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Special methods of analytic completion in field theory *)

R.F.Streater

CERN, Geneva

ABSTRACT

The Jost-Lehmann-Dyson representation is shown to provide a simple proof of the "edge of the wedge" theorem; it also gives an integral representation for the double commutator, which can be used to simplify some of the work needed to find the domain of regularity of the 3-fold Wightman functions. The Jacobi identity cannot be incorporated, and so the 3-fold case cannot be completely solved. The Jost-Lehmann-Dyson representation also provides some results in the 4-fold case; in particular, gives the envelope of holomorphy of the union of a certain four permuted extended tubes.

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I. Introduction.

In any theory of quantised fields occur "local field operators" $A(x)$, $B(x)$, ..., which, for each space-time point x , are operators in the Hilbert space formed by all physical states $|0\rangle$, ..., (where $|0\rangle$ is the vacuum). In attempts to avoid inconsistencies in the theory, attention has in recent years been directed to the Wightman functions (W-functions), that is, the vacuum expectation values of the products of field operators, such as $\langle 0 | A(x_1) B(x_2) \dots C(x_n) | 0 \rangle$, a function of n space-time points $x_1, \dots, x_n$. Because of the translational invariance of the theory, this function can depend only on the relative coordinates $y_1 = x_1 - x_2$, $y_2 = x_2 - x_3, \dots$. The functions may be "singular", that is, may contain the δ -function (Dirac 1958) and its derivatives; so in general we consider the W-functions to be Schwartz distributions (Schwartz 1951). Further, we are also interested in the Fourier transforms of these functions, since these are closely related to the form factors, scattering amplitudes, and other physical quantities of the theory. In order that the Fourier transform be definable, the W-functions must behave at most as a polynomial at infinity, and we say that they are "tempered". In this way a meaning can be given to the operator distribution $A(x)$. It has been shown (Wightman 1956) that if all the vacuum expectation values $\langle 0 | A(x_1) \dots C(x_n) | 0 \rangle$, for all permutations of $A, B, \dots, C$ are given, then the operators $A(x) \dots$ are determined. Then by prescribing properties for the W-functions, a field theory satisfying given properties can be constructed. Thus, for example, the postulates of unitarity and that every state has positive mass and energy lead to the conclusion that

$$\int \langle 0 | A(x_1) \dots C(x_n) | 0 \rangle \exp i \left\{ p_j (x_j - x_{j+1}) \right\} d^4(x_j - x_{j+1})$$

is zero unless the momentum variables p_i lie in the forward light cone ($i = 1, \dots, n-1$), and with similar conditions for the permuted functions. Conversely, a set of W-functions with this property specifies a field theory with positive mass and energy. Thus we are led to examine the general properties of functions whose Fourier transforms are zero except in light-cones. This property can be completely characterised by a domain of regularity in configuration space, the exact statement being given in Section II, as a lemma (Schwartz 1952).

Notation :

In the following, x_i or $x, y, \dots$, will denote four-vectors in momentum space. We shall denote $x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3$ by $x \cdot y$ and $x \cdot x$ by x^2 . If $(x-y)^2 > 0$ and $x_0 > y_0$ we shall write $x > y$ or $y < x$. If $(x-y)^2 < 0$ we shall write $x \sim y$. The vector $(x_1, \dots, x_n)$ of n four-vectors will be abbreviated to $\underline{x}$ or x_j , and $\underline{x} \cdot \underline{p} = x_i p_i = x_1 \cdot p_1 + x_2 \cdot p_2 + \dots + x_n \cdot p_n$. If $f(\underline{x})$ is a tempered distribution over R_{4n} , we define

$$\bar{f}(\underline{p}) = \int e^{i \underline{p} \cdot \underline{x}} d^4 x f(\underline{x}) = Ff; \left(f(x) = \bar{F} \bar{f} \right)$$

The set of real points $\underline{x}$ of R_{4n} for which

$$\sum_{i=1}^n \alpha_i x_i \sim 0 \quad \text{for all } \alpha_i > 0 \quad (\text{I.1})$$

has a special importance; these points have been called Jost points (Wightman 1958), shortened here to J.P. We shall be using the fundamental Lorentz invariant functions $\delta_s(p) \equiv \delta(p^2 - s)$ and $\epsilon(p) \delta_s(p)$ where $\epsilon(p) \equiv \text{sign}(p_0)$, and δ is the usual δ -function (Dirac 1958). Also we define

$$\begin{aligned} \theta(p) &= 1, & p_0 > 0 \\ &= 0, & p_0 < 0. \end{aligned}$$

The function $\theta(p) \delta_s(p)$ will be denoted by $\delta_s^r(p)$, and $\theta(-p) \delta_s(p)$ by $\delta_s^a(p)$. If $\underline{z} = \text{Re } \underline{z} + i \text{Im } \underline{z}$, then the set of $\underline{z}$ satisfying $\text{Im } z_i > 0$ is called the "forward tube" (Hall 1958), and is denoted by $\mathcal{T}_n(\underline{z})$ or $\mathcal{T}_n(z_i)$. The following lemma will be of use to us later. ("Convexity of the light cone"). If $x > 0$ and $y > 0$, then $\mu x + \lambda y > 0$ for all $\mu, \lambda > 0$.

