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CURRENT ALGEBRAS AND MESON SYSTEMS

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A B S T R A C T

We apply the algebra of compact $U(6) \times U(6)$, obtained from integrated current components, to meson systems. If we limit the sums over intermediate states to the lowest lying 36 $SU(3)$ invariant pseudoscalar and vector meson physical states at rest, with the masses of the pseudoscalar and vector mesons non-degenerate, then we obtain a prediction for the mean-squared radius of the pseudoscalar meson

$$\langle r_\pi^2 \rangle \approx 5.4 \frac{1}{M_V^2}$$

where M_V is the vector meson mass. This value is almost an order of magnitude too large and represents a real discrepancy. On the other hand if we consider all the masses of the meson 36-plet equal then we find the reasonable prediction $\langle r_\pi^2 \rangle \approx 1/\bar{m}^2$, where $\bar{m}$ is the mean meson mass.

The poor result obtained with the 36-plet of physical states suggests that either the assumed algebra must be modified to include the effects of mass splittings or that the algebra is not "saturated" by a single irreducible representation of the algebra and therefore higher intermediate states must be considered.

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I. INTRODUCTION

Following a proposal of Gell-Mann ¹⁾, the applications of current algebras to symmetry problems have been studied by several authors ²⁾. Although these methods have not yet reached full development, they seem to offer considerable promise. In particular, Dashen and Gell-Mann ²⁾ and B. Lee ²⁾ have used the algebra of non-chiral $U(6) \times U(6)$ to study baryon systems at rest, where they recovered several of the well-known predictions of $SU(6)$ in addition to new ones. In this method the algebra does not generate a symmetry of the theory, but rather provides a way of finding relationships between matrix elements. States required to "saturate" the algebra should form irreducible (or reducible) representations of the algebra, but since the algebra does not necessarily imply a symmetry, the states should be physical states with their physical masses.

Here we apply the algebra $U(6) \times U(6)$ to meson systems at rest, with particular attention to possible difficulties caused by mass differences within an irreducible representation of the algebra. For the example studied, we find a poor prediction for $\langle r_\pi^2 \rangle$ if the pseudo-scalar mesons and vector mesons are not degenerate in mass, however, this difficulty disappears if these masses are taken to be degenerate. This suggests that if one limits the sum over intermediate states to the lowest meson 36-plet of physical states, then one will have to modify the algebra to account for mass splittings. Another way of stating this result is that the sum over intermediate states is not "saturated" by a single irreducible representation of the algebra and therefore higher intermediate states must be included. This type of difficulty is not completely new ³⁾, however it is instructive to see it again in the meson system.

II. MATRIX ELEMENTS BETWEEN MESON STATES

We wish to consider the algebra $U(6) \times U(6)$, generated by the equal time commutation of integrated current components, and its application to meson systems. Following previous authors ^{1),2)}, we consider the space integrals of vector and axial vector currents

$$\begin{aligned} V_{\mu}^{(\lambda)}(t) &= -i \int d^3x j_{\mu}^{(\lambda)}(\vec{x}, t), \\ A_{\mu}^{(\lambda)}(t) &= -i \int d^3x j_{\mu}^{5(\lambda)}(\vec{x}, t), \end{aligned} \quad (1)$$

which generates a $U(6) \times U(6)$ algebra at equal times. It will be convenient to use the notation of B. Lee ²⁾; the reader is urged to consult that paper for additional notational details. Furthermore, the phase conventions of de Swart ⁴⁾ are followed throughout. The commutation relation between the space components of two $A_{\mu}^{(\lambda)}$ is given by ²⁾

