STATISTICAL THERMODYNAMICS OF STRONG INTERACTIONS

AT HIGH ENERGIES - II

(Momentum spectra of particles produced in high energy pp collisions)

R. Hagedorn and J. Ranft

CERN - Geneva
Our predictions are based on a combination of "thermodynamics of strong interactions at high energies" with some simple kinematical considerations. One can describe this model as follows:

- "fireballs" are highly excited hadrons, which decay according to thermodynamical laws; in this thermodynamics exists a "highest temperature" $T_0 \approx 150-160$ MeV.

- In a collision with given impact parameter the incoming particles overlap partly. In the overlapping region the highest temperature $T_0$ is (almost) attained; in the non-overlapping parts $T < T_0$, $T$ decreases in the outward direction from the overlapping part. Similarly, in the overlapping part the excited matter has come (almost) to rest, whereas the outer parts conserve more and more of their initial velocity the farther they are away from the overlap region.

- We assume therefore each volume element to be a virtual fireball with some temperature $T(x)$ and some Lorentz factor (representing its velocity) $\gamma(x)$ being capable of emitting in its rest frame isotropically and according to a thermodynamical momentum spectrum. Transformations from all these individual rest frames to the CM frame and superposition yields the total CM spectrum with the usual jet structure.
The distributions $T(x)$ and $f'(x)$ [$x = \text{distance from the collision centre}$] describe a geometrical situation and are therefore energy independent [except for a trivial dependence of $\gamma(x)$ from $\gamma_{\text{CM}}$] so that, having them fitted to experiments at one energy, we can safely extrapolate to any higher energy. $T(x)$ and $f'(x)$ have been fitted to 31 GeV experiments. Apart from an over-all normalization factor only four parameters (per kind of particle) are sufficient to obtain an excellent fit, which then automatically represents very well the available experimental data at 18 and 23 GeV with the same four parameters. We show the corresponding curves.

On this basis we present spectra for 300 GeV pp collisions in form of curves.

The model is not yet in its final shape. We try to reduce the number of parameters further. It is believed that our numerical predictions remain invariant under the intended improvements of the model.
ERRATA

Page 8  Line 4: read "...practically brake down...".
        Line 27: read Fig. 3 instead of Fig. 2.

Page 14 Line 17: read "...experimental spectra at (only)
                   one sufficiently..."

Page 18 In formula (8) read $L_{\chi_1(x)}$

Page 25 Line 16: omit the letter "e" at the end of that line.
INTRODUCTION

Among the various attempts \(^1,2,3,4\) to find formulae representing momentum spectra of particles produced in high-energy collisions, no one is the straightforward result of a physical model. In fact, all of them result in formulae which interpolate between experimental observations, to which combinations of polynomials and exponentials in the variables \(p_0\), \(p\) and \(\Theta\) are fitted. The number of parameters necessary to obtain good fits is rather large (up to 10). This would be no disadvantage if only one could be sure that by extrapolating these fits to much higher energies one would still have a sensible formula. This is in most cases not so – just because the physical insight is lacking.

We present here a preliminary report on another approach to the problem, in which we try to use as much physics and as few arbitrary parameters as possible.

The present report is preliminary in so far as we hope to reduce the still persisting arbitrariness of some parameters still further and to exhibit more clearly the underlying physical picture. We do not consider this report preliminary as far as the predicted spectra for higher energies are concerned: we believe that these numerical predictions will hardly be changed by our intended conceptual improvements. We shall in this text indicate at several places what we wish to change and how; in fact, during the time of preparation of this report, this development was already in full course and the present report has been written at this stage primarily for the meeting of ECFA working group 2 on October 6, 1966.