The condition that the theory has only states with positive mass and energy can be stated as the condition that $\langle 0 | A(x_1) \dots C(x_n) | 0 \rangle$ be regular in $\mathcal{T}_n(y_i)$, and all permuted functions are regular in domains obtained by permuting the x -variables. The imposition of Lorentz invariance implies (Hall 1958) that the W -functions can be analytically continued into the "extended tube" $\mathcal{T}'_n(y_i)$ obtained from $\mathcal{T}_n(y_i)$ by all complex Lorentz transformations connected to unity. The domain $\mathcal{T}'_1(y_1)$ is the y_1^2 -plane except for

the points $y_1^2 > 0$. The boundary of $\mathcal{T}'_2(y_1, y_2)$ has been calculated explicitly by Hall (Hall 1956, Källén 1958) in terms of the invariants $y_1^2, y_1 \cdot y_2, y_2^2$. Summarising, the postulates of positive mass and energy, and relativity, are entirely equivalent to the postulate that each W-function is regular in the appropriate permuted extended tube as a function of the scalar products $y_i \cdot y_j$.

The postulate of causality is

$$\left[A(x_1), B(x_2) \right] = 0, \quad \text{i.e. } A(x_1)B(x_2) = B(x_2)A(x_1), \quad \text{if } x_1 \sim x_2 \quad (\text{I.2})$$

The points (I.1) are significant because they are real regular points of $\mathcal{T}'_n(x_1, \dots, x_n)$. Condition (I.2) certainly implies that the various permuted W-functions of a given order are equal at points (I.1), and so continue one another as one regular function in the union of the $n!$ permuted tubes of order n , which actually coincide in pairs. It is an interesting question (Wightman 1958) whether (I.2) is implied by the (apparently) weaker condition (I.3)

$$(1-P) \left\langle 0 \left| A(x_1) \dots C(x_n) \right| 0 \right\rangle = 0 \quad \text{at J.P.} \quad (\text{I.3})$$

where P is any permutation of the order $A \dots C$. Källén and Wightman (Källén 1958) arrive at a result which shows that as far as the 2- and 3-fold vacuum expectation values are concerned, the conditions (I.2) and (I.3) are equivalent. In later sections we shall state theorems in the form (I.3), while proving them assuming condition (I.2) ^{*)}. The union of the $\frac{1}{2}n!$ permuted extended tubes of order n is not a domain of holomorphy. This means that any function regular in this union is also regular in a larger domain, called its envelope of holomorphy. If the envelope of holomorphy is known the function may be expressed in terms of its boundary values. The problem of finding this envelope of holomorphy is one of the analytic completion, and is closely connected with the problem of representing functions which vanish in a certain region, and whose Fourier transforms vanish in a given region. A solution of a typical problem of the latter type is the Jost-Lehmann-Dyson lemma (Dyson 1958a), which has simplified several problems in the general theory, especially the proof of

^{*)} D. Ruelle - (Thèse, Brussels Free University, 1959) has proved now that condition (I.3) implies (I.2).

dispersion relations (Lehmann 1958). In this paper Dyson's theorem is shown to provide a simple treatment for some other problems of the general theory of quantised fields. A proof of the "Edge of the Wedge" theorem is presented which covers all the cases of physical interest; some of the work needed to perform the analytic completion in the 3-fold case can be simplified; and a start can be made on the study of the 4-fold and n-fold vacuum expectation values.

II. The Edge of the Wedge Theorem.

Before stating the theorem we give a lemma in the theory of Laplace transforms of distributions (Schwartz 1952).

Lemma If $f(z)$ is regular, $z \in J_1$, and is bounded at ∞ by a polynomial, then the limit

$$\lim_{\epsilon \rightarrow 0} f(x+i\epsilon) = f(x)$$

exists in the sense of distributions, and $Ff = 0$ unless $p > 0$. In the lemma the limit need not exist in the usual sense; $f(x)$ will in general be a tempered distribution.

Lemma (Dyson 1958a). If $f(x)$ is a tempered distribution with the properties

$$f = 0 \quad \text{if} \quad x \sim 0; \quad \bar{f} = 0 \quad \text{if} \quad p \in S,$$

then for a large class of sets $S, \exists \phi(u, k^2)$ such that

$$\bar{f}(p) = \int \phi(u, k^2) \delta_k(u-p) \epsilon(u-p) d^4 u dk^2 \quad (\text{II.1})$$

where $\phi(u, k^2) = 0$ if $\exists p$ such that $p \in S, u-p^2 = k^2$.

Theorem (Edge of the Wedge) (Bremmermann 1958, Taylor 1958).

If $\bar{f}_1(z)$, $z = p+iq$ is regular for $z \in \mathcal{J}_1$, and $\bar{f}_2(z)$ regular for $-z \in \mathcal{J}_1$, and is bounded by a polynomial at ∞ , and the limits (see lemma) $\bar{f}_1(p) = \lim_{\varepsilon \rightarrow 0} \bar{f}_1(z+i\varepsilon)$, $\bar{f}_2(p) = \lim_{\varepsilon \rightarrow 0} \bar{f}_2(z-i\varepsilon)$ satisfy

$$\bar{f}_1(p) - \bar{f}_2(p) = 0, \quad \text{if } p \in S,$$

then there exists a function $\bar{f}(z)$ coinciding with $\bar{f}_1(z)$ in $\mathcal{J}$, and $\bar{f}_2(z)$ if $-z \in \mathcal{J}$, and regular in a complex neighbourhood of S .

