General construction of noiseless networks detecting entanglement with help of linear maps

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We present the general scheme for construction of noiseless networks detecting entanglement with the help of linear, hermiticity-preserving maps. We show how to apply the method to detect entanglement of unknown state without its prior reconstruction. In particular, we prove there always exists noiseless network detecting entanglement with the help of positive, but not completely positive maps. Then the generalization of the method to the case of entanglement detection with arbitrary, not necessarily hermiticity-preserving, linear contractions on product states is presented.

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The problem and general overview.- It has been known that entanglement can be detected with help of special class of maps called positive maps [1, 2, 3]. In particular there is an important criterion [1] saying that (positivity-preserving) map \( \Lambda \) of un–physical map \( \tilde{\Lambda} \): usual equation

\[
[I \otimes \Lambda](\varrho) \geq 0,
\]

where \( I \) is the identity map acting on \( B(H_A) \). Since any positivity-preserving map is also hermiticity-preserving, it makes sense to speak about eigenvalues of \( X_A(\varrho) \). However, it should be emphasized that there are many \( \Lambda \)s (and equivalently the corresponding criteria) and to characterize them is a hard and still unsolved problem (see e.g. 4 and references therein).

For a long time the above criterion has been treated as purely mathematical. One used to take matrix \( \varrho \) (obtained in some prior state estimation procedure) and then put it into the formula 1. Then its spectrum was calculated and the resulting conclusion was drawn. However it can be seen that for, say states acting on \( H_A \otimes H_B \cong C^d \otimes C^d \) and maps \( \Lambda : B(C^d) \rightarrow B(C^d) \), the spectrum of the operator \( X_A(\varrho) \) consists of \( n_{\text{spec}} = d^2 \) elements, while full prior estimation of such states corresponds to \( n_{\text{est}} = d^4 - 1 \) parameters.

The question was risen 3 whether one can perform the test 1 physically without need of prior tomography of the state \( \varrho \) despite the fact that the map \( I \otimes \Lambda \) is not physically realizable. The corresponding answer was 2 that one can use the notion of structural physical approximation \( I \otimes \Lambda \) (SPA) of un–physical map \( I \otimes \Lambda \) which is physically realizable already, but at the same time the spectrum of the state \( \tilde{X}_A(\varrho) = [I \otimes \Lambda](\varrho) \) is just an affine transformation of that of the (un–physical) operator \( X_A(\varrho) \). The spectrum of \( \tilde{X}_A(\varrho) \) can be measured with help of the spectrum estimator 3, which requires estimation of only \( d^2 \) parameters which (because of affinity) are in one to one correspondence with the needed spectrum of \( \tilde{\Lambda} \). Note that for \( 2 \otimes 2 \) systems (the composite system of two qubits), similar approaches leads to the method of detection of entanglement measures (concurrency 4 and entanglement of formation 5) without the state reconstruction 6.

The disadvantage of the above method is 10 that realizalization of SPA requires adition the noise to the system (we have to put some controlled ancillas, couple the system, and then trace them out). In ref. 10 the question was risen about the existence of noiseless quantum networks, i.e. those of which the only input data are: (i) unknown quantum information represented by \( \varrho^m \) (ii) controlled measured qubit (which reproduces us the spectrum moments - see 6). It has been shown that for at least one positive map (transposition) \( T \) the noiseless network exists 17. Such networks for two-qubit concurrency and three-qubit tangle have also been designed 11.

In the present paper we ask a general question is that so that noiseless networks work only for special maps (functions) or do they exist for any positive map test? In case of positive answer to the latter: is it possible to design general method for constructing them? Can it be adopted to any criteria other than the one defined in 11?

For this purpose we first show how to measure a spectrum of the matrix \( \Theta(\varrho) \), where \( \Theta : B(C^m) \rightarrow B(C^m) \) is an arbitrary linear, hermiticity-preserving map and \( \varrho \) is a given hermitian operator acting on \( C^m \), with the help of only \( m \) parameters estimated instead of \( m^2 - 1 \). For bipartite \( \varrho \) where \( m = d^2 \) this gives \( d^2 \) instead of \( d^4 - 1 \). Then we provide an immediate application in entanglement detection showing that for suitable \( \Theta \) the scheme constitutes just a right method for detecting entanglement without prior state reconstruction with the help of either positive map criteria 11 or linear contraction methods discussed later.

