Some Integral Representations in Field Theory*

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A B S T R A C T

It is shown that the properties of positive mass and energy and of causality lead in quantum field theory to problems in the theory of analytic completion. A representation for the double commutator previously proved the most general is stated, and its connection with analytic completion noted. A similar representation for the triple commutator is proved to satisfy causality and the positive mass and energy condition, but is not shown to give all the functions with these properties. A conjecture about the envelopes of holomorphy of domains occurring in the discussion of analytic properties of the four-point Green's function is made. Integral representations for the n -fold multiple commutator are proposed, and for some other combinations of Wightman functions. These representations lead to some results in analytic completion.

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1. Introduction

In a theory of quantised fields $A(x)$, $B(x)$ we can replace¹⁾ the postulate that every state has non-negative mass and energy by a requirement of regularity on the basic functions of the theory, the vacuum expectation values of products of field operators, such as $\langle 0 | A(x_1) B(x_2) \dots C(x_n) | 0 \rangle$. This must be regular in $J'_{n-1}(y_1, \dots, y_{n-1})$ where $y_i = x_i - x_{i+1}$ and J'_{n-1} is a domain of holomorphy known as the extended tube. Further, all the functions obtained from the order $A B \dots C$ by a permutation must be regular in the 'permuted' extended tube obtained from $J'_{n-1}(y_1, \dots, y_{n-1})$ by the appropriate permutation of the x variables. Because of invariance under translations the functions depend only on the differences $y_i = x_i - x_{i+1}$.

The postulate of causality (local commutativity) is that

$$[A(x_1), B(x_2)] = 0 \quad (1.1)$$

if $x_1 - x_2$ is space-like, written $x_1 \sim x_2$. This implies that the expectation values $\langle 0 | A(x_1) B(x_2) C(x_3) | 0 \rangle$ and $\langle 0 | B(x_2) A(x_1) C(x_3) | 0 \rangle$ are equal if $y_1 \sim 0$, and, since some of these points are regular points of both functions, they continue one another as one regular function of the complex variables y_1, y_2 , the functions ABC and BAC for real y being obtained as the limit to various parts of the boundary. Jost²⁾ proved that $J'_n(y_1, \dots, y_n)$ contains the real points $y_1, \dots, y_n$ if and only if

$$\left(\sum_i \alpha_i y_i \right) \sim 0 \quad \text{for all } \alpha_i > 0 \quad (1.2)$$

Such points are called Jost points (J.P.). A refinement of the theory which is becoming important is to note the difference between local commutativity (1.1) and Jost point commutativity (J.P.C.), which requires the various permutations of a given order to be equal somewhere where they are regular, say at Jost points (1.2). This is a rather less strong assumption than that they should be equal for all space-like differences, as in (1.1). It follows from the work of Källén and Wightman that, for products of three fields, J.P.C. implies commutativity.

We shall conjecture a theorem below which implies this for the vacuum expectation value of a product of any number of fields, and therefore gives the operator condition (1.1), since, as Wightman has shown, a field theory is completely specified by its set of vacuum expectation values.

The postulate of J.P.C. implies that the $n!$ products of n fields are limits of one regular function in the union of the $n!$ permuted extended tubes, which are equal in pairs by the CPT theorem²⁾. Conversely, any function regular in this union satisfies J.P.C. But it can be shown that any function regular in such a domain D is also regular in a larger domain, the envelope of holomorphy of D (E.H. of D). Before progress in the theory can be made this envelope of holomorphy must be found. The methods available for finding envelopes of holomorphy, when applied to this problem¹⁾, lead to rather difficult manipulations in which there is no systematic way of proceeding; moreover the method used in¹⁾ is insufficient in principle to solve the problem for products of four operators. In a previous paper³⁾ the author showed that, by using the method of integral representations, a technique more suited to the special problem can be developed. Although the method is not sufficiently general to solve the whole problem in the three-fold case the treatment is simplified in certain parts. More important, the integral representation provided a systematic method for finding the envelope of holomorphy of certain domains.

The representation for the double commutator proved³⁾ previously can be generalised almost on sight to n -fold multiple commutators⁴⁾, but without the proof that the representation is the most general. It is hoped that, nevertheless, the formulae will prove to be of use as a guide to solving the problem in the general case.

2. The Double Commutator

In this section we state the representation for the double commutator proved in^{3),4)}, to be the most general.

