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ONE-PARTICLE SINGULARITIES OF GREEN'S FUNCTIONS
IN QUANTUM FIELD THEORY. II

W. Zimmermann

Institut für theor. Physik
der Universität, Hamburg

and

CERN, Geneva

ABSTRACT

The time ordered Green's functions are expanded with respect to the singularities arising from intermediate one-particle states.

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Introduction.

In a preceding paper ¹⁾ the one-particle singularities of Green's functions in quantum field theory have been investigated. It has been shown

- (i) that the Fourier transform $\tau(p_1, \dots, p_n)$ of the τ -function $\langle TA(x_1) \dots A(x_n) \rangle_0$ has a pole along
- $$q^2 = -m^2$$

for any $q = p_{\alpha_1} + \dots + p_{\alpha_r}$ if m is a discrete mass of the system;

- (ii) that the residuum of this pole is a product of two τ -functions.

These results will be applied here in order to expand the τ -functions with respect to the singularities

$$(q_1^2 + m^2 - i\xi)^{-1} \dots (q_l^2 + m^2 - i\xi)^{-1} \quad \xi \rightarrow +0$$

where q_i denote arbitrary partial sums of the momenta $p_1, \dots, p_n$. The residua of these singularities will turn out to be either zero or products of $r+1$ factors. So the expansion expresses the complete structure of the one-particle singularities of an arbitrary τ -function.

In section 1 the general assumptions and results of I are summarized so far as needed in this paper. For a model of one scalar field and one discrete mass the expansion of $\tau(p_1, \dots, p_n)$ with respect to the one-particle singularities is derived in section 2. Examples for $n=3, 4$ and 5 are given in section 3. In section 4 is demonstrated how to generalize to the case of several masses, including bound states: the τ -functions corresponding to various two-particle scattering processes are considered for the model of a neutral and a charged scalar field having one stable bound state of charge two.

We finally mention two possible applications to the S-matrix theory not being considered in this paper. The first application refers to the physical domain of the S-matrix elements. The one-particle structure of $\tau(p_1, \dots, p_n)$ leads to certain relations among S-matrix elements first found by M. Fierz ²⁾ as a characteristic property of causal S-matrices. For details we refer to ²⁾ and to the investigations by V. Glaser ³⁾.

The second application concerns the unphysical domain of S-matrix elements. In the two-particle scattering amplitude, for instance, no one-particle singularity occurs so long as the momenta are real four vectors (physical domain). By analytical continuation into the unphysical domain, however, one-particle poles should be reached. A report on this programme has been given by Chew and Low⁴⁾. Further references are contained in that paper.

1. General Results of Part I.

In this section we repeat some of the definitions and results given in I so far as necessary for the investigations that follow.

We consider the model of one scalar field $A(x)$ with one discrete mass m . The field operator is assumed to be invariant, causal, complete, etc. Asymptotic field operators $B_{\text{in}}^{\text{out}}(x; \xi_1 \dots \xi_{r-1})$ are defined by

$$B_{\text{in}}^{\text{out}}(x; \xi_1 \dots \xi_{r-1}) = B(x; \xi_1 \dots \xi_{r-1}) + \int dx' \Delta_{\text{Ret}}^{\text{Adv}}(m, x-x'); (x'; \xi_1 \dots \xi_{r-1}) \quad (1)$$

with

$$B(x; \xi_1 \dots \xi_{r-1}) = TA(x_1) \dots A(x_r) \quad r \geq 1$$

$$j(x; \xi_1 \dots \xi_{r-1}) = (\square_x - m^2) B(x; \xi_1 \dots \xi_{r-1})$$

$$x = \frac{1}{r} (x_1 + \dots + x_r)$$

$$\xi_i = x_i - x_{i+1}; \quad i = 1, \dots, r-1$$

For the Fourier transform of $B_{\text{in}}^{\text{out}}$ we introduce the notation

$$\delta(q^2 + m^2) \tilde{B}_{\text{in}}^{\text{out}}(q; \xi_1 \dots \xi_{r-1}) = \frac{1}{(2\pi)^{5/2}} \int dx e^{-iqx} B_{\text{in}}^{\text{out}}(x; \xi_1 \dots \xi_{r-1}) .$$

Investigating the one-particle singularities of the τ -function ⁵⁾

$$\delta(p_1 + \dots + p_n) \tau_n(p_1, \dots, p_n) = \frac{1}{(2\pi)^{5n/2}} \int dx_1 \dots \int dx_n e^{-i(p_1 x_1 + \dots + p_n x_n)} \langle TA(x_1) \dots A(x_n) \rangle_0 \quad (2)$$

we obtained in I the following results :

(i) for $q = p_1 + \dots + p_r$ the relation

$$\tau(p_1, \dots, p_n) = \lim_{\xi \rightarrow +0} \frac{\{(q^2 + m^2) \tau(p_1, \dots, p_n)\}}{q^2 + m^2 - i\xi} \quad (3)$$

holds. Due to the (assumed) existence of the asymptotic operators ⁽¹⁾ the residuum $(q^2 + m^2) \tau(p_1, \dots, p_n)$ is finite ⁶⁾ for $q^2 = -m^2$ as follows from [see I, Eq.(33)]:

$$\begin{aligned} \int dx e^{-iqx} K_x \langle TA(x_1) \dots A(x_n) \rangle_0 \Big|_{q^2 = -m^2} = \\ = i(2\pi)^{3/2} \left\{ \langle \tilde{B}_{out}(q; \xi_1 \dots \xi_{r-1}) TA(x_{r+1}) \dots A(x_n) \rangle_0 + \right. \\ \left. + \langle TA(x_{r+1}) \dots A(x_n) \tilde{B}_{in}(q; \xi_1 \dots \xi_{r-1}) \rangle_0 \right\} \end{aligned} \quad (4)$$

(ii) in case of $2 \leq r \leq n-2$ this leads to the product relation ^{7),8)} [see I, Eq(35)]:

$$\begin{aligned} (q^2 + m^2) \tau_n(p_1, \dots, p_n) \Big|_{q^2 = -m^2} = \\ = 2\pi i \left\{ (q^2 + m^2) \tau_{r+1}(p_1, \dots, p_r, -q) \right\} \left\{ (q^2 + m^2) \tau_{n-r+1}(p_{r+1}, \dots, p_n, q) \right\} \end{aligned} \quad (5)$$