General scheme for construction of noiseless network detecting spectrum of \( \Theta(\varrho) \).- Since \( m \times m \) matrix \( \Theta(\varrho) \) is
hermitian its spectrum may be calculated using only m numbers
\[ \alpha_k \equiv \text{Tr}[\Theta(\rho)]^k = \sum_{i=1}^{m} \lambda_i^k \quad (k = 1, \ldots, m), \]
where \( \lambda_i \) are eigenvalues of \( \Theta(\rho) \). We shall show that
all these spectrum moments can be represented by mean values of special observable. To this aim let us consider
the permutation operator \( V^{(k)} \) defined by the formula
\[ V^{(k)}|e_1\rangle|e_2\rangle \otimes \ldots \otimes |e_k\rangle = |e_k\rangle|e_1\rangle \otimes \ldots \otimes |e_{k-1}\rangle, \]
where \( (k = 1, \ldots, m) \) and \( |e_i\rangle \) are vectors from \( \mathbb{C}^m \). One can see that \( V^{(1)} \) is just an identity operator \( \mathbb{I}_m \) acting
on \( \mathbb{C}^m \). Combining eqs. (2) and (3) we infer that \( \alpha_k \) may be expressed by relation
\[ \alpha_k = \frac{1}{2} \text{Tr} \left[ (V^{(k)} + V^{(k)})^\dagger \Theta(\rho)^{\otimes k} \right]. \]
which is generalization of the formula from [5, 6] where \( \Theta \) was (unlike here) required to be a physical operation.
Notice that these numbers may be obtained using hermitian its spectrum may be calculated using only
numbers
\[ | \text{tr} \left[ \Theta(\rho)^{\otimes k} \right] | \]
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which is generalization of the formula from [5, 6] where \( \Theta \) was (unlike here) required to be a physical operation.
Notice that these numbers may be obtained using hermitian conjugation of \( V^{(k)} \) which again is a permutation
operator but permutes states \( |e_i\rangle \) in the reversed order. Therefore all the numbers \( \alpha_k \) may be expressed as
\[ \alpha_k = \frac{1}{2} \text{Tr} \left[ (V^{(k)} + V^{(k)})^\dagger \Theta(\rho)^{\otimes k} \right]. \]
Let us focus for a while on the map \( \Theta \). Due to its positivity-preserving property it may be expressed as
\[ \Theta(\cdot) = \sum_j \eta_j K_j(\cdot) K_j^\dagger \]
with \( \eta_j \in \mathbb{R} \) and \( K_j \) being linearly independent matrices. By the virtue of this fact and some
well-known properties of trace, after rather straightforward algebra we may rewrite eq. (4) as
\[ \alpha_k = \frac{1}{2} \text{Tr} \left\{ \left[ \Theta^{\otimes k} \left( V^{(k)} \right)^\dagger \right] + \left[ \Theta^{\otimes k} \left( V^{(k)} \right)^\dagger \right] \right\}. \]
Here we have applied a map \( \Theta^{\otimes k} : \mathcal{B}(\mathbb{C}^m)^{\otimes k} \rightarrow \mathcal{B}(\mathbb{C}^m)^{\otimes k} \) on the operator \( V^{(k)} \) instead of applying it
on \( \rho^{\otimes k} \). This apparently purely mathematical trick with the aid of the fact that the square brackets in the above
contain hermitian operator allows us to express numbers \( \alpha_k \) as a mean values of some observables in state \( \rho^{\otimes k}. \)
Indeed, introducing
\[ 0^{(k)}_{\Theta} = \frac{1}{2} \left[ \left( \Theta^{\otimes k} \left( V^{(k)} \right)^\dagger \right) + \left( \Theta^{\otimes k} \left( V^{(k)} \right)^\dagger \right) \right] \]
we arrive at
\[ \alpha_k = \text{Tr} \left[ 0^{(k)}_{\Theta} \rho^{\otimes k} \right]. \]
In general, a naive measurement of all mean values
would require estimation of much more parameters that \( m \). But there is a possibility of building a unitary network
that requires estimation of exactly \( m \) parameters using the idea that we recall and refine below.