Notation :

$$x=x_1-x_2, y=x_2-x_3, z=x_3-x_4; \quad x^2=x_0^2-x_1^2-x_2^2-x_3^2.$$

$\bar{D}(p,q) = \int e^{ipx+iqy} d^4x d^4y D(x,y)$. If $x^2 > 0$, $x_0 > 0$ we shall write $x > 0$. If $x^2 > 0$, $x_0 < 0$ we write $x < 0$. If $x^2 < 0$ we write $x \sim 0$. $A(x_1)$, $B(x_2)$, $C(x_3)$, ... will be shortened to A, B, C ...

Theorem³⁾

Suppose $D(x,y) = \langle 0 | [A(x_1), [B(x_2), C(x_3)]] | 0 \rangle$ has the following properties :

$$D(x,y) \quad \text{is a Lorentz invariant tempered distribution} \quad (2.1)$$

$$D(x,y) = D(-x,-y) \quad (2.2)$$

$$D(x,y) = 0 \quad \text{if } y \sim 0 \quad (2.3)$$

$$\bar{D}(p,q) = 0 \quad \text{if } p \sim 0 \quad \text{or if } q \text{ and } p-q \sim 0. \quad (2.4)$$

Then $D(x,y)$ has the representation

$$D(x,y) = \int_0^\infty \phi(s,t,\lambda;K) \Delta_2(x,x+y; s,t,\lambda) \Delta_1(y;K) ds dt d\lambda dK \quad (2.5)$$

for some ϕ .

We remark that the double commutator has the properties (2.1) - (2.4), and therefore has the form (2.5). The double commutator, and also the representation (2.5), has the additional property

$$D(x,y) = 0 \quad \text{if } x \text{ and } x+y \sim 0. \quad (2.6)$$

Therefore (2.6) is implied by (2.1) - (2.4).

In (2.5), we define

$$\Delta_2(x,y; s,t,\lambda) = \int \delta(p^2-s^2) \delta(p,q-\lambda) \delta(q^2-t^2) \epsilon(p_0)_e^{-i(px+qy)} d^4 p d^4 q \quad (2.7)$$

That (2.5) has the property (2.6) is proved similarly to the proof of (3.5) below.

Let us for the moment assume that J.P.C. and commutativity are equivalent. It has been shown³⁾ that from the representation (2.5) we can construct the most general form for a function regular in $J'_2(x,y) \cup J'_2(x+y,-y)$ and this function turns out to be regular in $J'_2(x,x+y) \cap J'_1(y)$, but may have singularities on the boundary. Therefore, any function regular in $J'_2(x,y) \cup J'_2(x+y,-y)$ is regular also in $J'_2(x,x+y) \cap J'_1(y)$, but, in general, not in a larger domain. This is exactly what is meant by the statement 'The envelope of holomorphy of $J'_2(x,y) \cup J'_2(x+y,-y)$ ' (2.8)

is $J'_2(x,x+y) \cap J'_1(y)$.

As explained in³⁾ this result can be obtained also by the technique of¹⁾, where the assumption of the equivalence of J.P.C. and commutativity is not assumed, but is proved from (2.8) as follows: Suppose $F(x,y)$ is the regular function in question.

Then $ABC(x,y) = \lim_{\epsilon \rightarrow 0} F(x+i\epsilon, y+i\epsilon); \epsilon \text{ time-like}, \epsilon_0 > 0;$

$ACB(x,y) = \lim_{\epsilon \rightarrow 0} F(x+y+i\epsilon, -y-i\epsilon); \epsilon \text{ time-like}, \epsilon_0 > 0.$

Now if $\text{Im } x, \text{Im}(x+y) > 0, y$ real, the only singularities of the domain $J'_2(x,x+y) \cap J'_1(y)$ are when $y^2 > 0$. Hence if $y \sim 0$ we have $ABC(x,y) = ACB(x,y)$, $\text{Im } x > 0$.

Taking the limit $\text{Im } x \rightarrow 0$ we get $\langle 0 | A[B,C] | 0 \rangle = 0, y \sim 0$, that is, the fields appear to commute. In³⁾ integral representations were proved for

$$\langle 0 | AB \dots C[D,E] | 0 \rangle$$

which were used to prove that, assuming J.P.C. implies commutativity,

$$EH \left\{ J'_n(x_1, \dots, x_n) \cup J'_n(x_1, \dots, x_{n-1}+x_n, -x_n) \right\} = J'_n(x_1, \dots, x_{n-1}, x_{n-1}+x_n) \cap J'_1(x_n) \quad (2.9)$$

In turn if (2.9) is true, J.P.C. implies local commutativity.

