{3+4}}{2\sqrt{q}} & \text{for } \eta < \eta_- , \\ C_- e^{-i k_0} + C_+ e^{i k_0} & \text{for } \eta_- < \eta < \eta_+ , \\ S^{(+)} \frac{i l^2}{2 B \eta} + D^{(+)} \frac{B \eta^2}{2 l^2} & \text{for } \eta > \eta_+ . \end{cases} \quad (65) $$

Here we have denoted for simplicity $\tilde{k} := C k^{-\alpha}$ and

$$ A := \left( \frac{3 l^2}{2} \right)^{-1/2} a_0, \quad B := \left( \frac{3 l^2}{2} \right)^{-1/2} b_0. \quad (66) $$

Next we connect these solutions by requiring $u$ and $u'$ to be continuous at $\eta = \eta_{-}$ and $\eta_{+}$. We have

$$ \begin{pmatrix} u \\ u' \end{pmatrix} = Z_{\eta = \eta_{-}} \begin{pmatrix} S^{(-)} \\ D^{(-)} \end{pmatrix} $$

$$ = \left( \frac{\sqrt{q}}{A_{0}^{1/2}} \right) \begin{pmatrix} 3 \Lambda b_0^{2}|\eta_-|^{3+4} \\ \Lambda b_0^{2}|\eta_-|^{3+4} \end{pmatrix} \begin{pmatrix} S^{(-)} \\ D^{(-)} \end{pmatrix} $$

$$ = E_{\eta = \eta_{-}} \begin{pmatrix} C_- \\ C_+ \end{pmatrix} $$

$$ = \left( e^{-i k \eta_-} + e^{i k \eta_-} \right) \begin{pmatrix} C_- \\ C_+ \end{pmatrix}. \quad (67) $$

$$ \begin{pmatrix} u \\ u' \end{pmatrix} = Z_{\eta = \eta_{+}} \begin{pmatrix} S^{(+)} \\ D^{(+)} \end{pmatrix} $$

$$ = \left( \frac{\sqrt{q}}{A_{0}^{1/2}} \right) \begin{pmatrix} 3 \Lambda b_0^{2}|\eta_+|^{3+4} \\ \Lambda b_0^{2}|\eta_+|^{3+4} \end{pmatrix} \begin{pmatrix} S^{(+)} \\ D^{(+)} \end{pmatrix} $$

$$ = E_{\eta = \eta_{+}} \begin{pmatrix} C_- \\ C_+ \end{pmatrix} $$

$$ = \left( e^{-i k \eta_+} + e^{i k \eta_+} \right) \begin{pmatrix} C_- \\ C_+ \end{pmatrix}. \quad (68) $$

Thus

$$ \begin{pmatrix} S^{(+)} \\ D^{(+)} \end{pmatrix} = M \begin{pmatrix} S^{(-)} \\ D^{(-)} \end{pmatrix} $$

$$ = Z_{\eta = \eta_{-}}^{-1} E_{\eta = \eta_{+}} E_{\eta = \eta_{-}}^{-1} Z_{\eta = \eta_{-}} \begin{pmatrix} S^{(-)} \\ D^{(-)} \end{pmatrix}. \quad (69) $$

We need nontrivial $M_{21} \sim O(k^0)$ to get the scale-invariant spectrum since $D^{(+)} \sim M_{21} S^{(-)}$ and $S^{(-)} \sim O(k^{-3/2})$.

The leading term in the small $q$ expansion of the matrix element $M_{21}$ is given by

$$ M_{21} \approx \frac{\sqrt{\eta}}{3kAB|\eta_-|^{l_k}} \left( \tilde{k}(|\eta_-| + q \eta_+) \cos(\tilde{\eta}_n) \right. $$

$$ + \left. (-\tilde{k}^2 \eta_- |\eta_-| + q) \sin(\tilde{\eta}_n) \right). \quad (70) $$