Proof Using the lemma

$f(x) = f_1(x) - f_2(x) = 0$ if $x^2 < 0$, and we are given $\bar{f}(p) = \bar{f}_1(p) - \bar{f}_2(p) = 0$, $p \in S$. So by Dyson's lemma

$$\bar{f}(p) = \int \phi(u, k) \delta_k(u-p) \in (u-p) d^4 u dk^2.$$

Now $f_1(x) = \theta(x)f(x)$, and so

$$\bar{f}_1(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(p_0 - p'_0)x'_0} dx'_0 dp'_0 \theta(x'_0) \int \phi(u, k) \delta[(u_0 - p'_0)^2 - (u-p)^2 - k^2] \in (u_0 - p'_0) d^4 u dk$$

or

$$\bar{f}_1(p) = \frac{1}{2\pi i} \int \frac{\phi(u, k) d^4 u dk^2}{(u-p)^2 - k^2} \quad (\text{II.2})$$

The singular points of $\bar{f}_1(p)$ are where $(u-p)^2 - k^2 = 0$, $\phi(u, k) \neq 0$. But $\phi(u, k) = 0$ if $u-p^2 = k^2$, $p \in S$. Hence $\bar{f}_1(p)$ is regular in a complex neighbourhood of S .

Note that in the theorem we did not need to assume that $\bar{f}(p)$ was bounded in the neighbourhood of S ; this boundedness follows from the proof.

III. Integral Representation for a Double Commutator

Dyson (Dyson 1958b) has proposed an integral representation for the function $D = \langle 0 | [A(x_1), [B(x_2), C(x_3)]] | 0 \rangle$. However, the derivation has a mistake (Symanzik 1959) between Eqs. (17) and (6), and consequently the representation is not the most general. Eq. (17) states that

$$D(x, y) = \int_0^\infty dt \int_0^1 d\lambda \mathcal{V}(y, t, \lambda) \Delta_t(x + \lambda y) \quad \begin{array}{l} x = x_1 - x_2 \\ y = x_2 - x_3 \end{array}$$

where

- (a) $\mathcal{V}(y) = -\mathcal{V}(-y)$
- (b) $\mathcal{V}(y) = 0$ if $y^2 < 0$
- (c) $\mathcal{V}(y)$ is Lorentz invariant.

To complete the proof ^{*)} we require that $\mathcal{V}(y)$ have the representation

$$\mathcal{V}(y) = \int_0^\infty \phi(s) \Delta_s(y) ds. \quad (\text{III.1})$$

But (III.1) follows from (a), (b), and (c) only if $\mathcal{V}(y)$ is a tempered distribution. Symanzik (1959) has shown that a perturbation theory example gives

$$\mathcal{V}(y) \propto e^{i a(\lambda, t) \sqrt{y^2}} \Delta_t(y) \quad (\text{III.2})$$

which tends to ∞ rapidly for large y^2 .

We shall give a representation for $D(x, y)$ in a form in which it is most useful for application to the problem solved by Källén and Wightman (1958).

^{*)} This point is due to K. Symanzik (private communication).

Consider

$$f(p,y) = \int \left\langle A(x_1)B(x_2)C(x_3) - A(x_1)C(x_3)B(x_2) \right\rangle e^{ipx} d^4x \quad (\text{III.3})$$

$$= 0 \quad \text{if} \quad y^2 < 0.$$

If $\bar{f}(p,q) = \int f(p,y) e^{iqy} d^4y$, then $\bar{f}(p,q) = 0$ unless $p > 0$ and either $q > 0$ or $p-q > 0$. Then using Dyson's theorem

$$\bar{f}(p,q) = \int \phi(p,u,s) \delta_s(u-q) \epsilon(u-q) d^4uds^2 \quad (\text{III.4})$$

where $\phi(p,u,s) = 0$ unless the hyperbola in q -space

$$(u-q)^2 = s^2 \quad (\text{III.5})$$

lies entirely in $q > 0 \cup p-q > 0$. Now if $q > u$, then $q = (q-u)+u$ is > 0 for all q of the hyperbola (III.5) if and only if $u > 0$. If $q < u$ then $p-q = p+(u-q)-u$ is > 0 for all q of (III.5) if and only if $p-u > 0$. Hence $\phi(p,u) = 0$ except if $p,u \in (u > 0 \cap p-u > 0)$. The CPT theorem states

$$D(x,y) = f(x,y) + f(-x,-y)$$

and so

$$\bar{D}(p,q) = \int \psi(p,u,s) \delta_s(u-q) \epsilon(u-q) d^4uds^2 \quad (\text{III.6})$$

with $\psi(p,u) = -\psi(-p,-u) = 0$ unless $p,u,p-u > 0$ or $p,u,p-u < 0$.