3. A Representation for the Triple Commutator

The natural generalisation of $\Delta_2(x,y)$ is

$$\Delta_3(x,y,z;m_{ij}) = \int \prod_{i,j}^3 d^4 p_j e^{-ip_k \cdot x_k} \delta(p_i \cdot p_j - m_{ij}) \in (p_{10}) \quad (3.1)$$

a function of 3 four-vectors x_1, x_2, x_3 and six 'mass' parameters $m_{11}, m_{12}, \dots$ which sometimes will be collectively denoted by m . This function is zero somewhere in x -space, and has a Fourier transform which is zero somewhere in p -space. As a generalisation of (2.5) we are led to consider the following class of functions

$$T(x,y,z) = \int \delta(m_{ij}; s, t, \lambda; k) dm ds dt d\lambda \Delta_3(x, x+y, x+y+z; m_{ij}) \Delta_2(y, y+z; s, t, \lambda) \Delta_1(z; k) \quad (3.2)$$

Theorem

Functions of the form (3.2) satisfy the following properties

(3.3) - (3.6)

$$T(x,y,z) \text{ is Lorentz invariant} \quad (3.3)$$

$$T(x,y,z) = -T(-x, -y, -z) \quad (3.4)$$

$$T(x,y,z) = 0 \text{ if } z \sim 0 \text{ or if both } y \text{ and } y+z \sim 0 \quad (3.5)$$

$$\bar{T}(p,q,r) = 0 \text{ unless } p,q,r > 0 \text{ or } p,q,r < 0 \text{ or} \quad (3.6)$$

$$p,q,q-r > 0 \text{ or } p,q,q-r < 0 \text{ or } p,p-q, p-q+r > 0 \text{ or}$$

$$p,p-q, p-q+r < 0 \text{ or } p,p-q, p-r > 0 \text{ or } p,p-q, p-r < 0.$$

Before proving the theorem we note that these are properties of

$$T(x,y,z) = \langle 0 | [A(x_1), [B(x_2), [C(x_3), D(x_4)]]] | 0 \rangle$$

$$\text{if } x=x_1-x_2, \quad y=x_2-x_3 \quad \text{and} \quad z=x_3-x_4.$$

Proof of (3.3)

Each of the kernels $\Delta_3(x, x+y, x+y+z; m_{ij}), \Delta_2(y, y+z; s, t, \lambda)$ and $\Delta(z; k)$ is Lorentz invariant for any value of the parameters, and so the product is Lorentz invariant.

Proof of (3.4)

The kernels are antisymmetric under space-time reversal, for example $\Delta_3(x, y, z) = -\Delta_3(-x, -y, -z)$ and so the product is, for any value of the parameters.

Proof of (3.5)

The functions $\Delta_3(x, y, z; m_{ij}), \dots$ are simply related to the 'generalised singular functions' $\Delta^{(+)}$ of Källén and Wilhelmsson (see ref. ⁵), for example

$$\Delta(x, y, z; m_{ij}) = \Delta^{(+)}(x+i\epsilon, y+i\epsilon, z+i\epsilon; m_{ij}) - \Delta^{(+)}(x-i\epsilon, y-i\epsilon, z-i\epsilon; m_{ij})$$

where ϵ is time-like. If $m_{ij} > 0$ then $\Delta^{(+)}(p, q, r; m_{ij})$ is non-zero only if $p, q, r > 0$. Hence $\Delta^{(+)}(x, y, z)$ is regular in the forward tube. Hence we can apply the 'Edge of the Wedge' theorem ⁶, which says that

$$\Delta^{(+)}(x+i\epsilon, y+i\epsilon, z+i\epsilon; m_{ij}) - \Delta^{(+)}(x-i\epsilon, y-i\epsilon, z-i\epsilon; m_{ij}) = 0 \text{ at } x, y, z$$

if and only if $\Delta^{(+)}(x, y, z)$ is regular at x, y, z , and this for each m_{ij} .