Now we have a nonzero $M_{21}$. To avoid the oscillating factor in (70), we further assume that

$$ \tilde{k} \eta_+ \ll 1. \quad (71) $$

Then,

$$ M_{21} \approx \frac{\sqrt{\eta}}{3kAB|\eta_-|^{l_k}} (|\eta_-| + 2q \eta_+) $$

$$ = \frac{\sqrt{\eta}}{2a_0 b_0 |\eta_-|^{l_k}} (|\eta_-| + 2q \eta_+). \quad (72) $$
We obtain the scale invariant power spectrum
\[ P(k) = k^{3}|D^{(+)}|^2 = k^{3}|M_{21}S^{(-)}|^2 \]
\[ \sim \left( \frac{1}{\ln b} \right)^2 \left( \frac{a(q_+)}{a(q_-)} + 1 \right)^2 \sim \left( \frac{1}{b} \right)^2 . \]  

(73)

Here we have used the explicit form of \( S^{(-)} \). Since the observed CMB scale is about \( 10^{-5} \), this implies that the Hubble scale around which the bounce occurs should be at around \( 10^{-5} \) of the Planckian energy scale. This is quite reasonable. Since \( S^{(-)} \approx k^{-\frac{3}{2}-q} \), \( q \) correction makes the spectrum slightly shifted to red.

Finally, we comment on the requirement coming from [24]. Now
\[ \hat{k}\eta_+ = Ck^{-\alpha} \eta_+ \ll 1. \]  

(74)

Since we are hoping to get the scale-invariant power spectrum in the long wavelength limit, we only need this condition for very small \( k\eta_+ \). The constraint (74) can be easily satisfied by requiring \( 0 < \alpha << 1 \) and \( 0 < C\eta_+ << 1 \). One may think that \( \alpha < 0 \) also does the job, but this will contradict to the fact that during the bounce period the term \( Ck^{-\alpha} \) should dominate over the \( k^2 \) and the pumping term in (74) for the superhorizon modes. It is interesting to note that we will only need mild nonlocal effect (i.e., \( \alpha << 1 \)) instead of the canonical one in [24, 25] to have the scale-invariant spectrum. It deserves more efforts to derive such an effective theory from string/M theory.

In conclusion, we have shown that the nonlocal bounce and the corresponding matching condition is crucial in obtaining the scale-invariant CMB spectrum. For our calculation, we have assumed the equation for the fluctuation to be given by (57), which is based on a modified dispersion relation inspired by noncommutative field theory. It is important to derive the precise form of the nonlocality from a fundamental physical model such as string theory. We leave these issues for future study. We hope our result helps to shed light on the problem of bouncing type cosmologies by identifying the relevant kind of new physics that is needed.

V. CONCLUSIONS AND DISCUSSIONS

In this paper, we have studied some possible effect of physics beyond the general relativity on bouncing cosmologies. Under the constraint of the local causality condition, we have derived the general form of the matching condition that relate the physics before and after the bounce in the long wavelength limit. The possibility of mixing between the decaying mode and the constant mode is clarified. We find that the coefficient of the decaying mode after the bounce can be changed and can receive a correction coming from the constant mode. It can happen even in the context of the general relativity if there exist anisotropic stress during the phase transition.

On the other hand, the constant mode cannot be changed if \( k^2 S^{(-)} \) is sub-leading compared to \( D^{(-)} \), as in the case of the ekpyrotic scenario, for example. This conclusion is unaltered even considering effects coming from physics beyond general relativity as long as the local causality condition is satisfied. This rules out the possibility to obtain a scale-invariant spectrum for the pre-big bang and for the ekpyrotic scenarios, whenever local matching condition is employed. The identification of nonlocal effects as a possible outlet to achieve a scale invariant spectrum is one of the main results of this paper.

We have also studied the effects on the matching condition from a violation of the local causality during the bounce. With a toy model employing noncommutative geometry, we show that it is possible to obtain a scale invariant spectrum if the nonlocal effects enter the evolution equation of the fluctuation as \( k^{-\alpha} \) with \( \alpha \ll 1 \). Moreover, from the CMB constraint, we find that the bounce should occur at a few orders of magnitude below the Planck energy scale in our model. The demonstration that a suitable form of nonlocal effects does lead to a scale invariant spectrum is another main result of this paper.

It is an important task to understand the nature of spacetime in quantum gravity and to derive more precisely the form of the nonlocal effects from string theory or other theory of quantum gravity. It will be very interesting if the nonlocal effects we introduced in our toy model do appear. We will leave this for the future works.

Finally we comment on how nonlocal causality may play a role in the matching condition considered in other works.