The basis of our model is twofold: we combine thermodynamics of strong interactions as the guiding principle with some simple, purely kinematical mechanisms. We shall now give an outline of this model. Our units are \(\hbar = c = k\) (Boltzmann's constant) = 1; energies in MeV or GeV.
1. THERMODYNAMICS OF STRONG INTERACTIONS AT HIGH ENERGIES

Under this title a paper \(^5\) has been published, which we shall quote as (I) from now on. Its essential features are:

- The interaction region during the contact of the colliding particles is filled with highly excited hadronic matter in a state for which particle number operators are not diagonal. We call this state (or mixture) a "fireball".

- A fireball is considered to be a (not perfect; see below) statistical equilibrium of all kinds of hadrons, including all resonances.

- The high-lying resonances decay into many channels; the number of decay channels increases with the mass of the resonance (this is not an assumption, but an experimental fact). The weight of each channel would in the large majority of cases be determined by its phase space. In other words: for very large mass such a resonance would be described by statistics—just as the above fireball.

- If we adopt the name fireball for all hadrons (not only resonances of large mass) including stable ones, then we can say:

  We describe a fireball

  as a statistical equilibrium of an undetermined number of fireballs, which in turn we describe

- We have shown in (I) that this description can be carried through by the usual methods of statistical thermodynamics and that it leads, without any further assumption, to the following predictions:
i) the mass spectrum of hadrons, i.e., the number of resonant states between \( m \) and \( m + dm \), is given asymptotically by

\[
\frac{\zeta(m) \, dm}{m \to \infty} \rightarrow \frac{a}{\sqrt{m}} \, e^{-m/T_o} \, dm
\]

(1)

ii) when the energy of the collision becomes large (in fact: above a few GeV), then the temperature tends to a finite limit (highest temperature) \( T_o \); this \( T_o \) is the same as in the mass spectrum. Its value cannot (it seems) be determined in the framework of the model, but its existence can be proved.

These predictions can be compared to the experiment. The highest temperature \( T_o \) seems well established by the long-known fact that the transverse momentum distribution of secondaries is energy independent and of a form similar to a Boltzmann distribution [but not exactly equal; see Eq. (I.41)]. This is true for CM energies between a few GeV and the highest energies observed in cosmic-ray events. Deviations from the (unknown) true value \( T_o \) can be interpreted as due to imperfect statistical equilibrium of the fireball (see below), and as due to kinematical effects (smearing out of spectra) induced by the decay chain fireball \( \rightarrow \) fireballs \( \rightarrow \ldots \rightarrow \) resonances \( \rightarrow \) pions (nucleons, etc.). In any case the experimental value of \( T_o \) (derived from the mean transverse momentum \( \langle p_T \rangle \)) is never very different from 150 - 160 MeV. In (I) we derived from Orear's results that \( T_o = 158 \pm 3 \) MeV. The determination from elastic scattering being somewhat doubtful, we shall here redetermine \( T_o \) from known particle spectra at 30 GeV. We find indeed a value of (or better: values scattering about) 150 - 160 MeV. The important assertion of the thermodynamic model is that behind this stays a universal and energy-independent highest temperature \( T_o \) (at least for all strong interactions), which, because of the mentioned reasons (imperfect equilibrium, smearing out of spectra by successive decays) might be difficult to determine accurately.
The other prediction, for the asymptotic behaviour of the mass spectrum, is supported by the experiment; in fact the experimental mass spectrum has an exponential form in that region where by now all resonances are fairly known (i.e., up to \( \sim 1200 \text{ MeV} \)) and its slope is well represented by \( T_0 \approx 158 \text{ MeV} \) as Eq. (1) requires (see Fig. 1). The constant \( a \) in Eq. (1), if fitted numerically to the experimental spectrum, has a value which lies within the rather narrow limits which one obtains from a priori considerations [For details, see (I)].