$$\begin{aligned} [A_i^{(\lambda)}, A_j^{(\mu)}] &= -i \varepsilon_{ijk} \frac{4}{\sqrt{3}} \begin{pmatrix} 8 & 8 & 1 \\ \lambda & \mu & 0 \end{pmatrix} A_k^{(0)} \\ &\quad + i \varepsilon_{ijk} \sqrt{\frac{5}{3}} \begin{pmatrix} 8 & 8 & 8_s \\ \lambda & \mu & \nu \end{pmatrix} - \delta_{ij} \sqrt{3} \begin{pmatrix} 8 & 8 & 8_a \\ \lambda & \mu & \nu \end{pmatrix} V_0^{(\nu)}, \end{aligned} \quad (2)$$

where $i=1,2,3$ and $\lambda, \mu, \nu = 0, \dots, 8$. Another useful commutation relation ²⁾ can be obtained from the magnetic moment operators

$$\begin{aligned}
\mathcal{M}_i^{(\lambda)} &= \frac{1}{2} \int d^3x \varepsilon_{ijk} \chi_j \dot{\chi}_k^{(\lambda)}(\vec{x}, t). \\
4[\mathcal{M}_i^{(\lambda)}, \mathcal{M}_j^{(\mu)}] &= i \varepsilon_{ijk} \int d^3x \chi_k \dot{\chi}_\ell \dot{\chi}_\ell^{(v)} \sqrt{\frac{5}{3}} \begin{pmatrix} 8 & 8 & 8_s \\ \lambda & \mu & \nu \end{pmatrix} \\
&\quad - \sqrt{3} \begin{pmatrix} 8 & 8 & 8_a \\ \lambda & \mu & \nu \end{pmatrix} \left\{ \frac{2}{3} \int d^3x x^2 \delta_{ij} \dot{\chi}_0^{(v)} \right. \\
&\quad \left. + \frac{1}{3} \int d^3x [\chi^2 \delta_{ij} - 3\chi_i \chi_j] \dot{\chi}_0^{(v)} \right\} + \dots, \tag{3}
\end{aligned}$$

with an omitted term depending on $A_k^{(0)}$. Relations following from Eq. (3) may be rather interesting since they are outside the closed $U(6) \times U(6)$ algebra.

Let us begin by considering matrix elements of equation (3) between an octet of pseudoscalar mesons at rest. In the sum over intermediate states we insert a 36-plet of pseudoscalar and vector meson states. The matrix elements that we have to consider are

$$\begin{aligned}
\langle M_\alpha | \mathcal{M}_i^{(\lambda)} | M_\beta \rangle &= 0, \\
\langle M_\alpha | \int d^3x \chi_k (\dot{\chi}_\ell \dot{\chi}_\ell^{(v)}) | M_\beta \rangle &= 0, \\
\langle M_\alpha | \int d^3x [\chi^2 \delta_{ij} - 3\chi_i \chi_j] \dot{\chi}_0^{(v)} | M_\beta \rangle &= 0, \tag{4} \\
\langle M_\alpha | \int d^3x \chi^2 \dot{\chi}_0^{(v)}(x) | M_\beta \rangle \\
&= \sqrt{3} \langle \pi^2 \rangle \begin{pmatrix} 8 & 8 & 8_a \\ \beta & \nu & \alpha \end{pmatrix} \delta^3(\vec{q}),
\end{aligned}$$

and

$$\begin{aligned}
\langle M_\alpha | \mathcal{M}_i^{(\lambda)} | V_\mu^{(\nu)} \rangle \\
= \varepsilon_i \delta^3(\vec{q}) \left[G_8 \begin{pmatrix} 8 & 8 & 8_s \\ \sigma & \lambda & \alpha \end{pmatrix} + G_1 \begin{pmatrix} 8 & 8 & 1 \\ \sigma & \lambda & \alpha \end{pmatrix} (-1)^{Q_\lambda} \right].
\end{aligned}$$