**Detecting mean of an observable by measurement on a single qubit revised.** Let \( A \) be a arbitrary observable
(it may be even infinite dimensional) which spectrum lies between finite numbers \( a_\text{min}^A \) and \( a_\text{max}^A \) and \( \sigma \) be a state
acting on \( \mathcal{H} \). In ref. [12] it has been pointed out that the mean value \( \langle A \rangle_\sigma = \text{Tr}(A\sigma) \) may be estimated in process
involving the measurement of only one qubit. This fact is in good agreement with further proof that single qubits
may serve as interfaces connecting quantum devices [13]. Below we recall the mathematical details of the measure-
ment proposed in ref. [12]. At the beginning one defines the following numbers
\[ a_\text{A}^{(-)} \equiv \max[0, -a_\text{A}^{\text{min}}], \quad a_\text{A}^{(+)} \equiv a_\text{A}^{(-)} + a_\text{A}^{\text{max}}, \]
and observe that the hermitian operators
\[ V_0 = \sqrt{a_\text{A}^{(-)} 1_{\mathcal{H}} + A \sigma}, \quad V_1 = \sqrt{1_{\mathcal{H}} - V_0^\dagger V_0} \]
satisfy \( \sum_{i=0}^{1} V_i^\dagger V_i = 1_{\mathcal{H}} \) and as such define generalized quantum measurement which can easily be extended
to a unitary evolution (see Appendix A of ref. [14] for detailed description). Consider a partial isometry on the
Hilbert space \( \mathbb{C}^2 \otimes \mathcal{H} \) defined by formula
\[ \tilde{U}_A = \sum_{i=0}^{1} |i\rangle \langle 0| \otimes V_i = \begin{pmatrix} V_0 & 0 \\ V_1 & 0 \end{pmatrix}. \]
The first Hilbert space \( \mathbb{C}^2 \) represents the qubit which shall be measured in order to estimate the mean value
\( \langle A \rangle_\sigma \). The partial isometry can always be extended to unitary \( U_A \) such that if it acts on \( |0\rangle \otimes \sigma \) then the final
measurement of observable \( \sigma \) on the first (qubit) system gives probabilities ”spin-up” (of finding it in the state \( |0\rangle \)) and ”spin-down” (of finding in state \( |1\rangle \)), respectively of the form
\[ p_0 = \text{Tr} \left( V_0^\dagger V_0 \sigma \right), \quad p_1 = \text{Tr} \left( V_1^\dagger V_1 \sigma \right) = 1 - p_0. \]
One of the possible extensions of \( \tilde{U}_A \) to the unitary on \( \mathbb{C}^2 \otimes \mathcal{H} \) is the following
\[ U_A = \begin{pmatrix} V_0 & -V_1 \\ V_1 & V_0 \end{pmatrix} = 1_{\mathcal{H}} \otimes V_0 - i \sigma_y \otimes V_1. \]
The unitarity of \( U_A \) follows from the fact that operators \( V_0 \) and \( V_1 \) commute. Due to the practical reasons instead
of unitary operation representing POVM \( \{V_0, V_1\} \) we shall consider
\[ U^{\text{det}}(A, U_{\mathcal{H}}) = (1_{\mathcal{H}} \otimes U_{\mathcal{H}}) U_A (1_{\mathcal{H}} \otimes U_{\mathcal{H}})^\dagger, \]
where \( 1_{\mathcal{H}} \) is an identity operator on one-qubit space \( \mathbb{C}^2 \) and \( U_{\mathcal{H}} \) is an arbitrary unitary operation that acts on \( \mathcal{H} \).
and simplifies the decomposition of $U_A$ into elementary gates. Now if we define mean value of measurement of $\sigma_z$ on the first qubit after action of the network (which sometimes may be called visibility):

$$v_A = \text{Tr} \left[ (\sigma_z \otimes 1_H) (I_2 \otimes U'_H) U_A P_0 \otimes \sigma U_A^\dagger (I_2 \otimes U'_H)^\dagger \right],$$

where $P_0 = |0\rangle \langle 0|$, then we have an easy formula for the mean value of the initial observable $A$:

$$\langle A \rangle_\sigma = a_A^{(+)} p_0 - a_A^{(-)} = a_A^{(+)} v_A + \frac{1}{2} - a_A^{(-)}.$$  

A general scheme of a network estimating the mean value $\langle A \rangle_\sigma$ is provided in Fig. 1.

We put an additional unitary operation on the bottom wire after unitary $U_A$ (which does not change the statistics of the measurement on control qubit) and divided identity operator into two unitaries acting on that wire which explicitly shows how simplification introduced in eq. 13 works in practice.