The existence of an energy-independent highest temperature \( T_0 \) is a consequence of the exponential growth of the mass spectrum, which itself is a necessary consequence of the assumption that fireballs are a statistical equilibrium of fireballs. That a highest temperature should exist may seem somewhat shocking; it could have been, however, anticipated with some fantasy by extrapolating from the light-quantum gas. Indeed: in a classical gas with a fixed number \( N \) of particles all energy is stored as kinetic energy of the particles, hence the temperature rises linearly with energy. The situation is different in the light-quantum gas: the particle number is not fixed and an energy input is answered by increasing both the kinetic energy and the number of particles. Hence the kinetic energy per particle gets only part of the energy input and therefore the temperature grows no longer linearly with the energy — the result is the Stefan-Boltzmann law \( T = \frac{1}{\sigma} E^{1/4} \). Now in our fireball gas not only one but an infinity of different kinds of particles can be freely created. In this situation, creation of new particles leads to a much larger increase of the number of quantum states than an increase of the kinetic energy of the existing particles would do; with increasing total energy the new particles then eat up more and more of the energy and in the limit the kinetic energy per particle tends to remain constant: \( T \to T_0 \) for \( E \to \infty \) (practically for \( E \gtrsim \) a few GeV). One could also compare the situation in a way to that of the boiling point of a liquid and interpret it more in the language of energy: at the boiling point particles are emitted from the liquid into the vapour and thereby must overcome a large potential barrier; for this so much energy...
is used that although energy flows constantly in, the temperature stays constant. If, in this argument, the "potential wall between liquid and vapour" is replaced by the "potential wall of height \( m \) between non-existence and existence of a particle of mass \( m \)\), then one may speak in a very loose analogy of \( T_0 \) being the "boiling point of hadronic matter". But this analogy must never be taken literally; the reasons for the existence of a highest temperature lie much deeper.

The model therefore

- predicts the shape of the transverse momentum distribution correctly,
- explains its energy independence,
- relates these two features in a rather unexpected way to the mass spectrum of hadrons which is predicted correctly in the region where experimental information seems rather complete.

Apart from these three main achievements, the model predicts also quite well how the mean transversal momentum depends on the mass of the particle considered (not published; it is one of the by-products of the present work); it gives correct predictions of particle multiplicities, in particular of rather rare particles and antiprotons, antideuterons, etc. Without going into any detail one can safely say that this model, if unable to describe finer features covers an extremely large set of experimental results obtained in collisions with energies ranging from a few GeV to hundreds of thousands of GeV and produces formulae (transverse momentum spectra, multiplicities) which agree with experiments within a factor two over ranges of up to 10 orders of magnitude. This gives us confidence enough that this model might be a powerful tool in predicting particle spectra. It is, however, incomplete in so far as it cannot yet predict longitudinal momentum distributions.
The point is that during the collision only a small part of the centre-of-mass energy (abbreviated: CME) will be transformed into heat; most of it will remain kinetic energy of a fast collective motion in the forward-backward direction. The heat motion can reveal itself only in the transverse momenta, whereas in the forward-backward component of the spectrum it is hidden under the much larger velocities belonging to the collective motion (that is why in (1) only the transversal momentum spectrum was calculated); it is hidden, but not ineffective: it should clearly stay out when the collective motion is Lorentz-transformed away. We shall take this for granted and proceed in the inverse sense: in the rest frame of any volume element of the collision region the momentum distribution is assumed to be thermodynamic and in any other Lorentz frame it is obtained by the corresponding Lorentz transformation. It will be the main task of our kinematic considerations to find an approximate and simple description for this collective motion. This will be done below. Here we only add the remark that owing to the small value of $T_0$ (namely 150-160 MeV), an extremely small fraction of the total CME is generally sufficient to heat up the hadronic matter to nearly that temperature. Therefore in a central collision $T$ becomes almost equal to $T_0$; whereas in more peripheral collisions, where not much energy is transformed into heat, $T$ may be distinctly smaller than $T_0$. The same holds for the overlapping and non-overlapping parts respectively of two colliding hadrons: inside it is hotter than outside (but nowhere $T \geq T_0$). We shall exploit this below.