1. In [26], it was argued that one can have a scale invariant spectrum by considering a linear matching condition for \( \epsilon_m = -\frac{4}{3}\mathcal{H}k^2\Phi \) and \( \dot{\epsilon}_m \) at the bounce. It is easy to see that in order to arrive at such a conclusion, nonlocal causality of the form we introduced in this paper is needed. This is because, as can be seen from (13) and (15), the definition of the gauge invariant quantity of \( \Phi \) includes the spatial integral of the fluctuations of the metric, and hence even if the matching condition looks local in terms of \( \Phi \) and \( \dot{\Phi} \), it is not the case when the matching condition is expressed in terms of the metric and the scalar fields.

2. In [8], negative surface tension was considered. By imposing a specific matching condition on the fluctuation of the surface tension, it was argued that one can obtain a scale invariant spectrum. Since the initial condition before the bounce is completely determined by the metric and the scalar fields, thus in the framework of our general analysis, the proposed form of fluctuation of the surface tension with the choice of the matching surface should correspond to a violation of the local causality condition in order for a scale invariant spectrum to be possible. It will be interesting to have a concrete framework where the negative tension arises physically and to see how the violation of local causality comes about.

3. In [27] (see also [28]), matching conditions in the five
dimensional context has been studied. In our analysis, we are free to choose the time of the matching \( \eta_- \) so long as \( k/aH(\eta_-) \ll 1 \). The hypersurface for the matching does not need to be the one on which the bouncing phase start to happen. Since any fluctuation in the five dimensional framework can be traced back to fluctuation in the four dimensional effective theory if we go back in time. Thus, unless the time interval during which the four-dimensional framework can be traced back to fluctuation phase start to happen. Since any fluctuation in the five dimensional framework, it would be interesting to see how the condition in the four dimensional framework, and non-four-dimensional picture is not applicable is comparable to the scale \( k^{-1} \), we can choose to apply the matching condition in the four dimensional framework, and non-local causality is required to obtain the scale invariant spectrum. Finally, it would be interesting to see how the mechanism of mixing of perturbation modes presented in the recent work \[29\] interconnect with the argument in

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**APPENDIX A: METRIC AND SCALAR FLUCTUATION ON THE MATCHING SURFACE**

In this appendix, we show that one can exploit the residual gauge symmetry of the synchronous gauge to choose the matching surface such that \( \phi \) is constant over it. In this gauge \( \delta g_{00} = \delta g_{0i} = 0 \) and \( \delta \phi = 0 \) at \( \eta = \eta_- \). We will also compute the metric fluctuation \[27\] around \( \eta = \eta_- \) in this gauge.

We start with the longitudinal gauge in which \( B = E = 0 \). It also follows that \( \phi = \psi = \Phi \) for perfect fluid. Here \( B, E, \phi, \psi \) are parameters defined in \[13\]. In this gauge, the metric perturbation and the scalar field fluctuation are given by

\[
\delta g_{\mu \nu} = a^2 \begin{pmatrix} 2\phi & 0 \\ 0 & 2\delta \phi_{ij} \end{pmatrix} \tag{A1}
\]

and

\[
\delta \phi = \left( \frac{3}{2} a^2 \phi' \right)^{-1} (\phi' + \mathcal{H}\phi). \tag{A2}
\]

Next, we perform the coordinate transformation

\[
\eta^{(l)} \rightarrow \tilde{\eta} = \eta^{(l)} + \xi^{(l)}(\eta^{(l)}, x^{(l)}) \quad \text{and} \quad x^{(l)i} \rightarrow \tilde{x}^i = x^{(l)i} + \delta^{ij} \partial_j \xi^{(l)}(\eta^{(l)}, x^{(l)}) \tag{A3}
\]

to the gauge in which \( \delta \phi = 0 \) and keeping \( \delta g_{0i} = 0 \). Here superscript \( l \) represents the longitudinal gauge. To do this, we need

\[
\xi^{(l)0} = -(\mathcal{H}' - \mathcal{H}^2)^{-1}(\phi' + \mathcal{H}\phi) \tag{A4}
\]

and

\[
\xi^{(l)} = \int d\eta^0. \tag{A5}
\]