In general one has for any partial sum

$$q = p_{\alpha_1} + \dots + p_{\alpha_r}$$

of the basic momenta $p_1, \dots, p_n$ the two relations

$$\tau_n(p_1, \dots, p_n) = \lim_{\xi \rightarrow +0} \frac{\{(q^2 + m^2) \tau_n(p_1, \dots, p_n)\}}{q^2 + m^2 - i\xi} \quad (6)$$

$$\begin{aligned} (q^2 + m^2) \tau_n(p_1, \dots, p_n) = 2\pi i \left\{ (q^2 + m^2) \tau_{r+1}(p_{\alpha_1} \dots p_{\alpha_r}, -q) \right\} \\ \left\{ (q^2 + m^2) \tau_{n-r+1}(p_{\alpha_{r+1}} \dots p_{\alpha_n}, q) \right\} \end{aligned} \quad (7)$$

if $2 \leq r \leq n-2$

2. Expansion of τ -Functions with respect to One-Particle Singularities.

Relation (7) indicates a certain product structure of τ_n which we now study in more detail. We first remove from τ_n all the singularities $\delta(p_{\alpha_1} + \dots + p_{\alpha_r})$ and $(p_i^2 + m^2 - i\varepsilon)^{-1}$ by forming ⁹⁾

$$\eta'_n(p_1, \dots, p_n) = (p_1^2 + m^2) \dots (p_n^2 + m^2) \tau_n(p_1, \dots, p_n) \quad (8)$$

$$p_1 + \dots + p_n = 0$$

where η_n is defined ¹⁰⁾ recursively by Eq. (I.14). As is well known, these functions η'_n give the S-matrix elements when all p_i are taken on the energy shell.

In terms of η'_n the two basic relations (6) and (7) are

$$\eta'_n(p_1, \dots, p_n) = \lim_{\varepsilon \rightarrow +0} \frac{\{(q^2 + m^2) \tau(p_1, \dots, p_n)\}}{q^2 + m^2 - i\varepsilon} \quad (9)$$

$$(q^2 + m^2) \eta'_n(p_1, \dots, p_n) = 2\pi i \eta'_{r+1}(p_{\alpha_1} \dots p_{\alpha_r}, -q) \eta'_{n-r+1}(p_{\alpha_{r+1}} \dots p_{\alpha_n}, q) \quad (10)$$

for

$$q = p_{\alpha_1} + \dots + p_{\alpha_r}, \quad 2 \leq r \leq n-2$$

if $q^2 = -m^2$.

All possible momentum squares q^2 are obtained by forming

$$q_{\alpha_1 \dots \alpha_r}^2 = \left(\sum_{\nu=1}^r p_{\alpha_\nu} \right)^2, \quad 1 \leq \alpha_i \leq n-1, \quad 2 \leq r \leq n-2$$

for any partial sum $q_{\alpha_1 \dots \alpha_r}$ of the $n-1$ independent four vectors $p_1, \dots, p_{n-1}$. Accordingly we write η'_n as a function ¹¹⁾ of $p_1^2, \dots, p_n^2$ and all $q_{\alpha_1 \dots \alpha_r}^2$

$$\eta'_n = \eta'_n(p_1^2, \dots, p_n^2, q_{12}^2, \dots, q_{ij}^2, \dots, q_{ijk}^2, \dots).$$

We consider η'_n as defined only for those values of p_i^2 , $q_{(\alpha)}^2$ which can be obtained by inserting real four vectors for p_i . Then only $\binom{n}{2}$ of the coordinates p_i^2 , q^2 are independent.

From the relations (9), (10) we know that η'_n can have poles when one of the variables $q_{(\alpha)}^2$ is on the mass shell $q_{(\alpha)}^2 = -m^2$. η'_n will further have multiple poles of the form

$$(q_{(\beta_1)}^2 + m^2 - i\xi)^{-1} \dots (q_{(\beta_\tau)}^2 + m^2 - i\xi)^{-1}, \quad \xi \rightarrow +0$$

when the squares of the τ -momenta

$$q_{(\beta_1)} = q_{\beta_{11} \beta_{12}} \dots = p_{\beta_{11}} + p_{\beta_{12}} + \dots$$

$$q_{(\beta_2)} = q_{\beta_{21} \beta_{22}} \dots = p_{\beta_{21}} + p_{\beta_{22}} + \dots$$

.....

$$q_{(\beta_\tau)} = q_{\beta_{\tau 1} \beta_{\tau 2}} \dots = p_{\beta_{\tau 1}} + p_{\beta_{\tau 2}} + \dots$$

are on the mass shell. One would expect that η'_n can be expanded with respect to these singularities

$$\eta'_n = \lambda_n + \sum_{\tau} \sum_{(\beta_1) \dots (\beta_\tau)} \frac{\lambda_{n;(\beta_1) \dots (\beta_\tau)}}{(q_{(\beta_1)}^2 + m^2 - i\xi) \dots (q_{(\beta_\tau)}^2 + m^2 - i\xi)}, \quad \xi \rightarrow +0 \quad (11)$$

where

- (a) the coefficients $\lambda_{n;(\beta_1) \dots (\beta_\tau)}$ do not depend on the variables $q_{(\beta_1)}^2, \dots, q_{(\beta_\tau)}^2$ appearing in the denominator,
- (b) λ_n as well as $\lambda_{n;(\beta_1) \dots (\beta_\tau)}$ are regular when one or more of the variables are on the mass shell.

We now want to prove the possibility of the expansion (11) and to determine the coefficients. That will be done in two steps. We first define functions λ_n and study their properties. We then express η'_n in terms of the λ -functions, which automatically leads to the desired expansion.