Numerically, the relation between energy (strictly: the expectation value of the energy in the sense of statistical thermodynamics) and temperature is given by Eq. (I.34)

$$\langle E \rangle \rightarrow \propto \frac{T^2}{T_0^2 - T}$$

with $\propto$ not exactly known: $\propto \approx \frac{1}{5}$. 
In fact this is the energy contained in one nucleon volume \( V_0 \approx \frac{4\pi}{3}(m_\pi^2)^{-3} \) if this volume is filled with hadronic matter in perfect thermodynamic equilibrium. This being not the case in our present elaboration of the model, we apply the formula to each volume element separately. Then we write \( \mathcal{E} \) instead of \( E \), where now \( \mathcal{E} \) is the energy density measured in units \( \text{MeV}/V_0 \); we solve for \( T \) and equate the actual energy with the expectation value (which is no bad approximation):

\[
T(\mathcal{E}) \Rightarrow T_0 \left[ 1 - \frac{T_0}{\mathcal{E}} \right] ; \quad \alpha \approx \frac{1}{2}
\]

Now this formula is only asymptotically true for \( \mathcal{E} \to \infty \). We know that for \( \mathcal{E} \to 0 \) we should obtain \( T(\mathcal{E}) = 0 \), which this formula cannot yield. If we replace \( 1 - \frac{T_0}{\mathcal{E}} \) by \( \exp(-\alpha \frac{T_0}{\mathcal{E}}) \) then this formula coincides with the original one in the asymptotic region but it also gives the correct value for \( \mathcal{E} \to 0 \); it therefore has certainly a larger range of validity and may be used, without great mistake, through the whole region \( 0 \leq \mathcal{E} \leq \infty \) \([ \mathcal{E} \text{ is always to be understood as "excitation energy", that is, as the energy density (per } V_0 \text{) minus the mass density brought into the reaction} ]\).

We thus put

\[
T(\mathcal{E}) = T_0 \exp \left[ -\alpha \frac{T_0}{\mathcal{E}} \right] ; \quad \alpha \approx \frac{1}{2}
\]

\([ \mathcal{E} \text{ measured in MeV per } V_0 \approx \frac{4\pi}{3}m^{-3} ]\)

In Fig. 2, we have plotted this function for \( \alpha = \frac{1}{2} \) and with \( T_0 = 156 \text{ MeV} \), in order to show how rapidly values of the order of \( T_0 \) are reached, but how large \( \mathcal{E} \) must grow before one gets very near to \( T_0 \); for \( \mathcal{E} \approx 800 \text{ MeV}/V_0 \) one has \( T \approx 0.9T_0 \) but one needs \( \mathcal{E} \approx 8000 \text{ MeV}/V_0 \) for \( T \approx 0.99T_0 \) (etc.). One sees that for CME above a few GeV the temperature is always above \( 0.9T_0 \) in the overlapping part of the colliding particles, but one also sees that it may easily drop to rather small values in the non-overlapping regions (see Fig. 3).
2. COMBINATION OF THERMODYNAMICS WITH KINEMATICS

We picture the collision as shown in Fig. 3. The two nucleons penetrate each other with impact parameter \( \rho \). In the overlap region (1) they mix intimately; there the original velocity is practically broken down to zero and its kinetic energy transformed practically all into heat. Thus in the shaded region \( T \approx T_0 \), velocity \( \approx 0 \). Now consider the areas (2) with the arrows pointing up and down respectively. The arrows shall indicate that the shock which the nucleons receive in the overlap region travels outwards and heats up part of the non-overlapping region; however, the contact time is small and emission of particles from all parts of the colliding nucleons starts instantaneously — hence the shock wave affects only a part, but not all of the nucleons (roughly a region of extension equal to the contact time \( \Delta t \approx 2r/\gamma_{CM} \); \( r \) = nucleon radius, \( \gamma_{CM} = \) CM Lorentz factor). This part will have still some velocity and its temperature may be smaller than \( T_0 \). Finally there is the outer region (3) (left white in the figure) where during the contact time no effect of the interaction was felt; the shock wave would have reached this part much later if it could have continued to propagate; but in fact the whole has blown up into many smaller fireballs in the meantime and no wave can any more propagate. This part would then be heated only very little (by its breaking off from the rest) and keep almost its original velocity.