Since $\bar{D}(p,q)$ is Lorentz invariant we can choose a ψ which has this property : then

$$\psi(p,u,s) = \int \phi(\alpha,\beta,\gamma,s) \delta(u^2-\alpha^2) \delta[u-(p-u)-\beta] \delta[(p-u)^2-\gamma^2] \epsilon(u_0) d\alpha d\beta d\gamma \quad (\text{III.7})$$

and taking Fourier transforms to x, y -space

$$\langle 0 | [A [B, C]] | 0 \rangle = \int \phi(\alpha, \beta, \gamma, s) \Delta_{\alpha\beta\gamma}(x, x+y) \Delta_s(y) dx^2 dy^2 ds^2 \quad (\text{III.8})$$

where

$$\Delta_{\alpha\beta\gamma}(x, y) = \int e^{ipx+iqy} \delta(p^2 - \alpha^2) \delta(q^2 - \gamma^2) \delta(p \cdot q - \beta) \epsilon(p_0) d^4 p d^4 q \quad (\text{III.9})$$

The function $\Delta_{\alpha\beta\gamma}(x, x+y) \Delta_s(y)$ vanishes if $y \sim 0$, or if both x and $x+y \sim 0$. The latter property was not assumed in the proof, so we have shown that

1) Lorentz invariance of	$D(x, y)$	
2) CPT theorem	$D(x, y) = D(-x, -y)$	
3) $\underline{p}$ -space support	$\bar{D}(p, q) = 0$	unless p, q or $p, p-q$ timelike
4)	$D(x, y) = 0$	if $y^2 < 0$

imply

$$D(x, y) = 0 \quad \text{if } x \text{ and } x+y \sim 0.$$

This result was found by Symanzik (1959).

The proof given here follows

from the property of $\Delta_{\alpha\beta\gamma}(x, y)$ of being zero at Jost points.

With ϕ arbitrary, (III.8) does not satisfy the Jacobi identities.

That is, if we use the theorem for the three double commutators $D_A = [A [B, C]]$, $D_B = [B [C, A]]$ and $D_C = [C [A, B]]$, then in general $D_A + D_B + D_C$ is not identically zero for arbitrary ϕ_A, ϕ_B and ϕ_C .

IV. Applications to the n-fold Case

Consider

$$\bar{f}(p_1, \dots, p_{n-1}) = \int \langle 0 | A(x_1) \dots B(x_{n-2}) [C(x_{n-1}), D(x_n)] | 0 \rangle e^{i p \cdot y} d^4 y$$

$y_1 = x_1 - x_2$; $y_2 = x_2 - x_3$; etc. Then

$$\int e^{-i p_{n-1} \cdot y_{n-1}} \bar{f}(p) d^4 p_{n-1} = 0$$

if $y_{n-1} \sim 0$, and so as before there exists a $\emptyset$ such that

$$\bar{f}(p) = \int \emptyset(p_1, \dots, p_{n-2}; u, k) \delta_k(p_{n-1} - u) \epsilon(p_{n-1} - u) d^4 u dk^2 \quad (\text{IV.1})$$

where $\emptyset(p, u, k)$ can be chosen to be Lorentz invariant, and zero unless $p_1, \dots, p_{n-2}, p_{n-2} - u, u > 0$.

This theorem can be immediately generalised to any number of brackets; for example, if we define

$$\bar{f}(p, q, r) = \int \langle 0 | [A(x_1), B(x_2)] [C(x_3), D(x_4)] | 0 \rangle e^{i p y_1 + i q y_2 + i r y_3} d^4 y, \quad (\text{IV.2})$$

then $f(y_1, q, y_3) = 0$ if $y_3 \sim 0$, and if $y_1 \sim 0$; hence

$$f(y_1, q, r) = \int \emptyset(y_1, q, v, t) \delta_t(v - r) \epsilon(v - r) d^4 v dt$$

where $\emptyset(y_1, q, v, t) = 0$ if $y_1 \sim 0$, and unless $q, v, q - v > 0$. We can apply the theorem to $\emptyset(y_1, q, u, t)$ regarded as a function of y_1

$$\emptyset(p, q, v, t) = \int \psi(u, q, v, s, t) \delta_s(p - u) \epsilon(p - u) d^4 u ds$$

where $\psi = 0$ unless $q > 0, u > 0, q - u > 0$. We obtain finally

$$\bar{f}(p, q, r) = \int \psi(u, q, v, s, t) \delta_s(p - u) \delta_t(r - v) \epsilon(p - u) \epsilon(r - v) du dv ds dt \quad (\text{IV.3})$$

where $\psi = 0$ unless $u, v, q, q - u, q - v > 0$.

H. Lehmann (1958) used this formula, with the properties of mass spectrum included, to prove some analytic properties of the imaginary part of the scattering amplitude.

V. Applications to Analytic Completion

In this section we follow through the paper of Källén and Wightman (1958), making use of (III.4) to shorten some of the work. This technique is not powerful enough to solve the problem completely, we can deal with the three functions ABC , ACB , and BAC only two at a time. This, then, is the effect of not being able to make (III.4) satisfy the Jacobi identity.