We start defining λ_n by

$$\lambda_n = \eta'_n + \sum_{\tau} (-1)^{\tau} \sum_{(\beta_1) \dots (\beta_{\tau})} \frac{\lambda_{n; (\beta_1) \dots (\beta_{\tau})}}{(q_{(\beta_1)}^2 + m^2 - i\varepsilon) \dots (q_{(\beta_{\tau})}^2 + m^2 - i\varepsilon)}, \quad \varepsilon \rightarrow +0 \quad (12)$$

Here $\eta'_{n; (\alpha_1) \dots (\alpha_6)}$ denotes

$$\eta'_{n; (\alpha_1) \dots (\alpha_6)} = (q_{(\alpha_1)}^2 + m^2) \dots (q_{(\alpha_6)}^2 + m^2) \eta / q_{(\alpha_1)}^2 = \dots = q_{(\alpha_6)}^2 = -m^2 \quad (13)$$

for

$$q_{(\alpha_1)}^2 = \dots = q_{(\alpha_6)}^2 = -m^2.$$

$\eta'_{n; (\alpha_1) \dots (\alpha_6)}$ is a function of the p_i^2 and of the $q_{(\alpha)}^2$, except for $q_{(\alpha_1)}^2, \dots, q_{(\alpha_6)}^2$. Sometimes it is more convenient to write $\eta'_{n; (\alpha_1) \dots (\alpha_6)}$ as a function

$$\eta'_{n; (\alpha_1) \dots (\alpha_6)} = \eta'_{n; (\alpha_1) \dots (\alpha_6)}(p_1, \dots, p_n)$$

of the n four-vectors p_i restricted by

$$p_1 + \dots + p_n = 0$$

$$(p_{\alpha_{i1}} + p_{\alpha_{i2}} + \dots)^2 = -m^2, \quad i = 1, \dots, 6$$

It is easy to see that λ_n is regular when one or several of the variables are on the mass shell. We consider for instance the dependence on $q_{(\alpha)}^2$ and show that λ_n has no singularities $p \frac{1}{q_{(\alpha)}^2 + m^2}$ of $\delta(q_{(\alpha)}^2 + m^2)$. We have¹²⁾

$$\begin{aligned} \lambda_n &= \eta'_n + \sum_{\tau} (-1)^{\tau} \sum_{(\beta_1) \dots (\beta_{\tau})} \frac{\eta'_{n; (\beta_1) \dots (\beta_{\tau})}}{(q_{(\beta_1)}^2 + m^2 - i\varepsilon) \dots (q_{(\beta_{\tau})}^2 + m^2 - i\varepsilon)} \\ &= \eta'_n - \frac{\eta'_{n; (\alpha)}}{q_{(\alpha)}^2 + m^2 - i\varepsilon} + \sum_{\tau} (-1)^{\tau} \sum_{(\beta_1) \dots (\beta_{\tau})} \frac{\eta'_{n; (\beta_1) \dots (\beta_{\tau})}}{(q_{(\beta_1)}^2 + m^2 - i\varepsilon) \dots (q_{(\beta_{\tau})}^2 + m^2 - i\varepsilon)} \\ &\quad - \sum_{\tau} (-1)^{\tau} \sum_{(\beta_1) \dots (\beta_{\tau})} \frac{\eta'_{n; (\alpha) (\beta_1) \dots (\beta_{\tau})}}{(q_{(\alpha)}^2 + m^2 - i\varepsilon) (q_{(\beta_1)}^2 + m^2 - i\varepsilon) \dots (q_{(\beta_{\tau})}^2 + m^2 - i\varepsilon)} \end{aligned}$$

where the sum $\sum'$ is taken over all $(\beta_1), \dots, (\beta_\tau)$ which are different from (α) . Using relation (9), we obtain

$$\lambda_n = \frac{(q(\alpha)^2 + m^2) \eta'_n - \eta'_{n;\alpha}}{q(\alpha)^2 + m^2 - i\varepsilon} + \sum_{\tau} (-1)^{\tau} \sum_{(\beta_1) \dots (\beta_\tau)} \frac{(q(\alpha)^2 + m^2) \eta'_{n;(\beta_1) \dots (\beta_\tau)} - \eta'_{n;(\alpha)(\beta_1) \dots (\beta_\tau)}}{(q(\alpha)^2 + m^2 - i\varepsilon)(q(\beta_1)^2 + m^2 - i\varepsilon) \dots (q(\beta_\tau)^2 + m^2 - i\varepsilon)} \quad (14)$$

This relation shows that λ_n has no singularities of the form $\delta(q(\alpha)^2 + m^2)$ because

$$\begin{aligned} (q(\alpha)^2 + m^2) \eta'_n &= \eta'_{n;(\alpha)} \\ \text{for } q(\alpha)^2 &= -m^2 \\ (q(\alpha)^2 + m^2) \eta'_{n;(\beta_1) \dots (\beta_\tau)} &= \eta'_{n;(\alpha)(\beta_1) \dots (\beta_\tau)} \end{aligned}$$

There are also no poles $P(q(\alpha)^2 + m^2)^{-1}$, because

$$(q(\alpha)^2 + m^2) \lambda_n \Big|_{q(\alpha)^2 = -m^2} = 0$$

follows from (14).

We can show further that λ_n is finite¹³⁾ for $q(\alpha)^2 = -m^2$ if we assume that the derivative

$$\frac{\partial}{\partial q(\alpha)^2} \{ (q(\alpha)^2 + m^2) \eta'_n \} \Big|_{q(\alpha)^2 = -m^2}$$

is continuous at $q(\alpha)^2 = -m^2$. Then it follows from (14)

$$\lambda_n \Big|_{q(\alpha)^2 = -m^2} = F + \sum_{\tau} (-1)^{\tau} \sum_{(\beta_1) \dots (\beta_\tau)} \frac{F(\beta_1) \dots (\beta_\tau)}{(q(\beta_1)^2 + m^2 - i\varepsilon) \dots (q(\beta_\tau)^2 + m^2 - i\varepsilon)}$$

with

$$F = \frac{\partial}{\partial q(\alpha)^2} \{ (q(\alpha)^2 + m^2) \eta'_n \} \Big|_{q(\alpha)^2 = -m^2}$$

$$F(\beta_1) \dots (\beta_\tau) = (q(\beta_1)^2 + m^2) \dots (q(\beta_\tau)^2 + m^2) F \Big|_{q(\beta_1)^2 = \dots = q(\beta_\tau)^2 = -m^2}$$