We expect therefore that velocity and temperature vary from the inner part to the outer part qualitatively as depicted in Figs. 4a, b (4a : central ; 4b : peripheral collision).

We characterize the velocity by the corresponding Lorentz factor \( \gamma = (1-v^2)^{-\frac{1}{2}} \); \( x \) is the distance from the centre of collision (see Fig. 2). The velocity distribution (we neglect any transversal component) would depend (for two given colliding particles) only on \( \gamma_{CM} \) (i.e., on \( E_{CM} \)) and on the impact parameter \( \rho \); that is, \( \gamma (x) \) will be a universal function — which we do not know —

\[
\gamma = \gamma_{CM} (x; \gamma_{CM})
\]
The temperature \( T \) would now have to be described as a function of \( \gamma \); namely, if the velocity has been reduced from the initial value corresponding to \( \gamma_{CM} \) to the actual \( \gamma(x) \), then a corresponding amount of kinetic energy is transformed adiabatically into heat and then Eq. (2) allows us to calculate the corresponding temperature \( T(\gamma) \). This will be done at a later stage of development of the model. In the present report we shall treat \( T = T_1(x) \) as an independent function which decreases as shown in Figs. 4. The numerical forms of functions \( \gamma_1(x) \) and \( T_1(x) \), which are related in a not too transparent way to certain complicated averages over \( \gamma \) of the above \( \gamma_\xi(x) \) and \( T_\xi(x) \), have been determined by fitting experimental spectra. Although these functions \( \gamma_1(x) \) and \( T_1(x) \) cannot fulfill the expected relation \( T = T(\gamma) \) - because they are different; see below, Section 4 - we hope that it indeed will turn out that we can disperse with \( T_\xi(x) \) as an independent function, leaving us with one single "geometric" function \( \gamma_\xi(x; \gamma_{CM}) \); the numerical behaviour of the actually used \( T_1(x) \) and \( \gamma_1(x) \) is not discouraging.

Now our thermodynamic model, if taken literally, would require that from each volume element fireballs are emitted, which in turn emit smaller fireballs and so on until we arrive at particles which are stable against strong decay. This highly complicated process with its many Lorentz transformations and thermodynamic momentum spectra folding repeatedly into each other is far beyond our computing abilities.

We shall make the following (partly too simple) assumptions:

- each volume element (labelled by \( x \)) has, for a given impact parameter \( \xi \), a certain temperature \( T_\xi(x) \) and a certain velocity in (or opposite to) the direction of the collision axis; this velocity is measured by \( \gamma_\xi(x) \);

- each volume element \( (x) \) emits (isotropically in its rest frame \( K'_x \)) directly the final particles we are after (pions, nucleons, kaons,

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antinucleons, etc.) this omission obeys the thermodynamical momentum distribution (I, p. 163)

\[ \omega_i^{(q)}(\mathbf{p'}; m_i; T_\xi(x)) = \frac{n_i^{(q)}(x; T_\xi(x))}{\exp \left[ \frac{1}{T_\xi(x)} \sqrt{p'^2 + m_i^2} \right] - 1} \quad \{ \begin{align*} \text{bosons} \\ \text{fermions} \end{align*} \} \quad (4) \]

Note that using this equation with the temperature \( T_\xi(x) \) implies that we believe that most particles produced in the collision are emitted from the adiabatically heated volume elements before any over-all thermodynamic equilibrium (by means of whatever mechanism of heat transport) is approached. We only require a local thermodynamic equilibrium, i.e., an equilibrium of modes of existence of hadronic matter (particle number distribution, momentum distribution) in each volume element.