Having the general network estimating $v_A$, one needs to decompose an isometry $U_A$ onto elementary gates. One of possible ways to achieve this goal is, as we shall see below, to diagonalize the operator $V_0$. Hence we may choose $U_H'$ (see eq. 13) to be $U_H' = \sum_k |k\rangle \langle \phi_k|$, with $|\phi_k\rangle$ being normalized eigenvectors of $V_0$ indexed by a binary number with length $2^k$. Since $V_0$ and $V_1$ commutes, this operation diagonalizes $V_1$ as well. By virtue of these facts, eq. 13 reduces to

$$U_{\text{det}}(A, U_{H'}) = \sum_k U_k \otimes |k\rangle \langle k|,$$

with unitaries (as previously indexed by a binary number) $U_k = \sqrt{\lambda_k} k_2 - i \sqrt{1 - \lambda_k} \sigma_z$, where $\lambda_k$ are eigenvalues of $V_0$. So in fact we have a combination of operations on the first qubit controlled by $2^k$ wires. All this combined gives us the network shown in the Fig. 2.

Now we are in the position to combine all the elements presented so far an show how, if put together, they provide the general scheme for constructing noiseless network for spectrum of $\Theta(\varrho)$ for a given quantum state $\varrho$.

For the sake of clarity below we itemize all steps necessary to obtain the spectrum of $\Theta(\varrho)$: (i) Take all observables $\mathcal{O}^{(k)}$ defined by eq. 6. (ii) Construct unitary operations $U_{\varrho}^{(k)}$ according to the given prescription. Consider the unitary operation $U_{\text{det}}(A, U_H')$ ($U'_H$ arbitrary). Find decomposition of the operation into elementary quantum gates and minimize the number of gates in the decomposition with respect to $U'_H$. Build the (optimal) network found in this way. (iii) Act with the network on initial state $P_0 \otimes \varrho^{\otimes k}$. (iv) Measure the „visibilities“ $v_{\mathcal{O}^{(k)}}$ according to 13. (v) Using 13 calculate the values of $\alpha_k$ representing the moments of $\Theta(\varrho)$.

**Detecting entanglement with networks.** The first obvious application of the presented scheme is entanglement detection via positive but not completely positive maps. In fact for any bipartite state $\varrho \in \mathcal{B}(H_A \otimes H_B)$ we only need to substitute $\Theta$ with $I_A \otimes \Lambda_B$ with $\Lambda_B$ being some positive map. Then application of the above scheme immediately reproduces all the results of the schemes from 5 but without additional noise added (presence of which required more precision in measurement of visibility).

As an illustrative example consider $\Lambda_B = T$, i.e. $\Theta$ is partial transposition on the second subsystem (usually denoted by $T_B$), in $2 \otimes 2$ systems. Due to normalization of $\varrho$ we need only three numbers $\alpha_k$, ($k = 2, 3, 4$) measurable via observables $\mathcal{O}^{(2)}_{T_B} = V_1^{(2)} \otimes V_2^{(2)}$ and

$$\mathcal{O}^{(3,4)}_{T_B} = \frac{1}{2} \left( V_1^{(3,4)} \otimes V_2^{(3,4)} + V_1^{(3,4)} \otimes V_2^{(3,4)} \right),$$

where subscripts mean that we exchange first and second subsystems respectively. For simplicity we show only the network measuring second moment of $\varrho^{T_B}$. General scheme from Fig. 2 reduces then to the scheme from Fig. 6. Note that the network can also be regarded as a one measuring purity of state as $\text{Tr}(\varrho^{T_B})^2 = \text{Tr} \varrho^2$. Note that the this network is not optimal since alternative network measuring $\text{Tr} \varrho^2$ requires two controlled swaps.

**Extension to linear contractions criteria.** The above approach may be generalized to the so-called linear contractions criteria. To see this let us recall that the powerful criterion called computable cross norm (CCN) or ma-
matrix realignment criterion has recently been introduced [15, 16]. This criterion is easy to apply (involves simple permutation of matrix elements) and has been shown [17, 16] to be independent on PPT test [2]. It has been further generalized to linear contractions criterion [17] which we shall recall below. If by \( R_{\alpha}(\cdots \otimes H_{A_n}) \) we denote density matrices acting on Hilbert spaces \( H_{A_i} \) and by \( H \) certain Hilbert space, then for some linear map \( \mathcal{R} : \mathcal{B}(H_{A_1} \otimes \cdots \otimes H_{A_n}) \rightarrow \mathcal{B}(H) \) we have the following

**Theorem 17.** If some \( R \) satisfies

\[
\| \mathcal{R}(\varphi_{A_1} \otimes \varphi_{A_2} \otimes \cdots \otimes \varphi_{A_n}) \|_{\text{Tr}} \leq 1,
\]

then for any separable state \( \varphi_{A_1 A_2 \cdots A_n} \in \mathcal{B}(H_{A_1} \otimes \cdots \otimes H_{A_n}) \) one has

\[
\| \mathcal{R}(\varphi_{A_1 A_2 \cdots A_n}) \|_{\text{Tr}} \leq 1.
\]

The maps \( \mathcal{R} \) satisfying (18) are linear contractions on product states and hereafter they shall be called, in brief, linear contractions. In particular, the separability condition (19) comprises the generalization of the realignment test to permutation criteria [17, 18] (see also [11]).

