The Double Commutator in Quantum Field Theory *)

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A representation is obtained for the most general function with the properties of the double commutator, including non-zero mass thresholds but not the Jacobi identities. The thresholds are proved to satisfy triangular inequalities (without using any more information) which are always true physically. The problem of incorporating a discrete level at the mass of a stable particle is not solved.

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The Double Commutator in Quantum Field Theory

In a previous paper¹⁾ we have obtained an integral representation for the double commutator $D(x,y) = \langle 0 | [A(x_1), [B(x_2), C(x_3)]] | 0 \rangle$, where $x = x_1 - x_2$, $y = x_2 - x_3$ and A, B, C are local fields. More exactly, the following theorem was proved.

Suppose $D(x,y)$ has the properties (1), (2), (3), (4) :

- (1) $D(x,y)$ is Lorentz invariant.
- (2) $D(x,y) = D(-x, -y)$.
- (3) $\bar{D}(p,q) = 0$ unless p, q are time-like or $p, p-q$ time-like
- (4) $D(x,y) = 0$ if $y^2 < 0$.

Then $D(x,y)$ has the representation

$$(5) \quad D(x,y) = \int_0^\infty e^{(\lambda - \sqrt{s}t)} \phi(s, t, \lambda; K) \Delta_2(x, x+y; s, t, \lambda) \Delta(y, k) ds dt d\lambda dK$$

for at least one ϕ .

$$\text{In (5), } \Delta_2(x, y; s, t, \lambda) = \int e^{-ipx - iqy} \delta(p^2 - s) \delta(p \cdot q - \lambda) \delta(q^2 - t) \epsilon(p_0) d^4 p dq^4.$$

The form (5) also has the additional property²⁾

- (6) $D(x,y) = 0$ if x and $x+y$ are space-like.

Eq.(6) therefore follows from (1)-(4), a result found by Symanzik³⁾ which is closely connected with analytic completion. The functions $ABC, ACB, \dots$ which make up D have some analytic properties¹⁾ which follow from (1), (2), (3), (4), leading to a result like (6).

Now we are interested in putting more information into the condition (3), namely, the knowledge of the thresholds of the mass spectra. Of the terms in the double commutator, ABC and CBA are zero unless $p^2 \geq m_1^2$, $q^2 \geq m_3^2$, and ACB, BCA are zero unless $p^2 \geq m_1'^2$, $(p-q)^2 \geq m_2^2$. We have given m_1 and m_1' different symbols, but later we prove that they must be the same. The masses are defined as

- (7) $m_1^2 = \min. \gamma^2$ such that $\langle 0 | A | \gamma \rangle \langle \gamma | BC | 0 \rangle \neq 0$
- (8) $m_1'^2 = \min. \gamma^2$ such that $\langle 0 | A | \gamma \rangle \langle \gamma | CB | 0 \rangle \neq 0$
- (9) $m_3^2 = \min. \gamma^2$ such that $\langle 0 | AB | \gamma \rangle \langle \gamma | C | 0 \rangle \neq 0$
- (10) $m_2^2 = \min. \gamma^2$ such that $\langle 0 | AC | \gamma \rangle \langle \gamma | B | 0 \rangle \neq 0$

If A, B and C are fields of particles, there will perhaps be discrete levels at these masses, the spectrum being zero again until the threshold of the continuum. We cannot deal with this property simply, and so ignore it. If A,B,C are local currents there is no discrete level. It is an interesting question whether derivability from a field places any restriction on the current. We take A,B,C to be local currents, and so replace (3) by

$$(11) \quad \bar{D}(p,q) = 0 \text{ unless } p^2 \gg m_1^2, q^2 \gg m_3^2 \text{ or } p^2 \gg m_1'^2, (p-q)^2 \gg m_2^2$$

$$(12) \quad \bar{D}(p,q) \neq 0 \text{ in this spectrum.}$$

It is conceivable that the properties (1), (2), (4) and (11) already imply that $\bar{D}(p,q)$ vanishes in a region larger than (12). We shall show that (12) is inconsistent with (1), (2), (4) and (11) unless

$$(13) \quad m_1 = m_1'$$

$$(14) \quad m_2 \leq m_1 + m_3; \quad m_3 \leq m_1 + m_2$$

Similarly using another commutator $[C, [A, B]]$ we could prove

$$(15) \quad m_1 \leq m_2 + m_3$$

This result is connected with a type of theorem first proved by Dyson⁴⁾, namely, that it is not possible to construct functions with arbitrary support and spectra.