throughout $|M_\alpha\rangle$ will indicate a pseudoscalar meson state with $SU(3)$ quantum numbers α , $|V_i^{(\gamma)}\rangle$ a vector meson state where ε_i is the vector meson polarization and γ its $SU(3)$ quantum numbers, $q_\mu = (p_2 - p_1)_\mu$ where p_2 and p_1 are the initial and final meson four-momenta, and Q_λ is the charge of state λ . The solution to such a system of equations is not unique. However, one may supplement these equations with the considerations of Gell-Mann ⁵⁾, and consider equation (3) as an equation for the effective magnetic moment operator. Then one finds the solution

$$\begin{aligned} G_8^2 &= \frac{5}{18} \langle \pi^2 \rangle, \\ g_1^2 &= \frac{8}{9} \langle \pi^2 \rangle. \end{aligned} \quad (5)$$

We can then give a prediction for the rate of the process $V \rightarrow M + \gamma$ by considering the Lagrangian

$$\begin{aligned} \mathcal{L} = & \frac{1}{2M_V} \varepsilon_{\alpha\beta\mu\lambda} F_{\alpha\beta}^{(\sigma)}(x) \partial_\mu B_\lambda^{(\tau)}(x) \phi^{(\kappa)}(x) \\ & \times \left[G_{V\gamma M} \begin{pmatrix} 8 & 8 & 8_s \\ \tau & \kappa & \sigma \end{pmatrix} + g_{V\gamma M} \begin{pmatrix} 8 & 8 & 1 \\ \sigma & \bar{\kappa} & \tau \end{pmatrix} (-1)^{Q_\kappa} \right], \end{aligned} \quad (6)$$

where $\phi^{(\kappa)}(x)$ is the meson field, $B_\lambda^{(\tau)}(x)$ is the vector meson field, $G_{V\gamma M}$ and $g_{V\gamma M}$ are coupling constants, M_V the vector meson mass, and

$$F_{\alpha\beta}^{(\sigma)} = \partial_\alpha A_\beta^{(\sigma)} - \partial_\beta A_\alpha^{(\sigma)}$$

which is in terms of the electromagnetic field $A_\mu^{(\sigma)}$ [σ being an $SU(3)$ index]. Furthermore [suppressing $SU(3)$ indices] we can write for the effective magnetic moment operator

$$\begin{aligned} \langle M | \mathcal{M}_{\alpha\beta} | V_\lambda \rangle &= \frac{1}{i} \int d^3x \langle M | \mu_{\alpha\beta}(\vec{x}, t) | V_\lambda \rangle \\ &= \frac{1}{i} (2\pi)^3 \delta^3(\vec{0}) \langle M | \mu_{\alpha\beta}(0, t) | V_\lambda \rangle. \end{aligned} \quad (7)$$

Using Equations (6) and (7), we find for states at rest

$$\langle M^{(\sigma)} | \mathcal{M}_K^{(\delta)} | V_K^{(\lambda)} \rangle = \frac{\delta^3(\vec{q})}{4\sqrt{M_V m}} \left[G_{VM}^{\gamma}(q^2) \begin{pmatrix} \delta & \delta & \delta_5 \\ \lambda & \sigma & \delta \end{pmatrix} + g_{VM}^{\gamma}(q^2) \begin{pmatrix} \delta & \delta & 1 \\ \delta & \sigma & \lambda \end{pmatrix} (-1)^{Q_0} \right] \Big|_{q^2 = (M_V - m)^2} \quad (8)$$

The notation is that $G_{VM}^{\gamma}(q^2)$ is the form factor of the $V \rightarrow M + \gamma$ vertex with the photon off the mass shell normalized so that $G_{V\gamma M} = G_{VM}^{\gamma}(0)$. [We will use a particle label as a superscript to indicate that the particle is off the mass shell for the vertex being considered.] The pseudoscalar meson mass is m .