\( n_i^{(q)}(x; T_\xi(x)) \) is the multiplicity of the kind \( i \) of particles; it will depend directly on \( x \) in so far as the internal structure of the colliding particles (e.g., the spatial distribution of the nucleonic charge, etc.) is still "visible" in Fig. 3. The frequent presence in high energy jets of a "leading particle" with quantum numbers equal to those of the incident particle is an indication for that partial conservation of structure; this feature is in accord with our over-all picture which explicitly admits and takes into account an incomplete thermodynamic equilibrium. On the other hand \( n_i^{(q)}(x; T_\xi(x)) \) will depend on the temperature at \( x \), because the emission of particles is also governed by thermodynamics; if only thermodynamics would be in, then the emission probability for a mass \( m_i \) would be given by

\[ n_i(\tau) \sim m_i^2 \tau K_2 \left( \frac{m_i}{\tau} \right) \quad (5) \]

\( (K_2 = \text{second modified Hankel function}) \)
but in reality this formula has to be modified by the effects of conservation laws 7) (for instance: the two nucleons brought into the reaction need not be newly created, hence the equation does not apply to them) and of successive decays of fireballs → fireballs → ...

It seems that presently we have practically no other choice than to treat \( n_i(\xi)(x; T_\Phi(x)) \) as an unknown function, which may then be simply written \( n_i(\xi)(x) \); we hope to be able at a later stage to remove this arbitrariness somewhat. In fact, we shall use already in this work Eq. (5) to fix up the relative normalization of the various \( n_i \) in our predictions at 300 GeV (see Section 5).

A last remark concerning Eq. (4) applies to the \( \mp 1 \) in the denominator. For \( p = 0 \) the denominator becomes \( \exp(\gamma / p) \mp 1 \) and one sees at once that for nucleons and larger masses the \( \mp 1 \) can be neglected; however, for pions \( m \approx T \), so that we obtain \( \mp e - 1 \) in the denominator. Here the \(-1\) cannot be neglected and we found indeed that the \(-1\) is essential for obtaining good fits to the low momentum part of pion spectra (we first had tried to neglect the \( \mp 1 \) also for pions). This may be taken as a further support of our belief that thermodynamic aspects are essential in high energy processes of hadrons.

- The spectrum \( W_i(\xi) (p, m_i, T_\Phi(x)) \) is then Lorentz transformed from the rest frame \( K' \) of the moving volume element (label \( x \)) to the CM frame \( K \) of the collision. If we denote by \( \chi_\Phi(x) \) the Lorentz transformation (taken as an operator in the space of \( w \) functions) which transforms from \( K' \rightarrow K \), then in the CM frame \( K \) we obtain the contribution coming from \( x \) for a given impact parameter \( \xi \):

\[
W_i(\xi) (p, m_i, x) = \gamma_i(\xi)(x) \cdot \exp \left[ \frac{1}{(T_\Phi(x))} \sqrt{p^2 + m_i^2} \right] \mp 1
\]
This has to be integrated over $x$ from 0 to $r + \frac{f}{2}$ for one direction of motion and from 0 to $-(r + \frac{f}{2})$ for the opposite direction of motion and the two integrals have to be added; finally one integrates over $\xi$ from 0 to $2r$ [see Fig. 3; $r$ is the nucleon radius $\approx \frac{m}{\pi}$].

\[
W_\xi(p, m) = \int dp \int dx \, n_\xi(p, x) \, L(\gamma_\xi(x)) \, \frac{1}{\exp \left[ \frac{\gamma_\xi(x)}{T_\xi(x)} \sqrt{p^2 + m^2} \right] + 1} 
\]

(7)