So that $T(x, y, z) = 0$ if $z \sim 0$, or if y, z is a real regular point of

$\Delta^{(+)}(y, y+z; s, t, \lambda)$ for all 'physical' s, t, λ . But the points y, z which lie in the domain of regularity of $\Delta^{(+)}(y, y+z; s, t, \lambda)$ for all physical s, t, λ make up the extended forward tube ⁷ $J_2'(y, y+z)$ and the real points of $J_2'(y, y+z)$ are known to be Jost points, such that $y + \lambda z \sim 0$ for all in the interval $[0, 1]$. We have therefore proved that

$$\Delta(y, y+z; s, t, \lambda) = 0 \text{ for all 'physical' } s, t, \lambda \text{ if and only if}$$

$$y + \lambda z \sim 0 \text{ for all } \lambda \text{ in } [0, 1].$$

We have to show that if y and $y+z \sim 0$, then $T(x,y,z) = 0$. Sufficient for this is $\Delta(y,y+z;s,t,\lambda) = 0$ for all s,t,λ , and so, from what we have just shown, $T(x,y,z) = 0$ if $y+\lambda z \sim 0$ for λ in $[0,1]$. We may take $z^2 > 0$, since otherwise $T(x,y,z) = 0$ because of the factor $\Delta_k(z)$. Suppose y and $y+z \sim 0$, and the contrary, that there exists λ in $[0,1]$ such that $(y+\lambda z)^2 > 0$. We will show that this leads to a contradiction. There are 4 cases :

$$(1) \quad z > 0, \quad y+\lambda z > 0. \quad \text{Then} \quad y+z = (y+\lambda z) + (1-\lambda)z > 0$$

$$(2) \quad z > 0, \quad y+\lambda z < 0. \quad \text{Then} \quad y = (y+\lambda z) - \lambda z < 0$$

$$(3) \quad z < 0, \quad y+\lambda z > 0. \quad \text{Then} \quad y = (y+\lambda z) - \lambda z > 0$$

$$(4) \quad z < 0, \quad y+\lambda z < 0. \quad \text{Then} \quad y+z = (y+\lambda z) + (1-\lambda)z < 0$$

Here we have used the 'Convexity of the Light Cone'. If $x > 0, y > 0$ then $x+\lambda y > 0$ for all $\lambda, \mu > 0$. In each case we obtain a contradiction of the property y and $y+z$ space-like. This proves (3.5)

Proof of (3.6)

We use the definitions of the antisymmetric invariant functions

$$\Delta_1(z;k) = \int \delta(\alpha^2 - k^2) \epsilon(\alpha_0) e^{-i\alpha z} d^4\alpha$$

$$\Delta_2(y,y+z;s,t,\lambda) = \int \delta(\beta_1^2 - s^2) \delta(\beta_1\beta_2 - \lambda) \delta(\beta_2^2 - t^2) \epsilon(\beta_{10}) e^{-i(\beta_1 y + \beta_2(y+z))} d^4\beta_1 d^4\beta_2$$

$$\Delta_3(x,x+y,x+y+z;m_{ij}) = \int \prod_{i \leq j}^3 \delta(\gamma_i \gamma_j - m_{ij}) \epsilon(\gamma_{10}) d^4\gamma_i e^{-i(\gamma_1 x + \gamma_2(x+y) + \gamma_3(x+y+z))}$$

$$\text{Then } \tilde{T}(p,q,r) = \int d^4m_{ij} ds dt d\lambda dk \int e^{ipx+iqy+irz} d^4x d^4y d^4z \int d^4\alpha d^4\beta_i d^4\gamma_i \times$$

$$\times \prod_{i \leq j}^3 \delta(\alpha^2 - k^2) \epsilon(\alpha_0) \delta(\beta_1^2 - s^2) \delta(\beta_1\beta_2 - \lambda) \delta(\beta_2^2 - t^2) \epsilon(\beta_{10}) \delta(\gamma_i \gamma_j - m_{ij}) \epsilon(\gamma_{10}) \times$$

$$\times \exp \left\{ -i(\alpha z + \beta_1 y + \beta_2(y+z) + \gamma_1 x + \gamma_2(x+y) + \gamma_3(x+y+z)) \right\} \quad (3.7)$$

The x, y, z integrations give δ^4 -functions which restrict the integration over the momenta $\alpha, \beta_i, \gamma_i$ to the following region

$$p = \gamma_1 + \gamma_2 + \gamma_3; \quad q = \gamma_2 + \gamma_3 + \beta_1 + \beta_2; \quad r = \gamma_3 + \beta_2 + \alpha \quad (3.8)$$