Now $\langle 0 | B(x_2)A(x_1)C(x_3) | 0 \rangle$ is regular in the extended tube $\mathcal{T}'_{BAC} = \mathcal{T}'_2(-y_1, y_1+y_2)$, and $\langle 0 | A(x_1)C(x_3)B(x_2) | 0 \rangle$ is regular in the permuted tube $\mathcal{T}'_{ACB} = \mathcal{T}'_2(y_1+y_2, -y_2)$. Since $BAC = ACB$ somewhere where they are both regular (at least at J.P.), they continue one another to be regular in the union $T = \mathcal{T}'_2(-y_1, y_1+y_2) \cup \mathcal{T}'_2(y_1+y_2, -y_2)$. We know the boundary of $\mathcal{T}'_2(x, y)$ from the work of Hall (Hall 1959; Källén 1958). Working in the complex space of scalar products $x^2=z_1$; $y^2=z_2$; $(x+y)^2=z_3$, $\mathcal{T}'(x, y)$ is bounded by cuts

$$z_1 > 0 ; \quad z_2 > 0$$

$$(\text{Im} z_1)(\text{Im} z_2) > 0 : F_{12} \text{ curve} : z_3 = z_1 + z_2 + \rho + \frac{z_1 z_2}{\rho}, \quad 0 \leq \rho \leq \infty$$

$$(\text{Im} z_1)(\text{Im} z_2) < 0 : S \text{ curve} : z_3 = z_1 + z_2 - k z_1 - \frac{1}{k} z_2, \quad 0 \leq k \leq \infty$$

Using this we can plot the domain T , as in Figs. 1, 2, 3 (taken, with slight modifications, from Ref. ⁵⁾, Figs. 9, 10 and 12). The domain depicted in Figs. 1-3 is not a natural domain of holomorphy (Källén 1958); in Fig. 1 the boundaries F_{12} and F_{13} exchange roles when $y_1=y_2$, and we can continue through the corner so formed if $\text{Im} z_3 > 0$ to the curve F'_{12} of Fig. 4; this is discussed on page 37 of Källén (1958). If $\text{Im} z_3 < 0$ and z_1 and z_2 become co-linear, the branches of S exchange roles. This is discussed on page 29 of Källén (1958), and it is shown there that continuation to the curve F'_{12} is possible. This result will be obtained below using (III.4).

Now $\overline{ABC} = 0$ unless $p > 0$, $q > 0$, and $\overline{ACB} = 0$ unless $p > 0$, $p-q > 0$. Hence

Fig.1

The domain T plotted in z_3 plane for $(\text{Im } z_1)(\text{Im } z_2) > 0$.

Fig.2

The domain T (unshaded) plotted for $(\text{Im } z_1)(\text{Im } z_2) < 0; \text{Re } z_2 < 0$.

Fig.3

The domain T (unshaded) plotted for $(\text{Im} z_1)(\text{Im} z_2) < 0; \text{Re} z_1, \text{Re} z_2 > 0$.

Fig.4

Envelope of holomorphy (EH) of T plotted if $(\text{Im} z_1)(\text{Im} z_2) > 0$.

$$\bar{f}(p,q) = \overline{ABC} - \overline{ACB} = \overline{ABC} \quad \text{if } p > 0, \quad p-q \not> 0, \quad q > 0$$

$$\bar{f}(p,q) = \quad = -\overline{ACB} \quad \text{if } p > 0, \quad p-q > 0, \quad q \not> 0.$$

$$= \int \left[\phi(p,u,s) \delta_s^r(q-u) - \phi(p,u,s) \delta_s^a(q-u) \right] d^4uds$$

Using the properties of ϕ and the convexity of the light cone it is easily proved that

$$(a) \quad \int \phi(p,u,s) \delta_s^r(q-u) d^4uds = 0 \quad \text{unless } p > 0, \quad q > 0$$

$$(b) \quad \int \phi(p,u,s) \delta_s^a(q-u) d^4uds = 0 \quad \text{unless } p > 0, \quad p-q > 0.$$

Hence

$$\overline{ABC} = \int \phi \delta_s^r(q-u) d^4uds \quad \text{unless } p, q, p-q > 0$$

$$\overline{ACB} = \int \phi \delta_s^a(q-u) d^4uds \quad \text{unless } p, q, p-q > 0,$$

and so

$$\begin{aligned} \overline{ABC} - \int \phi \delta_s^r(q-u) d^4uds &= \overline{ACB} - \int \phi \delta_s^a(q-u) d^4uds = \bar{f}(p,q) \quad \text{say} \\ &= 0 \quad \text{unless } p, q, p-q > 0. \end{aligned}$$

giving the theorem :

The most general pair of functions ($\overline{ABC}$, $\overline{ACB}$ say) which have the properties

$$(a) \quad \overline{ABC}(x,y) = \overline{ACB}(x,y) \quad \text{if } y \sim 0$$

$$(b) \quad \overline{ABC}(p,q) = 0 \quad \text{unless } p, q > 0$$

$$(c) \quad \overline{ACB}(p,q) = 0 \quad \text{unless } p, p-q > 0,$$

has the representation

$$\frac{\overline{ABC}}{\overline{ACB}} = \int \phi(p,u,s) \delta_s^{ra}(q-u) d^4uds + \bar{f}(p,q) \quad (V.1)$$

where $\phi = 0$ unless $p, u, p-u > 0$, and $\bar{f} = 0$ unless $p, q, p-q > 0$.