3. GENERAL DISCUSSION OF THE MODEL

Here is the place to make some general remarks about the spirit of this formula: it describes a continuous superposition of fireballs with continuously varying velocity and temperature; these fireballs decay isotropically according to thermodynamics, but owing to their own movements the whole emission spectrum assumes the well-known jet structure if observed from the CM frame of the collision. If we look closer at the functions $n_1(\gamma, x)$, $\gamma_\xi(x)$ and $T_\xi(x)$, then we see that they mainly describe a geometrical situation. First of all, we know that in reality $T_\xi(x)$ is no independent function (only for technical reasons we presently treat it as such) but is in fact a function of $\gamma_\xi(x)$. This leaves us with only two independent functions, $n_1(\gamma, x)$ and $\gamma_\xi(x)$. Now $\gamma_\xi(x)$ varies from $\gamma \approx 1$ at $x = 0$ to $\gamma \approx \gamma_{CM}$ at the maximum value of $x$; this latter value $\gamma_{CM}$ is a function of the CM but the way in which $\gamma$ changes from 1 to $\gamma_{CM}$ is determined purely by geometry and does no longer depend on energy: in view of Figs. 4a, 4b one would say that the width and general form of the $\gamma$ curve depends only on $\gamma_\xi$ except that the end of the curve is lifted up and down with $\gamma_{CM}$. Next consider $n_1(\gamma, x)$. This function depends on the internal structure of the colliding particles and is in so far purely geometric, i.e., energy independent. Note that $x > 0$.
refers to one direction of motion, $x < 0$ to the opposite one (Fig. 3); similarly $n_1^f(x)$ may be thought of not being an even function of $x$ when the two colliding particles are unequal (e.g., $\pi - \tau$ collision); in that case it would be easy to describe the forward-backward asymmetry of the production spectra. On the other hand it depends in some way or other [as indicated by Eq. (5)] also on the temperature. However, the temperature distribution is much less energy dependent then $\chi(x)$ [because in the middle it never can increase beyond $T_0$ and in the outer parts it drops small; in fact if one writes down the $T(x)$ formula then one sees that for $\gamma_{\text{CM}} \to \infty$ $T(\gamma(x))$ tends to a limiting curve]. Therefore we may neglect in $n_1^f(x)$ the indirect energy dependence through $T(x)$ and consider $n_1^f(x)$ also to be essentially a geometric and energy independent function [an over-all energy-dependent normalizing factor $N_1(x)$ must be kept open; it can be determined from energy conservation together with Eq. (5); details will be discussed in Section 5].

We have then the following situation: if we (only to quote a possibility) represent $\chi_\rho(x)$ by a Gauss function

$$\chi_\rho(x) = \chi_{\text{CM}} + (1 - \chi_{\text{CM}}) \exp \left( -\frac{x^2}{\alpha^2} \right)$$

then the width $\alpha_\rho$ depends only on the impact parameter $\rho$ [compare Figs 4a and 4b] and not on the energy: a $\rho$ is "purely geometric" and all the energy dependence is trivially contained in $\chi_{\text{CM}}$. We do not assert that this is exactly so, but we believe it to be a good approximation to consider a $\rho$ (or corresponding parameters in other functional descriptions) energy independent. The other function, $n_1^f(x)$, may be characterized also by (nearly) energy independent parameters. $T$ is in fact a function of $\chi$, and can be thought of being explicitly expressed as such a function (in which no energy-dependent parameters do appear).
Now, with all this in mind, we look again at formula (7) for the total spectrum in the CM frame:

the only energy dependence entering into this formula is via \( \gamma_{CM} \); all the rest is "pure geometry". Combining this with the experimental fact that all total cross-sections of strong interactions at high energy tend to become constant (or, at least, change extremely slowly) - which, in this context, may be reformulated as: "at high energies geometry becomes energy-independent" - we arrive at the conclusion that formula (7) may safely be extrapolated to any large energy (above, say, 10-20 GeV) once the constants describing the geometrical shape of \( \gamma_{CM}(x) \) and \( n_1(\gamma, T)(x) \) have been determined by fitting formula (7) to experimental spectra at (only) sufficiently high energy.

We add a technical remark about further simplification of Eq. (7); what we describe now is our present programme (not finished).