By comparing Eqs. (5) and (8) we obtain

$$\left[\frac{G_{VM}^{\gamma}(q^2)}{4\sqrt{M_V m}} \right]_{q^2 = (M_V - m)^2}^2 = \frac{5}{18} \langle r_{\pi}^2 \rangle. \quad (9)$$

Since one believes that $\langle r_{\pi}^2 \rangle \simeq 1/M_V^2$, one would have a prediction of this rate if one ignored the dependence of $G_{VM}^{\gamma}(q^2)$ on q^2 . However, this is probably a dubious thing to do. If one considers the form factor to be dominated by vector mesons, then we can estimate

$$G_{VM}^{\gamma}(q^2) = \frac{G_{MVV} \gamma_{V\gamma}}{M_V^2 - q^2}, \quad (10)$$

which then gives

$$G_{VM}^{\gamma}(q^2) = \frac{M_V^2}{M_V^2 - q^2} G_{V\gamma M}, \quad (11)$$

where $G_{V\gamma M}$ is the coupling constant determined experimentally by radiative decays. The experimental situation for $V \rightarrow M + \gamma$ is sufficiently vague, so that an accurate comparison of the prediction of (9) with experiment must await more accurate determinations of the rates ⁶⁾.

6.

By using the conserved vector current (CVC) hypothesis, one can convert (9) into a prediction for $V \rightarrow M+V$. Assuming vector mesons are universally coupled,

$$G_{VM}^{\gamma}(q^2) = \left(\frac{M_V^2}{M_V^2 - q^2} \right) G_{VVM}, \quad (12)$$

where G_{VVM} is the coupling constant for $V \rightarrow M+V$ in the Lagrangian

$$\mathcal{L} = \frac{1}{2M_V} \epsilon_{\alpha\beta\mu\lambda} (\partial_{\alpha} B_{\beta}^{(\pi)}(x) (\partial_{\mu} B_{\lambda}^{(\pi)}(x)) \phi^{(\kappa)}(x) \\ \times \left[G_{VVM} \begin{pmatrix} 8 & 8 & 8_S \\ 7 & 8 & \sigma \end{pmatrix} + g_{VM} \begin{pmatrix} 8 & 8 & 1 \\ 6 & \bar{K} & 7 \end{pmatrix} (-1)^{q_K} \right]. \quad (13)$$

Thus, from (5), (9), (12), and (13), we can obtain a prediction for the rates $V \rightarrow V+M$. With the identification

$$\phi = \sqrt{\frac{2}{3}} \omega_8 - \sqrt{\frac{1}{3}} \omega_1, \\ \omega = \sqrt{\frac{1}{3}} \omega_8 + \sqrt{\frac{2}{3}} \omega_1, \quad (14)$$

the rate $\Gamma(\phi \rightarrow \rho + \pi) \equiv 0$. The other decay $\omega \rightarrow 3\pi$ can be related to $\omega \rightarrow \rho + \pi$ by the Gell-Mann, Sharp, and Wagner model ⁷⁾. We prefer to avoid this comparison with experiment as it involves both $(\omega - \phi)$ mixing and a model for $\omega \rightarrow 3\pi$. The reader can check these predictions as an exercise if he so desires.

We now show that we can get an independent prediction of G_{VVM} if we consider the matrix elements of equation (2) between vector meson states at rest and use the idea of partially conserved axial vector currents (PCAC) ⁸⁾. We will then be able to eliminate the coupling constant from (9) and arrive at a prediction for $\langle r_{\pi}^2 \rangle$ only in terms of masses. We again insert a 36-plet of meson states in the intermediate sum of (2). The matrix elements required are

$$\langle M_8 | A_j^{(\mu)} | V_j^{(\beta)} \rangle = f \varepsilon_j \begin{pmatrix} 8 & 8 & 8_a \\ \beta & \mu & \gamma \end{pmatrix} \delta^3(\vec{k}),$$

$$\langle V_\ell^{(\alpha)} | V_0^{(\nu)} | V_m^{(\beta)} \rangle = \sqrt{3} \begin{pmatrix} 8 & 8 & 8_a \\ \beta & \nu & \alpha \end{pmatrix} \delta_{\ell m} \delta^3(\vec{k}),$$