The right hand side of (12) can be simplified considerably by using the product relation (10). Thus we obtain

$$\eta'_{n;(\alpha_1 \dots \alpha_r)}(p_1, \dots, p_n) = 2\pi i \eta'_{r+1}(p_{\alpha_1}, \dots, p_{\alpha_r}, -q_{\alpha_1 \dots \alpha_r}) \eta'_{n-r+1}(p_{\alpha_{r+1}} \dots p_{\alpha_{n-1}} p_n q_{\alpha_1 \dots \alpha_r}) \quad (15)$$

$$(q_{\alpha_1 \dots \alpha_r}^2 = -m^2)$$

i.e. $\eta'_{n;(\alpha_1 \dots \alpha_r)}$ is a product of two η' -functions with $r+1$ and $n-r+1$ variables. The first factor depends only on the variables $q_{\beta_1 \dots \beta_s}^2$ (except $q_{\alpha_1 \dots \alpha_r}^2$) with each β_μ being one of the indices $(\alpha_1, \dots, \alpha_r)$. The second factor depends only on the variables $q_{\beta_1 \dots \beta_s}^2$ and $(q_{\beta_1 \dots \beta_s + p_n})^2$ (except $(q_{\alpha_{r+1} \dots \alpha_{n-1} + p_n})^2$) with each β_i being one of the indices $(\alpha_{r+1}, \dots, \alpha_{n-1})$. Hence the function $\eta'_{n;(\alpha_1 \dots \alpha_r)}$ depends only on those variables $q_{\beta_1 \dots \beta_s}^2$ belonging to one of the three groups :

- (i) $q_{\beta_1 \dots \beta_s}$ is a partial sum of $q_{\alpha_1 \dots \alpha_r}$, i.e. each p_{β_i} occurs in $(p_{\alpha_1}, \dots, p_{\alpha_r})$ and $s < r$.
- (ii) the partial sums $q_{\alpha_1 \dots \alpha_r}$ and $q_{\beta_1 \dots \beta_s}$ have no common summand, i.e. $\alpha_\nu \neq \beta_\mu$ for any ν, μ .
- (iii) $q_{\alpha_1 \dots \alpha_r}$ is a partial sum of $q_{\beta_1 \dots \beta_s}$.

As a consequence of this result

$$\eta'_{n;(\alpha_1 \dots \alpha_r)}(\beta_1 \dots \beta_s) = (q_{\beta_1 \dots \beta_s}^2 + m^2) \eta'_{n;(\alpha_1 \dots \alpha_r)} \Big|_{q_{\beta_1 \dots \beta_s}^2 = -m^2}$$

vanishes identically when $q_{\beta_1 \dots \beta_s}$ does not satisfy one of the conditions (i), (ii) or (iii). In any other case, however, $q_{\beta_1 \dots \beta_s}^2$ appears as coordinate in one of the factors of $\eta'_{n;(\alpha_1 \dots \alpha_r)}$. Hence the product relation can be applied again and leads to a product of three η' -functions for $\eta'_{n;(\alpha_1 \dots \alpha_r)}(\beta_1 \dots \beta_s)$.

We now want to determine $\eta'_{n;(\alpha_1)\dots(\alpha_G)}$ for an arbitrary set of momenta $q(\alpha_1), \dots, q(\alpha_G)$. We call $q(\alpha_1), \dots, q(\alpha_G)$ a normal system when between any two $q(\alpha_i), q(\alpha_j)$ one of the three relations (i), (ii) or (iii) holds. Repeated application of the product relation (10) leads to the result that $\eta'_{n;(\alpha_1)\dots(\alpha_G)}$ is a product of $G+1$ η' -functions when the $q(\alpha_1), \dots, q(\alpha_G)$ form a normal system. When the $q(\alpha_1), \dots, q(\alpha_G)$ do not form a normal system, however, $\eta'_{n;(\alpha_1)\dots(\alpha_G)}$ vanishes identically. For formulating precisely the product structure of an arbitrary $\eta'_{n;(\alpha_1)\dots(\alpha_G)}$ it is convenient to use some combinatorial tools which we introduce now. To an arbitrary normal system $q(\alpha_1), \dots, q(\alpha_G)$ we attribute a Feynman graph of the following type :

- (a) the graph has n external lines, corresponding to the basic momenta $p_1, \dots, p_n$;
- (b) the graph has G internal lines, corresponding to the partial sums $q(\alpha_i)$;
- (c) the graph is simply connected and has no closed loops. The number of vertices is $G+1$.

Given an arbitrary normal system $q(\alpha_1), \dots, q(\alpha_G)$ we define the corresponding graph by the following construction. We start drawing the external line p_n which leads to a vertex denoted by V_0 . Let $q(\beta_1), \dots, q(\beta_r)$ denote the maximal elements within the system $q(\alpha_1), \dots, q(\alpha_G)$; by maximal elements we mean those $q(\alpha_i)$ which are not partial sums of any other $q(\alpha_j)$ of the system. Let further $p_{b_1}, \dots, p_{b_t}$ denote all basic momenta ($1 \leq b_i \leq n-1$) which do not occur in any one of the maximal $q(\beta_i)$. Then we draw from the vertex V_0 the internal lines $q(\beta_1), \dots, q(\beta_r)$ and the external lines $p_{b_1}, \dots, p_{b_t}$.

We proceed constructing the graph as follows. Each internal line $q(\beta_i)$ connects V_0 with another vertex V_{oi} . Let $q(\beta_{i1}), q(\beta_{i2}), \dots$ denote all maximal elements among the partial sums of $q(\beta_i)$. Let further $p_{b_{i1}}, p_{b_{i2}}, \dots$ denote all momenta occurring in the sum $q(\beta_i)$ but not occurring in any $q(\beta_{ij})$. Then we draw from the vertex V_{oi} the internal lines $q(\beta_{i1}), q(\beta_{i2}), \dots$ and the external lines $p_{b_{i1}}, p_{b_{i2}}, \dots$.

Proceeding in this way until any p_i is used up we finally obtain a Feynman graph of the described type. Apparently there is a one-to-one correspondence between the normal systems of partial sums from $n-1$ vectors and the Feynman graphs satisfying (a), (b) and (c).

In terms of the defined graphs it is now easy to formulate the explicit product form of $\eta'_{n;(\alpha_1)\dots(\alpha_\tau)}$ when $q(\alpha_1), \dots, q(\alpha_\tau)$ is an arbitrary system. It is

$$\eta'_{n;(\alpha_1)\dots(\alpha_\tau)}(p_1, \dots, p_n) = (2\pi i)^\tau \prod_{\nu=1}^{\tau+1} \eta_{n_\nu}(u_{\nu 1} u_{\nu 2} \dots u_{\nu n_\nu}) \Big|_{q_{(\alpha_i)}^2 \neq \dots = q_{(\alpha_\tau)}^2 = -m^2} \quad (16)$$

where each factor η_{n_ν} corresponds to a vertex V_ν of the graph corresponding to $q(\alpha_1), \dots, q(\alpha_\tau)$. n_ν is the number of lines joining at V_ν . The variables $u_{\nu 1}, \dots, u_{\nu n_\nu}$ denote (up to a sign) the momenta attributed to the lines joining at V_ν . The sign of the $u_{\nu i}$ has to be taken appropriately so that $\sum_{i=1}^{n_\nu} u_{\nu i} = 0$: all external and internal $u_{\nu i}$ get a positive sign, except that internal $u_{\nu j}$ which contains the other internal $u_{\nu i}$ as partial sums. Finally all squares $q_{(\alpha_i)}^2$ of internal momenta are set equal to $-m^2$.