To obtain \[A4\], we have used the transformation law of the scalar quantity

\[
\delta \phi \rightarrow \delta \phi + \phi' \xi^0 \tag{A6}
\]

and

\[
\phi'^2 = \frac{2}{3a^2}(\mathcal{H}^2 - \mathcal{H}'). \tag{A7}
\]

Using \[A5\], \[A4\] and \[A6\], the metric fluctuation in this gauge can be computed to give

\[
\delta g_{\mu \nu}^{(\cdot -)}(\tilde{\eta}, k) = 2a^2 \begin{pmatrix} \tilde{\phi} \\ 0 \\ +k_i k_j(S^{(-)} l^2 \int d\eta a^{-2} - D^{(-)} \int d\eta (a^{-2} \int d\eta a^2 + O(k^{n \phi + 2})) \end{pmatrix} (D^{(-)} + O(k^{n \phi + 2})) \delta_{ij} \tag{A8}
\]

Note that \( \tilde{\phi} := (2a^2)^{-1} \delta g_{00}^{(-)} = O(k^{n \phi + 2}) \).

To achieve \( \delta g_{00}^{(-)} = 0 \) to arrive synchronous gauge while keeping \( \delta \varphi = 0 \) on the surface of the matching, we further...
perform the coordinate transformation
\[ \bar{\eta} \rightarrow \eta = \tilde{\eta} + \tilde{\eta}^0(\tilde{\eta}, \tilde{x}) \]
\[ \bar{x}^i \rightarrow x^i = \tilde{x}^i + \delta^{ij} \partial_j \tilde{\xi}(\tilde{\eta}, \tilde{x}) \]
with
\[ \tilde{\xi}^0 = a^{-1} \int_{\eta_-}^{\eta} d\eta \partial \tilde{\phi} \]  
(A10)

\[ \delta g_{\mu\nu}(\eta, k) = 2a^2 \begin{pmatrix} 0 \\ 0 \\ +k_i k_j \left( S^{(-)l^2} \int d\eta a^{-2} - D^{(-)} \int d\eta (a^{-2} \int d\eta a^2 + O(k^{n_s+2})) \right) \end{pmatrix} \]  
(A12)

\[ \delta \varphi(\eta, k) = \varphi' \tilde{\xi}^0 = O(k^{n_s+2}). \]  
(A13)

Because of (A13), \( \tilde{\xi}^0 = 0 \) at \( \tilde{\eta} = \tilde{\eta}_- \) and hence \( \delta \varphi = 0 \) at \( \eta = \tilde{\eta}_- \). In section III, \( \tilde{\eta}_- \) is denoted as \( \eta_- \).

**APPENDIX B: COMMENT ON THE CHOICE OF THE MATCHING SURFACE**

In section III we claim that one can choose any surface of the matching defined by \( \mathcal{O}_-(\eta, x) = \mathcal{O}^0 = \) constant instead of the surface \( \varphi(\eta, x) = \) constant without changing the matching condition at the order of \( k \) we considered. We will show this fact in this appendix. For simplicity, we omit the subscript \(-\) and write \( \mathcal{O} = \mathcal{O} \). First we will show that the leading power of \( k \) dependence of the fluctuation \( \delta \mathcal{O}(\eta, x) \) is \( k^{n_s+2} \) on \( \varphi = \) constant surface. To see this, consider possible scalar quantities contributing to \( \delta \mathcal{O}(\eta, x) \) by taking contraction of spatial indices from the metric. For example, these are
\[ \delta(g_{ij}g^{ij}), \delta g_{ij} k^i k^j, \ldots. \]  
(B1)

Note that we do not consider contributions from scalar field because \( \varphi \) and \( \varphi' \) has no fluctuation up to the order \( k^{n_s} \) in our gauge. In [B1] it is trivial that \( \delta g_{ij} k^i k^j = O(k^{n_s+2}) \). As for \( \delta(g_{ij}g^{ij}) \), noting \( g_{ij} = a^2 \partial_{ij} + 2a^2 \left( D \partial_{ij} + O(k^{n_s+2}) \right) \) from [B2],
\[ g_{ij}g^{ij} = \{ a^2 \left( 1 + 2D \partial_{ij} + O(k^{n_s+2}) \right) \}' \times a^{-2} \left( 1 - 2D \partial_{ij} + O(k^{n_s+2}) \right) \]
\[ = 6H + O(k^{n_s+2}). \]  
(B2)

Thus these terms contribute to \( \delta \mathcal{O}(\eta, x) \) at order \( O(k^{n_s+2}) \). Similarly one can verify this for any other quantity and thus \( \delta \mathcal{O}(\eta, x) \) is suppressed with a factor of \( k^2 \) compared to \( \Phi \). This is similar to the suppression of the entropy perturbation.