(15)

and

$$\langle V_\ell^{(\alpha)} | A_i^{(\lambda)} | V_n^{(\beta)} \rangle =$$

$$i \delta^3(\vec{k}) \varepsilon_{i\ell n} (\varepsilon_\ell^+ \varepsilon_n) \left[G_{VVA} \begin{pmatrix} 8 & 8 & 8_s \\ \gamma & \lambda & \alpha \end{pmatrix} + g_{VVA} \begin{pmatrix} 8 & 8 & 1 \\ \alpha & \lambda & \gamma \end{pmatrix} (-1)^{Q_\lambda} \right],$$

where $k = (p_f - p_i)$.

A solution to the equations is given by

$$\begin{aligned} G_{VVA}^2 &= \frac{5}{3/6} \\ g_{VVA} &= \frac{1}{3} \\ f^2 &= 3 \end{aligned} \quad (16)$$

The coupling G_{VVA} can now be related to G_{VVM} by PCAC. From (1) :

$$\langle \beta | A_i | \alpha \rangle = \frac{1}{i} (2\pi)^3 \delta^3(\vec{k}) \langle \beta | j_i^5(0) | \alpha \rangle \quad (17)$$

We assume that the axial vector current satisfies ⁸⁾

$$\partial_\mu j_\mu^5(\lambda) = c \phi(\lambda) \quad (18)$$

Using (13), the matrix elements of the meson current are

8.

$$\begin{aligned} \langle V_\beta^{(\sigma)} | J_M^{(\tau)} | V_\alpha^{(\rho)} \rangle &= \frac{1}{(2\pi)^3} \frac{\epsilon_{\alpha\beta\gamma\lambda}}{\sqrt{2\omega_2 2\omega_1}} \frac{\epsilon_{\rho\sigma\mu\nu}}{M_V} \epsilon_\beta^\dagger \epsilon_\alpha (p_{1\lambda} p_{2\mu}) \\ &\times \left[G_{VV}^M(k^2) \begin{pmatrix} 8 & 8 & 8_S \\ \pi & \gamma & \sigma \end{pmatrix} + g_{VV}^M(k^2) \begin{pmatrix} 8 & 8 & 1 \\ \beta & \bar{\sigma} & \alpha \end{pmatrix} (-1)^{\alpha\beta} \right] \end{aligned} \quad (19)$$

where we normalize $G_{VV}^M(k^2=m^2) \equiv G_{VVM}$.

In the usual way :

$$G_{VV}^A(k^2) = \frac{i C G_{VV}^M(k^2)}{2 M_V (k^2 - m^2)} \quad (20)$$

From (16) and (20) :

$$[G_{VV\pi}]^2 = - \frac{20}{3} \left[\frac{G_{VV}^M(k^2=m^2)}{G_{VV}^M(0)} \right]^2 \frac{(M_V m^2)^2}{C^2} \quad (21)$$

However, from (9) and (12) we also have

$$\langle \pi^2 \rangle = \frac{9}{40} \left(\frac{M_V^2}{2 M_V m - m^2} \right)^2 \frac{(G_{VV\pi})^2}{M_V m} \quad (22)$$

so that

$$\langle \pi^2 \rangle = \frac{3}{2} \frac{1}{M_V m} \left(\frac{M_V^2}{2 M_V m - m^2} \right)^2 \frac{(M_V m^2)^2}{(-C^2)} \left[\frac{G_{VV\pi}}{G_{VV}^M(0)} \right]^2 \quad (23)$$

We evaluate C by taking matrix elements of (18) between baryon states at rest. This calculation is quite well known⁸⁾. If we take the $SU(6)$ value for the axial vector coupling constant²⁾, and take $f_{\rho\pi\pi} = f_{\sigma\pi\pi}$ ⁹⁾, then we obtain

$$C = -i \frac{5}{3} M m^2 \left(\frac{G_{BBM}}{G_{BB}^M(0)} \right) \left(\frac{f_{\rho\pi\pi}}{g_{N\pi\pi}} \right), \quad (24)$$

where M is the baryon mass, $g_{N\pi\pi}$ the pion-nucleon coupling ($g_{N\pi\pi}^2/4\pi \simeq 15$), and $f_{\rho\pi\pi}$ the $\rho \rightarrow \pi + \pi$ coupling.