Then (V.1) exhibits $ABC(x,y)$ as the regular function :

$$ABC(x,y) = \int \delta(p,u,s) \delta_s^r(q-u) d^4 u ds e^{i(q-u)y+i(p-u)x+iu(x+y)} d^4(q-u) d^4(p-u) + \\ + \bar{F}(p,q) e^{ix(p-q)+iq(x+y)} d^4(p-q) d^4 q \quad (V.2)$$

The integrals in (V.2) converge if $\text{Im} x > 0$, $\text{Im}(x+y) > 0$; since the function is Lorentz invariant, (5.2) shows $ABC(x,y)$ to be regular in the extended tube $\mathcal{J}'_2(x,x+y)$ except for the cut $y^2 > 0$ coming from the function

$$\Delta_s^{(+)}(y) = \int \delta_s^r(q-u) e^{i(q-u)y} d^4(q-u).$$

The boundary of $\mathcal{J}'_2(x,x+y) = \mathcal{J}'_2(\xi, \eta, \zeta)$, $\xi = x^2$, $\eta = (x+y)^2$, $\zeta = (2x+y)^2$, can be written in terms of $z'_1 = x^2$; $z'_2 = y^2$; $z'_3 = (x+y)^2$, and is

$$\text{cuts} \quad z'_1 > 0, \quad z'_3 > 0$$

$$S' \text{ curve} \quad z'_2 = z'_3 + z'_1 + k z'_1 + \frac{1}{k} z'_3 \quad \text{if} \quad (\text{Im} z'_1)(\text{Im} z'_3) < 0; \quad 0 \leq k \leq \infty$$

$$F'_{13} \text{ curve} \quad z'_2 = z'_1 + z'_3 - \rho - z'_1 z'_3 / \rho \quad \text{if} \quad (\text{Im} z'_1)(\text{Im} z'_3) = 0; \quad 0 \leq \rho \leq \infty$$

and to this we must add the cut $y^2 = z'_2 > 0$.

In order to compare with Figs. 1, 2, 3, 4 we rewrite this as $z_3 = z'_2$, $z_1 = z'_1$, $z_2 = z'_3$ (to correspond with CAB, ACB instead of ABC, ACB), to arrive at the theorem : the envelope of holomorphy of T is bounded by the cuts $z_1 > 0$, $z_2 > 0$, $z_3 > 0$, and

$$(\text{Im} z_1)(\text{Im} z_2) < 0, \quad S : z_3 = z_1 + z_2 + k z_1 + \frac{1}{k} z_2 \quad 0 \leq k \leq \infty$$

$$(\text{Im} z_1)(\text{Im} z_2) > 0, \quad F'_{12} : z_3 = z_1 + z_2 - \rho - z_1 z_2 / \rho \quad 0 \leq \rho \leq \infty$$

These curves are exactly those in Figs. 2, 3, 4 shown by Källén and Wightman to be the correct envelope of holomorphy.

If we apply (III.4) to CAB, ACB, and to ABC, ACB and take the union of the domains, we arrive at Fig. 25 of Källén (1958). This domain is not a natural domain of holomorphy, and to complete the problem other methods must be used.

VI. Analytic Completion in the n-fold Case

We can generalise (V.1) using (IV.1), to give the following theorem :
The most general pair of functions $(A...BCD, \overline{A...BDC})$ say) with the property

- (a) $A...BCD = \overline{A...BDC}$ if $y_{n-1} \sim 0$
(b) $\overline{A...BCD} = 0$ unless $p_1, \dots, p_{n-1} > 0$
(c) $\overline{A...BDC} = 0$ unless $p_1, \dots, p_{n-2}, p_{n-2} - p_{n-1} > 0$

has the representation

$$\frac{\overline{A...BCD}}{\overline{A...BDC}} = \int \phi(p_1, \dots, p_{n-2}, u, s) \delta_s^r(p_{n-1} - u) d^4 u d^4 k \overline{F}(\underline{p}) \quad (VI.1)$$

where $\phi = 0$ unless $p_1, \dots, p_{n-2}, p_{n-2} - u, u > 0$, and $\overline{F}(\underline{p}) = 0$ unless $p_1, \dots, p_{n-2}, p_{n-2} - p_{n-1}, p_{n-1} > 0$. Taking the Laplace transform of (VI.1), the integral converges if $\text{Im}(y_1, \dots, y_{n-2}, y_{n-2} + y_{n-1}) > 0$, and we have proved the general theorem :

The envelope of holomorphy of $\mathcal{J}_n'(x_1, \dots, x_n) \cup \mathcal{J}_n'(x_1, \dots, -x_{n-1}, x_{n-1} + x_n)$ is $\mathcal{J}_n'(x_1, \dots, x_{n-1} + x_n) \cup \mathcal{J}_1'(x_n)$. The boundary of $\mathcal{J}_n'$ is not known explicitly except for $n=2$.

Using Eq. (IV.3) we can find the E.H. of the domain

$$\mathcal{J}'_{ABCD} \cup \mathcal{J}'_{ABDC} \cup \mathcal{J}'_{BACD} \cup \mathcal{J}'_{BADC} \quad (VI.2)$$

The problem is to extract the terms $ABCD, \dots$ from the commutator.

