DOUBLY SPECIAL RELATIVITY: A NEW RELATIVITY OR NOT?

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Abstract

Double Special Relativity theories are the relativistic theories in which the transformations between inertial observers are characterized by two observer-independent scales of the light speed and the Planck length. We study two main examples of these theories and want to show that these theories are not the new theories of relativity, but only are re-descriptions of Einstein’s special relativity in the non-conventional coordinates.

1 Introduction

It seems that the Planck length $l_p$ has a crucial role in the Quantum Gravity. In some scenarios of quantum gravity like loop quantum gravity the Planck length or Planck scales act as a threshold for quantum effects in the spacetime, beyond which the usual description of spacetime breaks down. Thus, it seems that the value of $l_p$ must have the same value in all inertial frames and this is in conflict with Einstein’s Special Relativity[1, 5]. Doubly Special Relativity (DSR) has proposed for solving this puzzle [2, 4, 5].

The Magueijo-Smolin (Ms) [2, 3] and Amelino-Camelia [4, 5] DSRs are two main examples of these theories that take much attractions recently. Here, we want to investigate these theories further.
Let’s now explain briefly the Magueijo - Smolin (Ms) transformations: Magueijo and Smolin have looked for a non-linear representation of the Lorentz group that remains the Planck length as an invariant. If we denote the ordinary Lorentz boosts by

$$L_{ab} = p_a \frac{\partial}{\partial p^b} - p_b \frac{\partial}{\partial p^a},$$  \hspace{1cm} (1)$$

then this representation can be obtained by using the modified generators of boosts as

$$K^i \equiv L^i_0 + l_p p^i D.$$  \hspace{1cm} (2)$$

Here, $D$ is the dilatation generator

$$D = p_a \frac{\partial}{\partial p_a}.$$$$

But, the rotation generators will be the unmodified $J^i \equiv \epsilon^{ijk} L_{jk}$. Also, the Lorentz Algebra remains intact:

$$[J^i, K^j] = \epsilon^{ijk} K_k, \quad [K^i, K^j] = \epsilon^{ijk} J_k, \quad [J^i, J^j] = \epsilon^{ijk} J_k.$$$$

By acting the modified generators of boosts on the momentum space, we obtain the Magueijo - Smolin (MS) transformations between two inertial systems which are in relative motion with constant speed along the common $x$-axis as:

$$p'_0 = \frac{\gamma (p_0 - \frac{v}{c} p_x)}{1 + l_p (\gamma - 1) p_0 - l_p \gamma \frac{v}{c} p_x},$$  \hspace{1cm} (3)$$

$$p'_x = \frac{\gamma (p_x - \frac{v}{c} p_0)}{1 + l_p (\gamma - 1) p_0 - l_p \gamma \frac{v}{c} p_x},$$  \hspace{1cm} (4)$$

$$p'_y = \frac{p_y}{1 + l_p (\gamma - 1) p_0 - l_p \gamma \frac{v}{c} p_x},$$  \hspace{1cm} (5)$$

$$p'_z = \frac{p_z}{1 + l_p (\gamma - 1) p_0 - l_p \gamma \frac{v}{c} p_x},$$  \hspace{1cm} (6)$$

These transformations have many new features [2, 3]. For example, they do not preserve the usual quadratic invariant on momentum space. But, there is a modified invariant:

$$\|p\|^2 = \frac{\eta_{ab} p_a p_b}{(1 - l_p p_0)^2}.$$  \hspace{1cm} (7)$$

Also, these transformations remain invariant the light speed "$c$" and the Planck length "$l_p$" as desired. This property can be seen from MS transformations and equation (7).