Properties (13), (14) and (15) can be proved easily using further properties usually ascribed to the fields and Hilbert space. For example, if $\langle \gamma_1 | AC | 0 \rangle \neq 0$, then $\langle \gamma_1 | CA | 0 \rangle \neq 0$; any γ occurring in (7) also occurs in (8), and vice-versa. This proves (13). To prove (15) note that if $|\gamma_2\rangle$ and $|\gamma_3\rangle$ are such that $\langle 0 | B | \gamma_2 \rangle \neq 0$ and $\langle 0 | C | \gamma_3 \rangle \neq 0$, then they also satisfy $\langle 0 | AC | \gamma_2 \rangle \neq 0$ and $\langle 0 | AB | \gamma_3 \rangle \neq 0$ (given that ABC is not identically zero). Then the state $|\gamma_2, \gamma_3\rangle$ has the same quantum numbers as A. By $|\gamma_2, \gamma_3\rangle$ we mean the state obtained by adding the particles in state $|\gamma_2\rangle$ to those in $|\gamma_3\rangle$: $|\gamma_2, \gamma_3\rangle = |\gamma_2\rangle \wedge |\gamma_3\rangle$ (5). These particles can form a state with γ_2 and γ_3 parallel to each other.

We can also find a state $|\gamma_2\rangle$ such that $\gamma_2^2 = m_2^2$, and a state $|\gamma_3\rangle$ such that $\gamma_3^2 = m_3^2$, in which case $|\gamma_2, \gamma_3\rangle$ has mass $m_2 + m_3$ and satisfies $\langle 0 | A |\gamma_2, \gamma_3\rangle \neq 0$. Hence $|\gamma_2, \gamma_3\rangle$ is a state $|\gamma\rangle$ occurring in (7), and so $m_1 \leq m_2 + m_3$. Similarly the other inequalities (14) can be proved using physical arguments.

If A, B and C are fields, the inequalities (14) and (15) do not imply that they must describe stable particles; m_1 , m_2 and m_3 are not necessarily the "masses" of the fields A, B and C. This is because the lowest state connected to the field need not be a one-particle state (and cannot be if the field describes an unstable particle). The inequalities (14) will be needed below in the method adopted to include the mass-spectrum (11) into the representation (5). It is therefore of interest that (13), (14) follow from (1), (2), (4), (11) and (12) without using the arguments about Hilbert space.

Proof of (13).

Consider $\Theta(P_0)\bar{D}(p, q)$. This is zero unless $p^2 \geq m_1^2$ and $q^2 \geq m_3^2$, $q_0 > 0$, or $p^2 \geq m_1'^2$, $(p-q)^2 \geq m_2^2$, $p_0 > q_0$, and so may be $\frac{d}{d\lambda}$ composed (non-uniquely) into two functions $\overline{ABC}(p, q) - \overline{ACB}(p, q)$ such that

$$\overline{ABC}(p, q) = 0 \text{ unless } p^2 \geq m_1^2, q^2 \geq m_3^2, p_0, q_0 > 0$$

$$\overline{ACB}(p, q) = 0 \text{ unless } p^2 \geq m_1'^2, (p-q)^2 \geq m_2^2, p_0, p_0 - q_0 > 0, \text{ and}$$

$$(16) \int \overline{ABC}(p, q) e^{-iqy} d^4q = \int \overline{ACB}(p, q) e^{-iqy} d^4q \text{ if } y^2 < 0 \text{ using (4).}$$

Then if $p^2 < \max(m_1^2, m_1'^2)$ one or other side of (16) is identically zero as a function of y . For a given p , $\overline{ABC}(p, y)$ is regular in the backward tube $\text{Im } y < 0$, and $\overline{ACB}(p, y)$ is regular in the forward tube $\text{Im } y > 0$, as is seen by examining their spectra in q . But by (16) they are equal when $y^2 < 0$. Using the "Edge of the Wedge" theorem⁶⁾ they continue one another and are regular for the points $y^2 < 0$. If either side of (16) is zero for $y^2 < 0$ it must be zero everywhere, and so must the other side. So both sides are zero unless $p^2 \geq \max(m_1^2, m_1'^2)$, and in order to avoid a contradiction with (12) we must have $m_1 = m_1'$.

