Equivalence of two mathematical forms for the bound angular momentum of the electromagnetic field

A. M. STEWART*

Department of Theoretical Physics,
Research School of Physical Sciences and Engineering,
The Australian National University,
Canberra, ACT 0200, Australia.
*E-mail: andrew.stewart@anu.edu.au

Abstract:
It is shown that the mathematical form, obtained in a recent paper, for the angular momentum of the electromagnetic field in the vicinity of electric charge is equivalent to another form obtained previously.

The angular momentum \( J(t) \) of the classical electromagnetic field in terms of the electric field \( \mathbf{E}(x,t) \) and magnetic field \( \mathbf{B}(x,t) \) [1]

\[
J(t) = \frac{1}{4\pi c} \int d^3 x \mathbf{x} \times [\mathbf{E}(x,t) \times \mathbf{B}(x,t)]
\]  

(Gaussian units, bold font denotes a three-vector) has been treated in a paper published recently [2] by means of the Helmholtz theorem on the decomposition of vector fields. It was shown there that the angular momentum associated with a volume of space far from the influence of electric charge may be expressed as a sum of three terms: a) a volume integral of angular momentum of spin-like character, b) a volume integral of angular momentum of orbital-like character and c) a surface integral. It was found to be essential [3, 4] to take account of the surface integral in order to understand the apparent paradox, from (1), that a plane wave appears to carry no angular momentum in the direction of propagation while experiment [5] shows that it does.

In [2] an expression for the "bound" angular momentum \( J_b \) associated with the presence of electric charge density \( \rho(x,t) \) was also obtained. It is given by

\[
J_b = \frac{1}{c} \int d^3 x \rho(x,t) \int \frac{d^3 y}{4\pi} \mathbf{y} \times [\nabla_y \times \frac{\mathbf{B}(y,t)}{|\mathbf{x} - \mathbf{y}|}]
\]  

(2)
where $\nabla_x$ is the gradient operator with respect to $x$.

We note that if the linear coordinate vector $y$ is replaced by $x$ in the integral (2) above, as shown immediately below,

$$J_b = \frac{1}{c} \int d^3 x \, \rho(x,t) \left[ \frac{d^3 y}{4\pi} \frac{x \times \nabla_x}{|x - y|} \right]$$

then this integral becomes the algebraically simpler expression

$$J_b = \frac{1}{c} \int d^3 x \, \rho(x,t) x \times A_t(x,t)$$

(4)

where the transverse component of the vector potential $A_t$ is given by [6, 7]

$$A_t(x,t) = \nabla_x \times \int d^3 y \, \frac{B(y,t)}{4\pi |x - y|}$$

(5)

Another expression for the bound angular momentum

$$J_b = \frac{q}{c} x \times A_t(x,t)$$

(6)

was obtained by Cohen-Tannoudji et al. [8] p 46 equ. (7) using a different method. They expressed their result in terms of a discrete charge distribution rather than the continuous charge distribution $\rho(x)$. Equation (4) is the continuum generalisation of equation (6).

These authors also used the abstract quantity $A_t$ for the transverse vector potential. They did not provide the explicit expression for the transverse vector potential given by (5).

Equation (2) will be mathematically equivalent to equations (3) and (4) only if the following vector integral $I$ vanishes

$$I = \int d^3 y (x - y) \times [\nabla_x \times \frac{B(y,t)}{|x - y|}]$$

(7)

In the remainder of this letter we show that the vector integral $I$ does indeed vanish and so (2) and (4) are equivalent forms of expressing the bound angular momentum of the electromagnetic field.

By using the vector identity

$$\nabla_x \times \frac{B(y,t)}{|x - y|} = \nabla_x \frac{1}{|x - y|} \times B(y,t)$$

(8)

we can express the integrand of (7) as a triple vector product.
\[ I = \int d^3y (x - y) \times [\nabla_x \left( \frac{1}{|x - y|} \right) \times B(y, t)] \]  

(9)

which may be expanded to give two terms \( I = I_1 + I_2 \) where

\[ I_1 = -\int d^3y B(y, t)[(x - y) \cdot \nabla_x \left( \frac{1}{|x - y|} \right)] \]  

(10)

By multiplying out the scalar product, this comes to

\[ I_1 = \int d^3y \frac{B(y, t)}{|x - y|} \]  

(11)

The second term of the expansion of the integral (9) is

\[ I_2 = \int d^3y \nabla_x \left[ \frac{1}{|x - y|} B(y, t) \cdot (x - y) \right] \]  

(12)

To simplify \( I_2 \) we introduce the (separable) tensor \( \mathbf{T} = B(y, t)(x - y) \), written in dyadic form,

\[ \mathbf{T} = \sum_{i,j=1}^{3} \hat{e}_j B_j (x^i - y^i) \hat{e}_i \]  

(13)

where \( \hat{e}_i \) is the unit vector in the \( i \) direction. Consider the divergence of \( \mathbf{T} \) with respect to \( y \)

\[ \nabla_y . \mathbf{T} = \sum_{k,i,j=1}^{3} \hat{e}_k \cdot \hat{e}_j \frac{\partial}{\partial y^k} [B_j (x^i - y^i)] \hat{e}_i \]  

(14)

Taking the derivative explicitly, and remembering that \( \nabla . B = 0 \), we get \( \nabla_y . \mathbf{T} = -B(y, t) \).

Next, we apply the identity

\[ \nabla_y \left( \frac{\mathbf{T}}{|x - y|} \right) = \nabla_y \left( \frac{1}{|x - y|} \right) . \mathbf{T} + \frac{1}{|x - y|} \nabla_y \cdot \mathbf{T} \]  

(15)

to (12) and, noting that the volume integral of a divergence gives a surface integral that vanishes at infinity, we get

\[ I_2 = -\int d^3y \frac{B(y, t)}{|x - y|} \]  

(16)
Equation (16) cancels (11) so it is proved that the integral \( \mathbf{l} \) vanishes and the two mathematical forms (2) and (4) for the "bound" angular momentum are equivalent.

The bound linear momentum comes to

\[
P_b = \frac{1}{c} \int d^3x \rho(x,t) A_i(x,t)
\]

This equation was printed incorrectly in equation (34) of [2]. Also, the second part of equation (4) of [2] should read

\[
F(x,t) = -\frac{\partial}{\partial t} \int d^3y \frac{B(y,t)}{4\pi c |x-y|}
\]

and the equation in the second line of text above equation (6) of that paper should read \( \nabla \cdot \mathbf{B} = 0 \).

References