Inserting this result into the definition of λ_n we obtain

$$\lambda_n = \eta'_n - \sum_G (-1)^\tau (2\pi i)^{\tau-1} \lim_{\varepsilon \rightarrow +0} \frac{\eta'_{n_1} \dots \eta'_{n_\tau}}{(q_{(\beta_1)}^2 + m^2 - i\varepsilon) \dots (q_{(\beta_{\tau-1})}^2 + m^2 - i\varepsilon)} \quad (17)$$

$(q_{(\beta_1)}^2, \dots, q_{(\beta_{\tau-1})}^2)$ are set equal to $-m^2$ in $\eta'_{n_1}, \dots, \eta'_{n_\tau}$.

The sum $\sum_G$ is taken over all Feynman graphs satisfying (a), (b) and (c). The product is constructed along the rules given to Eq. (16). In addition we have to any internal line $q(\alpha_i)$ a Feynman propagator $(q_{(\alpha_i)}^2 + m^2 - i\varepsilon)^{-1}$ in (17).

Finally we will show that η'_n can be expanded in the form ¹⁵⁾

$$\eta'_n = \lambda_n + \sum_G (2\pi i)^{\tau-1} \lim_{\varepsilon \rightarrow +0} \frac{\lambda_{n_1} \dots \lambda_{n_\tau}}{(q_{(\beta_1)}^2 + m^2 - i\varepsilon) \dots (q_{(\beta_{\tau-1})}^2 + m^2 - i\varepsilon)} \quad (18)$$

$(q_{(\beta_1)}^2, \dots, q_{(\beta_{\tau-1})}^2)$ are set equal to $-m^2$ in $\lambda_{n_1}, \dots, \lambda_{n_\tau}$.

when λ_n is given by (17). For the proof we assume that relation (18) is valid for all $n \leq m-1$ and derive (18) for $n = m$. Relation (17) gives

$$\eta'_m = \lambda_m + \sum_G (-1)^{\tau} (2\pi i)^{\tau-1} \lim_{\varepsilon \rightarrow +0} \frac{\eta'_{n_1} \dots \eta'_{n_\tau}}{(q_{(\alpha_1)}^2 + m^2 - i\varepsilon) \dots (q_{(\alpha_{\tau-1})}^2 + m^2 - i\varepsilon)}$$

As $n_{\alpha} \leq m-1$ we may use relation (18) in order to express $\eta'_{n_{\alpha}}$ by the λ -functions. We obtain the double sum

$$\eta'_m = \lambda_m + \sum_G (-1)^{\tau} \sum_{G' \supset G} (2\pi i)^{\tau-1} \lim_{\varepsilon \rightarrow +0} \frac{\lambda_{n_1} \dots \lambda_{n_\tau}}{(q_{(\alpha_1)}^2 + m^2 - i\varepsilon) \dots (q_{(\alpha_{\tau-1})}^2 + m^2 - i\varepsilon)}$$

where the summand corresponds to the graph G' . The sum $\sum_{G' \supset G}$ is taken over all graphs G' which can be obtained from G by inserting vertex parts.

Collecting all terms of the double sum with the same denominator we get

$$\eta'_m = \lambda_m + \sum_G c_G (2\pi i)^{\tau-1} \lim_{\varepsilon \rightarrow +0} \frac{\lambda_{n_1} \dots \lambda_{n_\tau}}{(q_{(\alpha_1)}^2 + m^2 - i\varepsilon) \dots (q_{(\alpha_{\tau-1})}^2 + m^2 - i\varepsilon)}$$

where the multiplicity c_G is

$$c_G = (-1)^{\tau} + (-1)^{\tau-1} c_{G_1} + (-1)^{\tau-2} c_{G_2} + \dots + (-1)^2 c_{G_{\tau-2}}$$

Here c_{G_i} denotes the number of all graphs with $\tau-i$ vertices from which G can be obtained by inserting vertex parts. It is

$$c_{G_i} = \binom{n-1}{i}$$

hence $c_G = 1$ for any graph G .

Relation (18) represents the desired expansion (11). We have seen that (18) is an identity when only λ_n is defined by (12). We finally have to check whether the coefficients of (18) satisfy the conditions (a), (b) which we have demanded in the beginning of this section. Condition (a) is fulfilled because all $q_{(\alpha_i)}^2$ are set equal to $-m^2$ in the coefficient

$$\lambda_{n;(\alpha_1) \dots (\alpha_{\tau-1})} = (2\pi i)^{\tau-1} \lambda_{n_1} \dots \lambda_{n_\tau}$$

That λ_n satisfies (b) was already shown. Then (b) is also true for the coefficients $\lambda_{n;(\alpha_1) \dots (\alpha_{\tau-1})}$ as they are either products of λ -functions or vanish identically.

3. Examples.

For illustrating the general investigations of the last section let us consider the one-particle structure of η' -functions depending on 3, 4 and 5 variables. The η -functions are defined by [see I, Eq.(14)]

$$\begin{aligned}\eta(x_1 x_2 x_3) &= \langle TA(x_1)A(x_2)A(x_3) \rangle_0 \\ \eta(x_1 x_2 x_3 x_4) &= \langle TA(x_1)A(x_2)A(x_3)A(x_4) \rangle_0 - \langle TA(x_1)A(x_2) \rangle_0 \langle TA(x_3)A(x_4) \rangle_0 - \text{perm. terms} \\ \eta(x_1 \dots x_5) &= \langle TA(x_1) \dots A(x_5) \rangle_0 - \langle TA(x_1)A(x_2)A(x_3) \rangle_0 \langle TA(x_4)A(x_5) \rangle_0 - \text{perm. terms}\end{aligned}$$