If the parameters $m_{ij}, s^2, t^2, \lambda, k$ are positive, then the functions

$$\delta(\alpha^2 - k^2) \epsilon(\alpha_0), \delta(\beta_1^2 - s^2) \delta(\beta_1 \cdot \beta_2 - \lambda) \delta(\beta_2^2 - t^2) \epsilon(\beta_{10}) \text{ and } \prod_{i \leq j}^3 (\gamma_i \gamma_j - m_{ij}) \epsilon(\gamma_{10})$$

are non-zero in two separate regions $\alpha > 0$ or $\alpha < 0$; β_1 and $\beta_2 > 0$ or β_1 and $\beta_2 < 0$;

γ_1, γ_2 and $\gamma_3 > 0$ or γ_1, γ_2 and $\gamma_3 < 0$ respectively. We have excluded the regions $\alpha \sim 0, \beta_i \sim 0$ etc. by choosing $k^2, s^2, t^2, m_{ii} > 0$.

The region $\beta_1 > 0, \beta_2 < 0$ is excluded because $\beta_1 \cdot \beta_2 = \lambda > 0$. The product

$$\left\{ \delta(\alpha^2 - k^2) \epsilon(\alpha_0) \right\} \left\{ \delta(\beta_1^2 - s^2) \delta(\beta_1 \cdot \beta_2 - \lambda) \delta(\beta_2^2 - t^2) \epsilon(\beta_{10}) \right\} \prod_{i \leq j}^3 \delta(\gamma_i \cdot \gamma_j - m_{ij}) \epsilon(\gamma_{10})$$

is therefore zero outside the union of the following sets

$$R_{(1)} \alpha > 0; \beta_i > 0; \gamma_i > 0 \quad R_{(2)} \alpha < 0; \beta_i > 0; \gamma_i > 0$$

$$R_{(3)} \alpha > 0; \beta_i < 0; \gamma_i > 0 \quad R_{(4)} \alpha < 0; \beta_i < 0; \gamma_i > 0$$

$$R_{(5)} \alpha > 0; \beta_i > 0; \gamma_i < 0 \quad R_{(6)} \alpha < 0; \beta_i > 0; \gamma_i < 0$$

$$R_{(7)} \alpha > 0; \beta_i < 0; \gamma_i < 0 \quad R_{(8)} \alpha < 0; \beta_i < 0; \gamma_i < 0$$

$$\text{Let } S_1 \equiv (p, q, r > 0); \quad S_2 \equiv (p, q, q-r > 0); \quad S_3 \equiv (p, p-q, p-q+r > 0);$$

$$S_4 \equiv (p, p-q, p-r > 0); \quad S_5 \equiv (p, p-q, p-r < 0); \quad S_6 \equiv (p, p-q, p-q+r < 0);$$

$$S_7 \equiv (p, q, q-r < 0); \quad S_8 \equiv (p, q, r < 0).$$

Using (3.8) we now show that if $\alpha, \beta, \gamma \in R_i$ then $p, q, r \in S_i$.

(1) $\alpha, \beta, \gamma \in R_1$. Then

$$p = \gamma_1 + \gamma_2 + \gamma_3 > 0; \quad q = \gamma_2 + \gamma_3 + \beta_1 + \beta_2 > 0; \quad r = \gamma_3 + \beta_2 + \alpha > 0.$$

Hence $p, q, r \in S_1$.

(2) $\alpha, \beta, \gamma \in R_2$. Then $q-r = \beta_1 + \gamma_2 - \alpha > 0$; so $p, q, r \in S_2$.

(3) $\alpha, \beta, \gamma \in R_3$; then $p > 0$; $p-q = \gamma_1 - \beta_1 - \beta_2 > 0$; $p-q+r = \gamma_1 + \gamma_3 - \beta_1 + \alpha > 0$;
so $p, q, r \in S_3$.

(4) $\alpha, \beta, \gamma \in R_4$; then $p > 0$, $p-q > 0$ as in (3); $p-r = \gamma_1 + \gamma_2 - \beta_2 - \alpha > 0$.

Hence $p, q, r \in S_4$.