The noisy network for entanglement detection with the help of the latter have been proposed in [20]. Here we improve this result in two ways, namely, by taking into account all maps \( \mathcal{R} \) of type (18) (not only permutation maps) and introducing the corresponding noiseless networks instead of noisy ones. For these purposes we need to generalize the lemma from ref. [20] formulated previously only for real maps \( S : \mathcal{B}(H) \rightarrow \mathcal{B}(H) \). We represent action of \( S \) on any \( \varphi \in \mathcal{B}(H) \) as

\[
S(\varphi) = \sum_{i,j,k,l} S_{i,j,k,l} \text{Tr}(\varphi P_{ij}) P_{kl},
\]

where in Dirac notation \( P_{xy} = |x\rangle \langle y| \). Let us define complex conjugate of the map \( S \) via complex conjugates of its elements, i.e., \( S^*(\varphi) = \sum_{i,j,k,l} S_{i,j,k,l}^\ast \text{Tr}(\varphi P_{ij}) P_{kl}^\ast \). Then we have the following lemma which is easy to proof by inspection:

**Lemma.** Let \( S \) be an arbitrary linear map on \( \mathcal{B}(H) \). Then the map \( S' = [T \circ S^* \circ T] \) satisfies \( S'(\varphi) = |S(\varphi)|^2 \).

Suppose then we have \( \mathcal{R} \) satisfying (18) and a given physical source producing copies of a system in state \( \varphi \) for which we would like to check (19). Let us observe that \( \| \mathcal{R}(\varphi) \|_{\text{Tr}} = \sum_{i} \gamma_i \) where \( \gamma_i \) are eigenvalues of the operator \( X_{\mathcal{R}}(\varphi) = \mathcal{R}(\varphi) \mathcal{R}(\varphi)^\dagger \). Below we show how to find the spectrum of \( \gamma_i \). We need to apply our previous scheme from to the special case. Let us define the map \( \mathcal{L}_{\mathcal{R}} = \mathcal{R} \otimes \mathcal{R}' \) where \( \mathcal{R} \) is our linear contraction and \( \mathcal{R}' \) is defined according to the prescription given in the Lemma above, i.e., \( \mathcal{R}' = [T^\ast \circ \mathcal{R}^* \circ T] \). Let us also put \( \varphi' = \varphi^\otimes 2 \) and apply the scheme presented above to detect the spectrum of \( \mathcal{L}_{\mathcal{R}}(\varphi') \). It is easy to see that the moments detected in that way are \( \text{Tr}[\mathcal{R}(\varphi')]^k = \text{Tr}[(\mathcal{R}(\varphi) \mathcal{R}(\varphi))^\dagger]^k = \sum \gamma_i^k \). From the moments one easily reconstructs \( \gamma_i \) and check violation of (19).

**Summary.** We have shown how to detect spectrum of the operator \( \Theta(\varphi) \) for arbitrary linear hermiticity-preserving map \( \Theta \) given the source producing copies of the system in state \( \varphi \). The network involved in the measurement is noiseless in sense of [10] and the measurement is required only on the controlled qubit. Further we have shown how to apply the method to provide general noiseless network scheme of detection detecting entanglement with the help of criteria belonging to one of two classes, namely, those involving positive maps and applying linear contractions on product states.

The structure of the proposed networks is not optimal and needs further investigations. Here however we have been interested in quite a fundamental question which is interesting by itself: Is it possible to get noiseless networks schemes for any criterion from one of the above classes? Up to now their existence was known only for special case of positive partial transpose (cf. [11]). Here we have provided a positive answer to the question.

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[21] Here $B(H_i)$ denotes the set of bounded operators sending elements of $H_i$ onto itself ($i = A, B$).
[22] By $1_H$ we denote an identity operator on $H$.
[23] We use standard Pauli matrices, i.e. $\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\sigma_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$, $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$. 