Apart from a δ -function the Fourier transforms depend only on the squares and inner products of the independent variables :

$$\delta(p_1+p_2+p_3) \eta_3(p_1^2, p_2^2, p_1 p_2) = \frac{1}{(2\pi)^{15/2}} \int dx_1 dx_2 dx_3 e^{-i(p_1 x_1 + p_2 x_2 + p_3 x_3)} \eta(x_1 x_2 x_3) \quad (20)$$

(correspondingly for η_4, η_5). The variables $p_i^2, q_{(\alpha)}^2$ used in the last section will here be denoted by

$$\begin{aligned}n=3 &: p_1^2, p_2^2, p_3^2 \\ n=4 &: p_1^2, p_2^2, p_3^2, p_4^2, \\ & q_{12}^2, q_{23}^2, q_{13}^2, \quad q_{ij} = p_i + p_j \\ n=5 &: p_1^2, p_2^2, p_3^2, p_4^2, p_5^2, \\ & q_{12}^2, q_{13}^2, q_{14}^2, q_{23}^2, q_{24}^2, q_{34}^2, \quad q_{ijk} = p_i + p_j + p_k \\ & q_{15}^2 = q_{234}^2, q_{25}^2 = q_{341}^2, q_{35}^2 = q_{124}^2, q_{45}^2 = q_{123}^2\end{aligned}$$

In these variables we write the functions η'_3, η'_4 and η'_5 (which are defined by (Eq.(8))

$$\begin{aligned}\eta'_3(p_1^2, p_2^2, p_3^2) &= (p_1^2 + m^2)(p_2^2 + m^2)(p_3^2 + m^2) \eta_3(p_1^2, p_2^2, p_1 p_2) \\ \eta'_4(p_1^2, p_2^2, p_3^2, p_4^2, q_{12}^2, q_{13}^2, q_{23}^2) &= (p_1^2 + m^2) \dots (p_4^2 + m^2) \eta_4(p_1^2, \dots, p_1 p_2 \dots) \\ \eta'_5(p_1^2, \dots, p_5^2, q_{12}^2, \dots, q_{45}^2) &= (p_1^2 + m^2) \dots (p_5^2 + m^2) \eta_5(p_1^2, \dots, p_1 p_2 \dots)\end{aligned} \quad (21)$$

According to (12), (17) we define λ -functions by

$$\lambda_3(p_1^2 p_2^2 p_3^2) = \eta_3'(p_1^2 p_2^2 p_3^2) \quad (22a)$$

$$\begin{aligned} \lambda_4(p_1^2 \dots p_4^2 q_{12}^2 q_{13}^2 q_{23}^2) &= \eta_4'(p_1^2 \dots p_4^2 q_{12}^2 q_{13}^2 q_{23}^2) - \\ &- \lim_{\xi \rightarrow +0} 2\pi i \frac{\eta_3'(p_1^2, p_2^2, -m^2) \eta_3'(-m^2, p_3^2, p_4^2)}{q_{12}^2 + m^2 - i\xi} \\ &- \lim_{\xi \rightarrow +0} 2\pi i \frac{\eta_3'(p_1^2, p_3^2, -m^2) \eta_3'(-m^2, p_2^2, p_4^2)}{q_{13}^2 + m^2 - i\xi} \\ &- \lim_{\xi \rightarrow +0} 2\pi i \frac{\eta_3'(p_2^2, p_3^2, -m^2) \eta_3'(-m^2, p_1^2, p_4^2)}{q_{23}^2 + m^2 - i\xi} \end{aligned} \quad (22b)$$

$$\begin{aligned} \lambda_5(p_1^2 \dots p_5^2 q_{12}^2 \dots q_{45}^2) &= \eta_5'(p_1^2 \dots p_5^2 q_{12}^2 \dots q_{45}^2) - \\ &- \lim_{\xi \rightarrow +0} 2\pi i \frac{\eta_3'(p_1^2, p_2^2, -m^2) \eta_4'(-m^2, p_3^2 p_4^2 p_5^2 q_{45}^2 q_{35}^2 q_{34}^2)}{q_{12}^2 + m^2 - i\xi} \\ &- \text{perm. terms} \end{aligned} \quad (22c)$$

$$\begin{aligned} &+ \lim_{\xi \rightarrow +0} (2\pi i) \frac{2 \eta_3'(p_1^2, p_2^2, -m^2) \eta_3'(-m^2, p_3^2, -m^2) \eta_3'(-m^2, p_4^2, p_5^2)}{(q_{12}^2 + m^2 - i\xi)(q_{45}^2 + m^2 - i\xi)} \\ &+ \text{perm. terms} \end{aligned}$$

(The corresponding graphs are drawn in Fig. 1, 2). These λ -functions do not have singularities at $q_{(\alpha)}^2 = -m^2$ any more. η_3' , η_4' , η_5' can be re-expressed in terms of the λ -function [Eq. (18)]

$$\eta_3'(p_1^2 p_2^2 p_3^2) = \lambda_3(p_1^2 p_2^2 p_3^2) \quad (23a)$$

$$\begin{aligned} \eta_4'(p_1^2 \dots p_4^2 q_{12}^2 q_{13}^2 q_{23}^2) &= \lambda_4(p_1^2 \dots p_4^2 q_{12}^2 q_{13}^2 q_{23}^2) + \\ &+ \lim_{\xi \rightarrow +0} 2\pi i \frac{\lambda_3(p_1^2, p_2^2, -m^2) \lambda_3(-m^2, p_3^2, p_4^2)}{q_{12}^2 + m^2 - i\xi} \\ &+ \lim_{\xi \rightarrow +0} 2\pi i \frac{\lambda_3(p_1^2, p_3^2, -m^2) \lambda_3(-m^2, p_2^2, p_4^2)}{q_{13}^2 + m^2 - i\xi} \\ &+ \lim_{\xi \rightarrow +0} 2\pi i \frac{\lambda_3(p_2^2, p_3^2, -m^2) \lambda_3(-m^2, p_1^2, p_4^2)}{q_{23}^2 + m^2 - i\xi} \end{aligned} \quad (23b)$$

$$\begin{aligned}
\eta_5'(p_1^2 \dots p_5^2 q_{12}^2 \dots q_{45}^2) = & \lambda_5(p_1^2 \dots p_5^2 q_{12}^2 \dots q_{45}^2) + \\
& + \lim_{\xi \rightarrow +0} 2\pi i \frac{\lambda_3(p_1^2, p_2^2, -m^2) \lambda_4(-m^2, p_3^2, p_4^2, p_5^2, q_{45}^2, q_{35}^2, q_{34}^2)}{q_{12}^2 + m^2 - i\xi} \\
& + \text{perm. terms} \\
& + \lim_{\xi \rightarrow +0} (2\pi i)^2 \frac{\lambda_3(p_1^2, p_2^2, -m^2) \lambda_3(-m^2, p_3^2, -m^2) \lambda_3(-m^2, p_4^2, p_5^2)}{(q_{12}^2 + m^2 - i\xi)(q_{45}^2 + m^2 - i\xi)} \\
& + \text{perm. terms.}
\end{aligned}$$

These formulae represent the expansions of $\eta_3', \eta_4', \eta_5'$ with respect to all singularities of the form $P(q_{ij}^2 + m^2)^{-1}$, $\delta(q_{ij}^2 + m^2)$ and products thereof.