To prove (14) we apply Dyson's lemma⁴⁾ to the function $\bar{f} = \Theta(P_0) \bar{D}(p, q)$. Because of (4), $\bar{f}$ has the representation

$$(17) \quad \bar{f}(p, q) = \int \delta(p, u, s) \delta((q-u)^2 - s^2) \epsilon(q-u) d^4 u ds,$$

where $\delta(p, u, s) = 0$ unless $p > 0$, and unless the hyperbola in q -space, for fixed (K, u) given by

$$(18) \quad (q-u)^2 = K^2$$

lies entirely in the spectrum of $\bar{f}(p, q)$. The hyperbola (18) has two branches, one with $q_0 \rightarrow \infty$ and the other with $q_0 \rightarrow -\infty$. The first branch must lie entirely in $p, q > 0$ and the second entirely in $p, p-q < 0$ (the sign > 0 means "lies in the forward light cone"). Now $q = (q-u) + u$, and to be > 0 for all $q-u$ on the positive branch of (18) we must have $u > 0$. Since $q^2 = (q-u)^2 + 2u \cdot (q-u) + u^2$, we have

$$(19) \quad q^2 \geq K^2 + u^2 + 2K \sqrt{u^2}, \quad (q \text{ on the positive branch}).$$

Equality can be obtained in (19) for some q of (18). So if the positive branch of (18) is to lie in the set $q^2 \geq m_3^2$, $q_0 > 0$ for all q , we must have

$$(20) \quad K^2 + u^2 + 2K \sqrt{u^2} \geq m_3^2.$$

We also require that the negative branch of (18) should lie in the set $(p-q)^2 \geq m_2^2$, $p_0 > q_0$. Now $p-q = (p-u) - (q-u)$, and is forward time-like for all $q-u$ on the negative branch of (18) only if $p-u > 0$. Then using $(p-q)^2 = (p-u)^2 + K^2 - 2(p-u) \cdot (q-u)$, we get

$$(21) \quad (p-q)^2 \geq (p-u)^2 + 2K \sqrt{(p-u)^2 + K^2}, \quad (q \text{ on the negative branch})$$

equality in (21) holding for some points of (18). Thus to ensure $(p-q)^2 \geq m_2^2$ we require

$$(22) \quad (p-u)^2 + K^2 + 2K \sqrt{(p-u)^2 + K^2} \geq m_2^2.$$

Solving (20) and (22) for K we get

$$(23) \quad K \geq m_3 - \sqrt{u^2}$$

$$(24) \quad K \geq m_2 - \sqrt{(p-u)^2}$$

Therefore we have shown that ϕ in (17) can be chosen zero unless $p > 0$, $u > 0$, $p-u > 0$ and $K \geq \max \{0; m_3 - \sqrt{u^2}; m_2 - \sqrt{(p-u)^2}\}$. Because of (19) and (21), the spectrum will be too small unless both the extrema of K : $m_3 - \sqrt{u^2}$, $m_2 - \sqrt{(p-u)^2}$, are attained for some u . Thus the condition that $\bar{f}(p, q)$ is non-zero when $q^2 = m_3^2$ and $(p-q)^2 = m_2^2$ imposes the inequalities

$$(25) \quad m_3 - \sqrt{u^2} \geq m_2 - \sqrt{(p-u)^2} \text{ for some } u \text{ such that } p > u > 0$$

$$(26) \quad m_2 - \sqrt{(p-u)^2} \geq m_3 - \sqrt{u^2} \text{ for some } u \text{ such that } p > u > 0$$

The best case of (25) is when $u=0$, and of (26) is $u=p$, giving

$$\begin{aligned} m_3 &\geq m_2 - \sqrt{p^2} \\ m_2 &\geq m_3 - \sqrt{p^2} \end{aligned}$$

If $p^2 = m_1^2$ is a value of p where $\bar{f}(p, q) \neq 0$, we get the inequalities (14). To sum up, causality imposes triangular inequalities on the thresholds of the spectra of three fields if their product has non-zero expectation value.

Imposing Lorentz invariance on (17), we see that $\phi(u, p, k)$ can be chosen as a Lorentz invariant function zero unless $u, p-u \geq 0$ and $p^2 \geq m_1^2$. In general this has the form

$$(27) \quad \phi(u, p, k) = \int \psi(s, t, \lambda; k) \delta(u^2 - \lambda) \delta(\overline{p-u}^2 - t) \delta(u, \overline{p-u} - \lambda) \theta(u_0) ds dt d\lambda.$$

Since $p^2 = (p-u)^2 + u^2 + 2u \cdot \overline{p-u}$, a ψ can be chosen zero if $\lambda < \frac{1}{2}(m_1^2 - s - t)$. Finally, using (2), we get for the double commutator the representation (5), with the limits of integration

$$\begin{aligned} (28) \quad s, t &> 0 \\ \lambda &\geq \max \left(\sqrt{st}; \frac{1}{2}(m_1^2 - s - t) \right) \\ K &\geq \max \left(0; m_2 - t; m_3 - s \right) \end{aligned}$$

The author has proposed²⁾ integral representation for triple commutators, without the proof that they give the most general function. In the same way as in the present paper it is possible to incorporate the mass-spectrum provided certain inequalities are satisfied by the thresholds. It is worth noting that all these inequalities are satisfied in practice.

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