Define (with the notation of (IV.3))

$$\begin{aligned} F_1(p, q, r) &= \overline{ABCD} - \int \phi(u, q, v, s, t) \delta_s^r(p-u) \delta_t^r(r-v) d^4 u d^4 v ds dt \\ F_2(p, q, r) &= \overline{ABDC} - \int \phi(u, q, v, s, t) \delta_s^r(p-u) \delta_t^a(r-v) d^4 u d^4 v ds dt \\ F_3(p, q, r) &= \overline{BACD} - \int \phi(u, q, v, s, t) \delta_s^a(p-u) \delta_t^r(r-v) d^4 u d^4 v ds dt \\ F_4(p, q, r) &= \overline{BADC} - \int \phi(u, q, v, s, t) \delta_s^a(p-u) \delta_t^a(r-v) d^4 u d^4 v ds dt \end{aligned} \quad (VI.3)$$

and let the sets in p, q, r space be denoted as follows :

$$S_1 = p, q, r > 0; \quad S_2 = p, q, q-r > 0; \quad S_3 = q-p, q, r > 0; \quad S_4 = q-p, q, q-r > 0.$$

Then if $\text{supp.} F_i$ denote the set $F_i(p, q, r) \neq 0$, we have

$$\text{Supp.} F_i \subseteq S_i \quad (\text{VI.4})$$

Now by (IV.3) the F_i satisfy

$$F_1 - F_2 - F_3 + F_4 = 0 \quad (\text{VI.5})$$

We deduce from (VI.4) and (VI.5) that

$$\text{Supp.} F_i \subseteq \bigcup_j (S_i \cap S_j) \quad (\text{VI.6})$$

Now

$$\begin{aligned} S_1 \cap S_2 &\supseteq S_1 \cap S_4, & S_1 \cap S_2 &\supseteq S_2 \cap S_3 \\ S_1 \cap S_3 &\supseteq S_2 \cap S_3, & S_3 \cap S_4 &\supseteq S_1 \cap S_4 \end{aligned}$$

and so (VI.6) becomes

$$\begin{aligned} \text{Supp.} F_1 &\subseteq (S_1 \cap S_2) \cup (S_1 \cap S_3) \\ \text{Supp.} F_2 &\subseteq (S_1 \cap S_2) \cup (S_2 \cap S_4) \\ \text{Supp.} F_3 &\subseteq (S_3 \cap S_1) \cup (S_3 \cap S_4) \\ \text{Supp.} F_4 &\subseteq (S_2 \cap S_4) \cup (S_3 \cap S_4) \end{aligned}$$

Hence there exist functions, to be denoted by (ij) , such that

$$\text{Supp}(ij) = S_i \cap S_j = S_{ij}, \quad \text{and}$$

$$\begin{aligned} F_1 &= (12) + (13) \\ F_2 &= (21) + (24) \\ F_3 &= (31) + (34) \\ F_4 &= (42) + (43) \end{aligned} \quad (\text{VI.7})$$

With the functions (ij) arbitrary, (VI.7) does not give F_i satisfying (VI.5).

Consider (VI.5) in the set $S_{12} \cap \overline{S_{13}}$, i.e. $p, q, r, q-r > 0$, $q-p < 0$. Then $p, q, r \notin S_4$. Then using (VI.7), Eq. (VI.5) becomes

$$(12) = (21) \quad \text{in } S_{12} \cap \overline{S_{13}} \quad (\text{VI.8})$$

i.e. everywhere except $S_{123} = S_1 \cap S_2 \cap S_3 = S_1 \cap S_2 \cap S_3 \cap S_4 = S_{1234}$ say.

Now consider (VI.5) in the set $\overline{S_{12}} \cap S_{24}$. Then $p \in S_2$, $p \notin S_1$, and so $r < 0$, and $p \notin S_3$; so (VI.5) becomes

$$(24) = (42) \quad \text{except in } S_{124} = S_{1234} \quad (\text{VI.9})$$

Now consider (VI.5) in the set $\overline{S_{12}} \cap S_{13}$; then $p \in S_1$, $p \notin S_2$, i.e. $r > q$. Hence $p \notin S_4$, and (VI.5) reads

$$(13) = (31) \quad \text{except in } S_{123} = S_{1234} \quad (\text{VI.10})$$

Finally consider (VI.5) in the set $\overline{S_{13}} \cap S_{34}$; then $p \in S_3$, $p \notin S_1$ and so $p < 0$; hence $p \notin S_2$, and (VI.5) reduces to

$$(34) = (43) \quad \text{except if } p \in S_{134} = S_{1234}. \quad (\text{VI.11})$$

Therefore there exist functions (1), (2), (3), (4), each with support in S_{1234} , such that

$$\frac{(12)-(21)}{2} = (1); \quad \frac{(24)-(42)}{2} = (2); \quad \frac{(13)-(31)}{2} = (3); \quad \frac{(34)-(43)}{2} = (4)$$

Define also $\frac{(ij)+(ji)}{2} = \overline{ij}$; then (VI.7) becomes

$$\begin{aligned} F_1 &= \overline{12} + \overline{13} + (1) + (3) \\ F_2 &= \overline{12} + \overline{24} - (1) + (2) \\ F_3 &= \overline{13} + \overline{34} - (3) + (4) \\ F_4 &= \overline{24} + \overline{34} - (2) - (4) \end{aligned} \quad (\text{VI.12})$$