And
\[ \tilde{\xi} = \int_{\eta_-}^{\eta} d\eta \left( a^{-1} \int_{\eta_-}^{\eta} d\eta \partial \tilde{\phi} \right). \]  
(A11)

Using [B5], it is easy to obtain the metric and the scalar field fluctuation in this gauge,

Next, we change the surface of the matching to one defined by \( \mathcal{O}(\eta, x) = \) constant and show that it only has an effect on the higher order terms in the matching condition. To change the surface of the matching, we perform the coordinate transformation
\[ \eta \rightarrow \tilde{\eta} = \eta + \tilde{\xi}^0(\eta, x) \]  
(B3)

and consider the matching condition on \( \tilde{\eta} = \tilde{\eta}_- \) surface. Demanding \( \delta \mathcal{O} \rightarrow \delta \tilde{\mathcal{O}} = \delta \mathcal{O} + O^0(\eta_-, x) = 0 \), we get
\[ \tilde{\xi}^0(\eta_-, x) = \frac{\delta \mathcal{O}}{\delta \mathcal{O}} \bigg|_{\eta_-=\eta_-} = O(k^{n_s+2}) \]  
(B4)

since \( \delta \mathcal{O} = O(k^{n_s+2}) \). To satisfy the synchronous gauge condition \( \delta g_{00} = \delta g_{i0} = 0 \), one needs
\[ \dot{\phi} = \phi - \left( a'/a \right) \xi^0 - (\xi^0)' = 0 \]  
(B5)

and
\[ \tilde{B} = B + \xi^0 - \xi' = 0. \]  
(B6)

\( \phi \) and \( B \) are zero as is read off from [B7] and \( \xi \) is the parameter for the diffeomorphism in the spatial direction
\[ x^i \rightarrow x^i + \delta^i \partial_j \xi(\eta, x). \]  
(B7)

[B5] and [B6] are solved by
\[ \xi^0 = Ca^{-1} \]  
(B8)

and
\[ \xi = C \int d\eta a^{-1}, \]  
(B9)

where \( C = -a \left( \frac{\delta \mathcal{O}}{\delta \mathcal{O}} \right) \bigg|_{\eta_-=\eta_-} = O(k^{n_s+2}) \) due to [B4]. Substituting these parameters in [B9], one obtains
\[ \delta g_{ij} \rightarrow \delta \tilde{g}_{ij} = \delta g_{ij} + 2a^2 \left( C \frac{a'}{a^2} \delta_{ij} + \int d\eta a^{-1} C_{ij} \right), \]  
(B10)
where $\delta g_{ij}$ in the RHS is given by [27]. Thus a different choice of the matching surface only affects the higher order terms in the fluctuation of the metric. It is also easy to verify that $\delta \varphi = O(k^{n+2})$ and $\delta \varphi' = O(k^{n+2})$ on the $\eta = \eta_-$ surface. Therefore the fluctuations on the $\mathcal{O} =$ constant surface take the same form as [27] and [28] up to the order we consider and hence the same matching condition is resulted.

APPENDIX C: EFFECTS OF ENTROPY PERTURBATION

In this appendix we comment on the entropy perturbation and its effect in modifying the matching conditions. Entropy perturbation is the perturbation seen by a local observer and can affect the local matching condition. Entropy perturbation is defined by (see e.g., [21])

$$S = a \mathcal{H} \left( \frac{\delta p}{p'} - \frac{\delta \rho}{\rho'} \right)$$  \hspace{1cm} (C1)

where $\rho$ and $p$ are energy density and pressure respectively. If there is no entropy perturbation, (C1) indicates that the fluctuation of pressure and the energy density can be reinterpreted as a local time delay. Thus for the purely adiabatic perturbation, every portion of the space experiences the same history.