Similarly if $\gamma_i < 0$ we can prove that if $\alpha, \beta, \gamma \in R_i$ then $p, q, r \in S_i, i=5, \dots, 8$. This proves that the spectrum of $T(x, y, z)$ is $\bigcup_{i=1}^8 S_i$, since the α, β, γ vary over $\bigcup_{i=1}^8 R_i$, subject to (3.8). This proves (3.6).

We now conjecture the converse to this theorem, namely, that every function with the properties (3.3) - (3.6) has the representation (3.2). This is a reasonable conjecture in view of (2.5) and the theorem which follows it. Functions of the form (3.2) also vanish in a larger region than (3.5). Because of the factor $\Delta(x, x+y, x+y+z; m_{ij})$, $T=0$ if $x, x+y, x+y+z$ are J.P. But functions of the form (3.2) do not in general²⁾ vanish if $x, x+y$ and $x+y+z$ are space-like, which is a property of the triple commutator. Also, these functions do not satisfy the algebraic relations corresponding to the Jacobi identities. If (2.9) is true, the incorporation of the Jacobi identities would automatically ensure that T satisfied the further property: $T(x, y, z) = 0$ if $x, x+y$ and $x+y+z$ are space-like.

Unlike the double commutator, the triple commutator has such a complicated overlap of spectra that the various terms ABCD, ABDC,, cannot be separated cleanly, and so that even if (3.2) is the most general form we do not obtain immediately the envelope of holomorphy of the four extended tubes of ABCD, ABDC, ACDB, ADCB. To do this we would have to find the most general form for a function regular in

$$T = J_3^I(x, y, z) \cup J_3^I(x, y+z, -z) \cup J_3^I(x+y, z, -y-z) \cup J_3^I(x+y+z, -y, -z).$$

We can construct examples of functions regular in T . Consider

$$f(x,y,z) = \int \phi(m_{ij}, s, t, \lambda; k) \Delta_3^{(+)}(x, x+y, x+y+z; m_{ij}) \Delta_2^{(+)}(y, y+z; s, t, \lambda) \Delta^{(+)}(z; k) \quad (3.9)$$

The function (3.9) is the Fourier transform of a function like (3.7), but with $\epsilon(\alpha_0) \epsilon(\beta_{10}) \epsilon(\gamma_{10})$ replaced by $\theta(\gamma_{10}) \theta(\pm\alpha_0) \theta(\pm\beta_{10})$

the sign $\pm$ depending on the sign of the imaginary parts of x, y, z . As was shown in the proof of (3.6), taking one of the signs $\pm\alpha_0, \pm\beta_{10}$ restricts $\bar{f}(p, q, r)$ to lie in one of the sets

$p, q, r > 0$; $p, q, q-r > 0$; $p, p-q, p-q+r > 0$; $p, p-q, p-r > 0$, and so (3.9) is regular in a tube conjugate to one of these. So we have constructed four functions, regular respectively in $J'_3(x, y, z)$, $J'_3(x, y+z, -z)$, $J'_3(x+y, z, -y-z)$ and $J'_3(x+y+z, -y, -z)$. The difference between, for example, the first two is the Fourier transform of (3.7) with $\theta(\gamma_{10}) \theta(\beta_{10}) \epsilon(\alpha_0)$ in place of $\epsilon(\gamma_{10}) \epsilon(\beta_{10}) \epsilon(\alpha_0)$ and this is zero somewhere in x -space. Hence the functions continue one another. This proves that (3.9) is regular in T . The singularities of (3.9) lie on the boundary of

$$J'_3(x, x+y, x+y+z) \cap J'_2(y, y+z) \cap J'_1(z) \quad (3.10)$$

We have proved that in general a function regular in T can have singularities on the boundary of (3.10), and so the envelope of holomorphy of T is contained in (3.10). By analogy with (2.8) we can conjecture that (3.10) is the actual EH of T i.e.

$$\begin{aligned} \text{EH} \left\{ J'_3(x, y, z) \cup J'_3(x, y+z, -z) \cup J'_3(x+y, z, -y-z) \cup J'_3(x+y+z, -y, -z) \right\} \\ = J'_3(x, x+y, x+y+z) \cap J'_2(y, y+z) \cap J'_1(z) \end{aligned} \quad (3.11)$$

Eq.(3.11) is a useful 'Ansatz' with which to start the analysis of the 4-fold expectation values.