4. Extension to Several Fields and Particles.

The investigations of the preceding sections can easily be extended to models with several fields and particles (including stable bound states). As an example we discuss here the one-particle structure of some Green's functions associated with various two-particle scattering processes. We consider the model of a charged scalar field $\psi(x)$ interacting with a neutral scalar field $A(x)$. The field operators are as usual assumed to be invariant, causal, complete, etc. We further demand that the system have the discrete (non-vanishing) mass values κ , m and M corresponding to the charge values 0, 1 and 2. The corresponding particles are called "meson", "nucleon" and "deuteron". All particles are assumed to be spinless.

We consider the three τ -functions

$$\begin{aligned}\tau^I &= \langle T \psi(x) \bar{\psi}(y) A(z_1) A(z_2) \rangle_0 \\ \tau^{II} &= \langle T \psi(x_1) \psi(x_2) \bar{\psi}(y_1) \bar{\psi}(y_2) \rangle_0 \\ \tau^{III} &= \langle T \psi(x_1) \psi(x_2) A(y) \bar{B}(z) \rangle_0\end{aligned}$$

with the deuteron field operator $B(x)$ defined by

$$B(x) = \lim_{\xi \rightarrow 0} (2\pi)^{3/2} \frac{T \psi(x+\xi) \psi(x-\xi)}{(\Phi, T \psi(\xi) \psi(-\xi) \Phi)}$$

where Φ is the deuteron state at rest. We split off the vacuum singularities by forming

$$\eta^I(x; y; z_1, z_2) = \langle T \psi(x) \bar{\psi}(y) A(z_1) A(z_2) \rangle_0 - \langle T \psi(x) \bar{\psi}(y) \rangle_0 \langle T A(z_1) A(z_2) \rangle_0 \quad (24a)$$

$$\begin{aligned}\eta^{II}(x_1, x_2; y_1, y_2) &= \langle T \psi(x_1) \psi(x_2) \bar{\psi}(y_1) \bar{\psi}(y_2) \rangle_0 - \\ &- \langle T \psi(x_1) \bar{\psi}(y_1) \rangle_0 \langle T \psi(x_2) \bar{\psi}(y_2) \rangle_0 - \\ &- \langle T \psi(x_1) \bar{\psi}(y_2) \rangle_0 \langle T \psi(x_2) \bar{\psi}(y_1) \rangle_0\end{aligned} \quad (24b)$$

$$\eta^{III}(x_1, x_2; y; z) = \langle T \psi(x_1) \psi(x_2) A(y) \bar{B}(z) \rangle_0 \quad (24c)$$

$$\delta(p_1 + p_2 + p_3 + p_4) \eta_4^{(i)}(p_1, p_2, p_3, p_4) = \frac{1}{(2\pi)^{10}} \int dx_1 dx_2 dx_3 dx_4 e^{-i(p_1 x_1 + \dots + p_4 x_4)} \eta^{(i)}(x_1, x_2, x_3, x_4)$$

We further remove all one-particle singularities in $p_1^2, p_2^2, p_3^2, p_4^2$ by applying the corresponding Klein-Gordon operators

$$\begin{aligned}\eta_4^{I'}(p; q; k_1, k_2) &= (p_1^2 + m^2)(q_1^2 + m^2)(k_1^2 + \kappa^2)(k_2^2 + \kappa^2) \eta_4^I(p; q; k_1, k_2) \\ \eta_4^{II'}(p_1, p_2; q_1, q_2) &= (p_1^2 + m^2)(p_2^2 + m^2)(q_1^2 + m^2)(q_2^2 + m^2) \eta_4^{II}(p_1, p_2; q_1, q_2) \\ \eta_4^{III'}(p_1, p_2; k; q) &= (p_1^2 + m^2)(p_2^2 + m^2)(k^2 + \kappa^2)(q^2 + M^2) \eta_4^{III}(p_1, p_2; k; q)\end{aligned} \quad (25)$$

The remaining one-particle singularities of these functions can now be studied in a manner analogous to that of section 2. We obtain the relations

$$\begin{aligned}
 \eta_4^{\text{I}}(p; q; k_1 k_2) &= \lambda_4^{\text{I}}(p; q; k_1 k_2) + \\
 &+ \lim_{\xi \rightarrow +0} 2\pi i \frac{\lambda_3^{\text{N}\bar{\text{N}}\text{M}}(p^2; -m^2; k_1^2) \lambda_3^{\text{N}\bar{\text{N}}\text{M}}(-m^2; q^2; k_2^2)}{(p+k_1)^2 + m^2 - i\xi} \\
 &+ \lim_{\xi \rightarrow +0} 2\pi i \frac{\lambda_3^{\text{N}\bar{\text{N}}\text{M}}(p^2; -m^2; k_2^2) \lambda_3^{\text{N}\bar{\text{N}}\text{M}}(-m^2; q^2; k_1^2)}{(p+k_2)^2 + m^2 - i\xi} \\
 &+ \lim_{\xi \rightarrow +0} 2\pi i \frac{\lambda_3^{\text{N}\bar{\text{N}}\text{M}}(p^2; q^2; -k^2) \lambda_3^{\text{M}\bar{\text{M}}\text{M}}(k_1^2, k_2^2, -k^2)}{(p+q)^2 + k^2 - i\xi}
 \end{aligned} \tag{26a}$$