But, looking closer at these transformations, we can see that changing 4-momentum $p_\mu$ to

$$\pi_\mu = \frac{p_\mu}{1 - l_p p_0}$$

2 Magueijo-Smolin DSR
Figure 1: Magueijo - Smolin (MS) spacetime diagram in terms of \( \pi \) coordinates. MS coordinates \( p_0 \) and \( p_1 \) is drawn as horizontal and radial lines.

then the MS transformations become

\[
\begin{align*}
\pi'_0 &= \gamma (\pi_0 - \frac{\nu}{c} \pi_x) \\
\pi'_x &= \gamma (\pi_x - \frac{\nu}{c} \pi_0) \\
\pi'_y &= \pi_y \\
\pi'_z &= \pi_z
\end{align*}
\]

These are the same ordinary Lorentz transformations for momentum space. Therefore, the MS transformations are probably only re-description of the usual Lorentz transformations in the non-conventional coordinates [6, 7]. This fact can also be seen from the MS momentum space diagram. In this figure the MS momentum \( p_0 \) and \( p_1 \) are drawn as horizontal and radial lines.

3 Amelino-Camelia DSR

The other main example of DRS theories is the Amelino-Camelia DSR. Amelino-Camelia was the first physicist that wanted to solve the mentioned puzzle: How
could $l_p$ play a role in the structure of spacetime without violating Special Relativity. He modified the basic postulates of the Einstein’s Relativity as

1. The laws of physics involve a fundamental velocity scale $”c”$ and a fundamental length $”l_p”$.
2. Each inertial observer can establish the value of $l_p$ (same value for all inertial observer) by determining the dispersion relation for photons, which takes the form $E^2 - c^2 p^2 + f(E, p; l_p) = 0$, where $f$ is the same for all inertial observers and in particular all inertial observers agree on the leading $l_p$ dependence of $f$: $f(E, p; l_p) \simeq l_p cp^2 E$.

If we find that the dispersion relation takes the form of

$$2E^2 [\cosh \left( \frac{E}{E_p} \right) - \cosh \left( \frac{m}{E_p} \right)] = \vec{p}^2 e^{E/E_p}$$

by some reasoning or by the experimental analysis.

The next step will be finding the deformed boost transformations which leave the above dispersion relation as an invariant.

In ordinary special relativity the boosts can be described by

$$B_a = i p_a \frac{\partial}{\partial E} + i E \frac{\partial}{\partial p_a}$$

By assuming that the modified generators also obey the same Lorentz algebra and they preserve the dispersion relation (10) as an invariant, he finds the following modified generators of boosts

$$B_a = i p_a \frac{\partial}{\partial E} + i \left( \frac{1}{2E_p} \vec{p}^2 + E_p \frac{1 - e^{-2E/E_p}}{2} \right) \frac{\partial}{\partial p_a} - i p_a E_p \left( p_b \frac{\partial}{\partial p_b} \right)$$

(11)

Please, note that the rotation generators remain intact. One can easily obtain the finite boost transformations that relate the observations of two observers by integrating the familiar differential equations

$$\frac{dE}{d\xi} = i[B_a, E], \quad \frac{dp_a}{d\xi} = i[B_a, p_b].$$

But, here as in the Magueijo-Smolin transformations case we can see that by defining the new variables $\epsilon$ an $\pi$ through the relations

$$\frac{\epsilon}{\mu} = \frac{E/E_p - \cosh \left( \frac{m}{E_p} \right)}{\sinh \left( \frac{m}{E_p} \right)}, \quad \frac{\pi}{\mu} = \frac{p}{E_p} \frac{e^{E/E_p}}{\sinh \left( \frac{m}{E_p} \right)}$$

(12)

all relations come back to the ordinary special relativity forms. For example, the modified boost will take the form of usual Lorentz boost:

$$B_a = i \pi_a \frac{\partial}{\partial \epsilon} + i \epsilon \frac{\partial}{\partial \pi_a}.$$  

Note that the transformations (12) are non-singular and we can define the variables $\epsilon$ and $\pi$.

So, it seems that the Amelino-Camelia DSR like the MS ones is only a re-description of the special relativity in the non-conventional coordinates.
4 Conclusion

From the above discussion it seems that Doubly Special theories are only re-descriptions of the special relativity in the non-Cartesian and non-conventional coordinates.

References