4. Other Integral Representations

The natural generalisations of the representations for $D(x,y)$ and $T(x,y,z)$ would be

$$F_n(y_1, \dots, y_{n-1}) = \int \delta(m_{ij}; \dots; k) dm_{ij} \dots dk \Delta(y_{n-1}, k) \dots \dots \Delta_{n-1}(y_1, y_1+y_2, y_1+y_2+y_3, \dots; m_{ij}) \quad (4.1)$$

where

$$F(y_1, \dots, y_{n-1}) = \langle 0 | [A(x_1), [B(x_2), \dots, [D(x_{n-1}), E(x_n)] \dots]] | 0 \rangle \quad (4.2)$$

It is a remarkable fact, proved in⁴⁾ that (4.1) has the spectrum of the multiple commutator (4.2). For $n=4$ this was proved in the previous section. The representation (4.1) does not satisfy all the consequences of causality, because it is not zero in x -space in as large a region as is the right hand side of (4.2). But (4.1) does satisfy the consequences of J.P.C., and would satisfy causality if the Jacobi identities could be incorporated. It is important to note that these results are merely algebraic, and that the analytic problem of showing that these are the most general has not been solved. In the same way as (3.11) was derived we can get a conjecture for the envelope of holomorphy of 2^{n-2} extended tubes at once in the n -fold case, the tubes being those which occur in the multiple commutator. In³⁾ it was shown that in the 3-fold case, by dealing with the three functions in pairs, and then combining the results, we can follow through the working of¹⁾ up to a certain point. To finish the problem by this method would require the Jacobi identities. In the four fold case we can deal with only 4 out of 12 at once, and these must be the special 4 occurring in a triple commutator. The effect of the Jacobi identity is therefore more important.

Spectral representations for a number of other forms for the commutator brackets were proposed in⁴⁾

$$\langle 0 | [[A, B], [C, D]] | 0 \rangle = \int \phi(k, m, n_j) \Delta_4(y+z, y, x+y+z, x+y; n_j) \times \\ \times \Delta(x; k) \Delta(z; n) dm dn_j dk \quad (4.3)$$

$$\langle 0 | [[A, B], [C, [D, E]]] | 0 \rangle = \int \phi \Delta(x; \ell) \Delta(w, n) \Delta_2(z, z+w; n) d\ell dm dn \times \\ \times \Delta_6(x+y, x+y+z, x+y+z+w, y, y+z, y+z+w) \quad (4.4)$$

Here A has argument x_1 , B has argument x_2 , etc. and $w = x_4 - x_5$.

In⁴⁾ we also proposed representations (not proved the most general) for the difference of any two 4-fold functions. As one example

$$ABCD - BDCA = P(x^2) \int \phi(m, n) \Delta_4(x, x+y, z, x+y+z; n) \times \Delta_2^{(+)}(y, y+z; m) dm dn \quad (4.5)$$

where $P(x^2)$ is some polynomial. Then consider the two functions

$$ABCD = P(x^2) \int \phi dm dn \Delta_2^{(+)}(y, y+z; m) \Delta_4^{(+)}(x+i\epsilon, x+y+i\epsilon, z+i\epsilon, x+y+z+i\epsilon; n) \\ BDCA = P(x^2) \int \phi dm dn \Delta_2^{(+)}(y, y+z; m) \Delta_4^{(+)}(x-i\epsilon, x+y-i\epsilon, z-i\epsilon, x+y+z-i\epsilon; n)$$

These are regular respectively in the extended tubes $J_3'(x, y, z)$ and $J_3'(x+y, -y-z, z)$, and since they are equal somewhere in x -space, they continue one another into the union

$$J_3'(x, y, z) \cup J_3'(x+y, -y-z, z) \quad (4.6)$$

But it is known that the singularities of $\Delta_2^{(+)}(y, y+z; m)$ and $\Delta_4^{(+)}(x, x+y, z, x+y+z; n)$ as m, n vary over the physical region, lie outside the domains $J_2'(y, y+z)$ and $J_4'(x, x+y, z, x+y+z)$ and so (4.5) defines functions regular in

$$J_4'(x, x+y, z, x+y+z) \cap J_2'(y, y+z) \quad (4.7)$$

Therefore

$$EH \left\{ J_3'(x, y, z) \cup J_3'(x+y, -y-z, z) \right\} \subset J_4'(x, x+y, z, x+y+z) \cap J_2'(y, y+z) \quad (4.8)$$

R E F E R E N C E S

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