$$\begin{aligned}
 \eta_4^{\text{II}}(p_1 p_2; q_1 q_2) &= \lambda_4^{\text{II}}(p_1 p_2; q_1 q_2) + \\
 &+ \lim_{\xi \rightarrow +0} 2\pi i \frac{\lambda_3^{\text{N}\bar{\text{N}}\text{D}}(p_1^2 p_2^2; -M^2) \lambda_3^{\text{N}\bar{\text{N}}\text{D}}(q_1^2 q_2^2; -M^2)}{(p_1+p_2)^2 + M^2 - i\xi} \\
 &+ \lim_{\xi \rightarrow +0} 2\pi i \frac{\lambda_3^{\text{N}\bar{\text{N}}\text{M}}(p_1^2, q_1^2; -k^2) \lambda_3^{\text{N}\bar{\text{N}}\text{M}}(p_2^2, q_2^2; -k^2)}{(p_1+q_1)^2 + k^2 - i\xi} \\
 &+ \lim_{\xi \rightarrow +0} 2\pi i \frac{\lambda_3^{\text{N}\bar{\text{N}}\text{M}}(p_2^2, q_1^2; -k^2) \lambda_3^{\text{N}\bar{\text{N}}\text{M}}(p_1^2, q_2^2; -k^2)}{(p_2+q_1)^2 + k^2 - i\xi}
 \end{aligned} \tag{26b}$$

$$\begin{aligned}
 \eta_4^{\text{III}}(p_1 p_2; k; q) &= \lambda_4^{\text{III}}(p_1 p_2; k; q) + \\
 &+ \lim_{\xi \rightarrow +0} 2\pi i \frac{\lambda_3^{\text{N}\bar{\text{N}}\text{D}}(p_1^2, p_2^2; -M^2) \lambda_3^{\text{D}\bar{\text{D}}\text{M}}(-M^2; q^2; k^2)}{(p_1+p_2)^2 + M^2 - i\xi} \\
 &+ \lim_{\xi \rightarrow +0} 2\pi i \frac{\lambda_3^{\text{N}\bar{\text{N}}\text{M}}(p_1^2, -m^2; k^2) \lambda_3^{\text{N}\bar{\text{N}}\text{D}}(-m^2, p_2^2; q^2)}{(p_1+k)^2 + m^2 - i\xi} \\
 &+ \lim_{\xi \rightarrow +0} 2\pi i \frac{\lambda_3^{\text{N}\bar{\text{N}}\text{M}}(p_2^2, -m^2; k^2) \lambda_3^{\text{N}\bar{\text{N}}\text{D}}(-m^2, p_1^2; q^2)}{(p_2+k)^2 + m^2 - i\xi}
 \end{aligned} \tag{26c}$$

Here λ_4 and the residua do not contain any one-particle singularities. The factors of the residua are identical with the so-called vertex functions :

$$\begin{aligned} \lambda_3^{ijk}(p_1^2 p_2^2 p_3^2) &= (p_1^2 + m_i^2)(p_2^2 + m_j^2)(p_3^2 + m_k^2) \eta_3^{ijk}(p_1^2 p_2^2 p_3^2) \\ \delta(p_1 + p_2 + p_3) \eta_3^{ijk}(p_1^2 p_2^2 p_3^2) &= \\ &= \frac{1}{(2\pi)^{15/2}} \int dx_1 dx_2 dx_3 e^{-i(p_1 x_1 + \dots + p_3 x_3)} \langle T A_i(x_1) A_j(x_2) A_k(x_3) \rangle_0 \end{aligned} \quad (27)$$

where $i, j, k = N, \bar{N}, M, D, \bar{D}$

$$A_N(x) = \psi(x), \quad A_{\bar{N}}(x) = \bar{\psi}(x), \quad A_M(x) = A(x),$$

$$A_D(x) = B(x), \quad A_{\bar{D}}(x) = \bar{B}(x),$$

$$m_N = m_{\bar{N}} = m, \quad m_M = k, \quad m_D = m_{\bar{D}} = M.$$

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- 2) M. Fierz, Helv. Phys. Acta 23, 731 (1950)
- 3) V. Glaser, private communication
- 4) G.F. Chew and F.E. Low, Phys. Rev. 113, 1640 (1959)
- 5) $\langle \rangle_0$ denotes the vacuum expectation value.
- 6) By finite we mean that $(q^2 + m^2) \tau_n(p_1 \dots p_n)$ is a well defined "distribution" in the remaining variables when q^2 is set equal to $-m^2$.
- 7) Provided that $A(x)$ is properly normalized.
- 8) In the derivation of (5) causality has been used, for instance, in order to justify that

$$\begin{aligned} & \int dx \int dz e^{-i(px + qz)} K_x K_y TA(x_1) \dots A(x_r) A(z) \\ &= \int dz \int dx e^{-i(px + qz)} K_x K_y TA(x_1) \dots A(x_r) A(z) \\ & \text{for } x = \frac{1}{r} (x_1 + \dots + x_r), \quad p^2 = q^2 = -m^2. \end{aligned}$$

- 9) The τ -function can be reobtained from η' by the formula (see I, equ.(26))

$$\eta_n(p_1 \dots p_n) = \lim_{\xi \rightarrow +0} \frac{\eta'_n(p_1 \dots p_n)}{(p_1^2 + m^2 - i\xi) \dots (p_n^2 + m^2 - i\xi)}$$

- 10) In perturbation theory η_n represents all those contributions to τ_n which come from simply connected Feymann graphs.
- 11) As an invariant function without singularities $d(p_{\alpha_1} + \dots + p_{\alpha_r})$ the function η'_n depends only on the squares p_i^2 and the products $p_i p_j$ of $n-1$ independent four vectors.
- 12) In all following formulae the limit $\xi \rightarrow +0$ has to be taken.
- 13) I.e. λ_n is a defined "distribution" in the remaining variables when $q_{(\omega)}^2 = -m^2$.

- 14) In general these variables are not independent from each other. As the first factor of $\eta'_n(\alpha_1, \dots, \alpha_r)$ depends on $\binom{r+1}{2} - 1$ and the second factor on $\binom{n-r+1}{2} - 1$ independent variables only $\binom{r+1}{2} + \binom{n-r+1}{2} - 2$ among the $q_{\beta_1 \dots \beta_s}^2$ are independent.

- 15) The notation in (18) is the same as in (16), (17).

Fig. 1 Graphs corresponding to equ. (22b) and (23b).

Fig. 2 Graphs corresponding to equ. (22c) and (23c).

Fig. 3 Graphs corresponding to equ. (26a)

Fig. 4 Graphs corresponding to equ. (26b)

Fig. 5 Graphs corresponding to equ. (26c)