Thus, from (23) and (24) :

$$\langle \Lambda_{\pi}^2 \rangle = \frac{27}{50} \frac{1}{M_V m} \left(\frac{M_V^2}{2M_V m - m^2} \right)^2 \left(\frac{M_V}{M} \right)^2 \left(\frac{g_{N\pi\pi}}{f_{\rho\pi\pi}} \right)^2 \left[\frac{G_{BB}^M(0)}{G_{BBM}} \right]^2 \left[\frac{G_{VVM}}{G_{VV}^M(0)} \right]^2 \quad (25)$$

It is probably reasonable to approximate

$$\left[\frac{G_{BB}^M(0)}{G_{BBM}} \right]^2 \left[\frac{G_{VVM}}{G_{VV}^M(0)} \right]^2 \simeq 1 \quad (26)$$

so finally

$$\langle \Lambda_{\pi}^2 \rangle \simeq \frac{27}{50} \frac{1}{M_V m} \left(\frac{M_V^2}{2M_V m - m^2} \right)^2 \left(\frac{M_V}{M} \right)^2 \left(\frac{g_{N\pi\pi}}{f_{\rho\pi\pi}} \right)^2, \quad (27)$$

One estimates $f_{\rho\pi\pi}$ from $\rho \rightarrow \pi\pi$ to be $f_{\rho\pi\pi}^2/4\pi \simeq 2.5$ ⁹⁾. If we choose $m = 470$ MeV, $M_V = 825$ MeV, and $M = 1120$ MeV, then

$$\langle \Lambda_{\pi}^2 \rangle \simeq 5.4 \frac{1}{M_V^2}. \quad (28)$$

This result is much too large and should be considered a real discrepancy. To illustrate how this depends on the physical masses rather let the 36 meson-plet have the same mass, setting $m = M_V = \bar{m}$ and take $M = \bar{M}$ as the mean mass of the 56-plet of baryons. Then

$$\langle r_\pi^2 \rangle \simeq \frac{2.7}{50} \frac{1}{\bar{m}^2} \left(\frac{\bar{m}}{\bar{M}} \right)^2 \left(\frac{g_{NN\pi}}{f_{\rho\pi\pi}} \right)^2 \quad (29)$$

and on using the same value of $(g_{NN\pi}/f_{\rho\pi\pi})^2$ one gets

$$\langle r_\pi^2 \rangle \simeq 3.8 \left(\frac{\bar{m}}{\bar{M}} \right)^2 \frac{1}{\bar{m}^2}.$$

Any reasonable estimate for $(\bar{m}/\bar{M})$ shows that $\langle r_\pi^2 \rangle \approx 1/\bar{m}^2$: a result which is much more plausible and in agreement with our a priori knowledge of $\langle r_\pi^2 \rangle$.

The difficulty we found is directly traceable to the assumption that we are taking matrix elements with physical states with their physical masses. Thus, the form factor in (9) is to be evaluated at the timelike momentum transfer $q^2 = (M_V - m)^2$. A pole estimate for the form factor (10) shows that $G_{VM}(q^2)$ is larger than its value at $q^2 = 0$. This is essentially the origin of the discrepancy. If this estimate of the q^2 behaviour of the form factor is correct, then one will have to either enlarge the algebra to include mass splitting terms or consider more terms in the sum over intermediate states ¹⁰⁾ if a consistent set of predictions is to be obtained when matrix elements are taken between physical states.

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