For a single scalar field model as considered in this paper, entropy perturbation is known to be $k^2$ order higher than $\Phi$ [30]. This is the reason why the entropy fluctuation can be ignored in our analysis.

For multi-fields case, one can decompose the fields into adiabatic field and entropy fields [21]. In the Minkowski space regime, entropy perturbations are $k^1$ order higher than $\Phi$. If the bounce occurs in a time period much shorter than the wavelength as in the ekpyrotic scenario with $p \ll 1$, one only has modification at the order of $k^{n+1}$ in the matching condition [17]. There is no modification in the lowest order and thus one cannot obtain a scale invariant mode from $D^{(+)}$. On the other hand, if there is enough time comparable to $k^{-1}$ prior to the bounce, one can change the order of $D^{(-)}$. Indeed, changing the parameter $p$ in the scalar potential will change the power of $k$ of $D^{(-)}$. There have been attempts to get scale invariant spectrum by having sufficient duration in the contracting phase [31, 32]. It was argued in [11, 31] that a scenario with $p = 2/3$ is able to provide a scale invariant spectrum. Unfortunately the solution turned out not to be a stable attractor point [22].

APPENDIX D: EFFECTS OF ANISOTROPIC STRESS

To see the effect of anisotropic stress, we assume the validity of the general relativity during the bounce and consider the metric fluctuation of the form (in the synchronous gauge)

$$\delta g_{\mu\nu} = 2a^2 \begin{pmatrix} 0 & 0 \\ 0 & D\delta i + F_{ij} \end{pmatrix}$$  \hspace{1cm} (D1)

with the initial condition given by [27] at $\eta = \eta_-$. The Einstein tensor for $i \neq j$ up to the order $k^{n+}$ becomes

$$G_{ij} = a^{-2} \left( k_i k_j D(x) + F''_{ij} + 2\mathcal{H} F'_{ij} \right).$$  \hspace{1cm} (D2)

Note the indices are not raised in RHS. Einstein equation $G_{ij} = 3l^2 T_{ij}$ can be solved by

$$F_{ij} = A_{ij} l^2 \int d\eta a^{-2} - k_i k_j D \int d\eta \left( a^{-2} \int d\eta a^2 \right) + B_{ij}(\eta)$$  \hspace{1cm} (D3)

with constants $A_{ij}$ and

$$B_{ij}(\eta) = 3l^2 \int d\eta \left( a^{-2} \int d\eta a^4 T^i_j \right).$$  \hspace{1cm} (D4)

Initial condition (27) can be satisfied by setting $A_{ij} = k_i k_j S^{(-)}$. Since $T^i_j \neq 0$ only in the region $\eta_- < \eta < \eta_+$, after $\eta = \eta_+$ becomes

$$F_{ij} = \left( k_i k_j S^{(-)} + 3 \int_{\eta_-}^{\eta_+} d\eta a^4 T^i_j \right) l^2 \int d\eta a^{-2}$$

$$- k_i k_j D \int d\eta \left( a^{-2} \int d\eta a^2 \right).$$  \hspace{1cm} (D5)

This results in a jump in $S$:

$$k_i k_j S^{(+)} = k_i k_j S^{(-)} + 3 \int_{\eta_-}^{\eta_+} d\eta a^4 T^i_j.$$

Since the background stress tensor is diagonal, the off-diagonal components of $T^i_j$ come from the fluctuation and should be linear in $S^{(-)}$ or $D^{(-)}$. Thus the contribution to $S^{(+)}$ is also linear in $S^{(-)}$ or in $D^{(-)}$ as in (17) but the details of the mixing depends on the physics during the bounce.

In conclusion, after taking into account the anisotropic stress during the bounce, the matching condition remains the most general one in the framework of general relativity. Physics beyond general relativity can only affect $C_S$ and $C_D$ quantitatively, and there is no contribution from $S^{(-)}$ to $D^{(+)}$ if local causality is assumed.


[33] In general relativity, junction conditions are used to match solutions to the equation of motion across a surface of discontinuity, for example, one caused by a sharp localization of matters. Some well-known junction conditions are the Lichnerowicz junction condition, the Israel junction condition and the O’Brien-Synge junction condition. However, in studies such as the bounce cosmologies where general relativity may break down around the bounce, junction condition may prove limited and generalization such as the matching condition are needed.