Then (VI.5) implies $(1) + (3) - (2) - (4) = 0$. Define

$$(1) + (3) = f = (2) + (4); \quad (1) - (2) = g = (4) - (3),$$

where $\text{supp } f = \text{supp } g = S_{1234}$. Then define $\widehat{12} + f/2 - g/2 = \widehat{12}$; $\widehat{13} + f/2 + g/2 = \widehat{13}$; $\widehat{24} - g/2 - f/2 = \widehat{24}$; $\widehat{34} - f/2 + g/2 = \widehat{34}$; then $\text{supp } \widehat{ij} = S_{ij}$, and (VI.12) reads :

$$\begin{aligned} F_1 &= \widehat{12} + \widehat{13} \\ F_2 &= \widehat{12} + \widehat{24} \\ F_3 &= \widehat{13} + \widehat{34} \\ F_4 &= \widehat{24} + \widehat{34} \end{aligned} \tag{VI.13}$$

This is the most general solution ($\widehat{ij}$ arbitrary) satisfying the supports of F_i , and (VI.5). We now impose the condition of commutativity on $ABCD \dots$, and this leads to

$$(F_1 - F_2)(p, q, z) = 0 \quad \text{if } z^2 < 0 \tag{VI.14}$$

$$(F_1 - F_3)(x, q, r) = 0 \quad \text{if } x^2 < 0 \tag{VI.15}$$

$$(F_2 - F_4)(x, q, r) = 0 \quad \text{if } x^2 < 0 \tag{VI.16}$$

$$(F_1 - F_4)(x, q, z) = 0 \quad \text{if } x, z \sim 0 \tag{VI.17}$$

(VI.14) and (VI.15) imply

$$\widehat{13}(p, q, z) = \widehat{24}(p, q, z) \quad \text{if } z \sim 0 \tag{VI.18}$$

$$\widehat{12}(x, q, r) = \widehat{34}(x, q, r) \quad \text{if } x \sim 0 \tag{VI.19}$$

Using Dyson's theorem, the most general solution of (VI.18) and (VI.19) is

$$\widehat{13} - \widehat{24} = \int \phi_2(p, q, u, s) \delta_s(u-r) \epsilon(u-r) d^4 u dr \tag{VI.20}$$

where $\phi_2 = 0$ unless $p, q, q-p, q-u, u > 0$, and

$$\hat{1}_2 - \hat{3}_4 = \int \phi_3(q, r, u, s) \delta_s(u-p) \epsilon(u-p) d^4 u dr \quad (\text{VI.21})$$

where $\phi_3 = 0$ unless $q > r > 0$, $q > u > 0$; on substituting (VI.20), (VI.21) we find that (VI.16) and (VI.17) are satisfied. We are now in a position to write down the most general function with the following properties :

- (a) the function $\overline{ABCD} = 0$ unless $p, q, r > 0$
- (b) there exists a function (ABDC say) such that $\overline{ABDC} = 0$ unless $p, q, q-r > 0$, and $ABCD = ABDC$ if $z \sim 0$
- (c) there exists a function BACD such that $\overline{BACD} = 0$ unless $q-p, q, r > 0$, and $ABCD = BACD$ if $x \sim 0$
- (d) there exists a function BADC such that $\overline{BADC} = 0$ unless $q-p, q, q-r > 0$, and $ABCD = BADC$ if both $x, z \sim 0$. Such a function has the representation

$$\begin{aligned} \overline{ABCD}(p, q, r) = & \int \phi_1(u, q, v, s, t) \delta_s^r(p-u) \delta_t^r(r-v) d^4 u d^4 v ds dt + \\ & + \int \phi_3(q, r, u, s) \delta_s^r(p-u) d^4 z ds + \int \phi_2(p, q, u, s) \delta_s^r(r-u) du ds + \bar{\phi}_4(p, q, r) \end{aligned} \quad (\text{VI.22})$$

where

- (a) $\phi_1 = 0$ unless $q > u > 0$; $q > v > 0$
 - (b) $\phi_3 = 0$ unless $q > u > 0$; $q > r > 0$
 - (c) $\phi_2 = 0$ unless $q > p > 0$; $q > u > 0$
 - (d) $\bar{\phi}_4 = 0$ unless $q > p > 0$; $q > r > 0$
- (VI.23)

Taking the Fourier transform of (VI.22) we find as usual that it gives a function regular where $\phi_4(x, y, z)$ is, except for the cuts $x^2 > 0$, $z^2 > 0$. $\bar{\phi}_4(p, q, r)$ is an arbitrary Lorentz invariant function satisfying (VI.23) (d). The domain of regularity $\mathcal{U}$ of $\phi_4(x, y, z)$ is not known at present, but is not too difficult to characterise. If we ignore the difference between (I.2) and (I.3) as discussed in the introduction, we have found the envelope of holomorphy of

$$\mathcal{J}'_{ABCD} \vee \mathcal{J}'_{ABDC} \vee \mathcal{J}'_{BACD} \vee \mathcal{J}'_{BADC} \quad \text{as}$$

$$\mathcal{U} \cap \mathcal{J}'_i(x) \cap \mathcal{J}'_i(z)$$

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