For the integral Eq. (7) we may draw, in the \( x, \gamma' \) plane, curves \( \gamma' = \gamma' (x, \gamma, \gamma_{CM}) \) along which \( \gamma' \) is kept constant. Instead of using the variables \( \{ x, \gamma' \} \), we may then, as well, use the variables \( \{ \gamma, \gamma' \} \). This has the advantage that everything following after \( n_1(\gamma', T)(x) \) in the integrand depends only on \( \gamma' \), if we assume \( T \) expressed as a function of \( \gamma \) as mentioned repeatedly. The function \( n_1(\gamma', T)(x) \) may be re-expressed by the variables \( \{ \gamma, \gamma' \} \) and then combined with the Jacobian of this transformation of variables; this gives a new function, which we call \( n_1(\gamma, \gamma') \). Now our integral becomes
\[ W_L(p, m) = \begin{cases} \frac{\gamma_{CM}}{d_\gamma L(\gamma)} \frac{1}{\exp \left[ \frac{1}{T(\gamma)} \sqrt{p^2 + m^2} \right] + 1} \left\{ \begin{array}{l} \frac{2}{x} \\ d_\gamma n_L(\gamma, \gamma_{CM}) \end{array} \right\} \bigg|_{x > 0} \\ + \left\{ \text{corresponding integral} \right\} \bigg|_{x < 0} \end{cases} \]

The limits of the \( \gamma \) integration depend, of course, in some complicated way on \( \gamma \) and \( \gamma_{CM} \); however, complicated this may be, the integral yields a function \( n_L(\gamma, \gamma_{CM}) \) which, according to our above discussion, is of "purely geometric" origin and depends on CM only through \( \gamma_{CM} \); this latter dependence is, however, trivial in so far as it has been brought in by our transformation of integration variables. It can be made explicit and separated off from the geometrical-physical contents of these functions. We expect that \( n_L(\gamma, \gamma_{CM}) \) will be essentially a function of \( \gamma/\gamma_{CM} \) only. Remembering that \( n_L(\gamma)(x) \) need not equal \( n_L(\gamma)(-x) \) and that the Lorentz transformations belonging to \( x \) and to \( -x \) go in opposite directions, we arrive finally at

\[ W_L(p, m) = \frac{\gamma_{CM}}{d_\gamma \left( \sum_{\gamma} \left[ \frac{n_L^+(\gamma, \gamma_{CM}) L(\gamma)^+ - n_L^-(\gamma, \gamma_{CM}) L(\gamma)^-} {\exp \left[ \frac{1}{T(\gamma)} \sqrt{p^2 + m^2} \right] + 1} \right] \right) d_\gamma} \]

(7a)

where the subscripts \((+\) and \((-\) on \( n_L(\gamma, \gamma_{CM}) \) and \( L(\gamma) \) refer to the forward and backward direction respectively. For collision of equal particles \( n^+(\gamma) = n^-(\gamma) \), of course. In this formula the only places where the CME enters is in the upper limit of the integral and in \( n_L(\gamma, \gamma_{CM}) \) — all the rest are (practically) energy-independent universal functions, which may be more or less well described by properly chosen trial functions fitted to experiments at any sufficiently large
energy. In formula (7a) the obvious geometrical interpretation which was apparent in (7) has disappeared and one may read this new formula in a new spirit:

It is a superposition of a continuum of decaying fireballs, each with some $\gamma$ and a corresponding temperature $T(\gamma)$; the functions $n_i(\gamma, \gamma_{CM})$ are weight functions which say how much each $\gamma$ will contribute to the total spectrum.

One is now no longer obliged to talk of spatial distributions, which are in any case hardly ever directly measurable. We could have written down this formula right at the beginning in this sense; the roundabout way via our discussion of the $x, \phi$ dependence has however taught us that the functions $n_i(\gamma, \gamma_{CM})$ are "of geometrical origin" and therefore only trivially energy dependent.

One obvious, but important final remark: formulae (7) and (7a) describe spectra in the thermodynamical sense, that is: spectra of processes where all channels are in free competition. Therefore they apply only to total production spectra of particles of kind $i$; they become more or less wrong if applied, for instance, to purely elastic processes or to any other selected set of final states (as, e.g., a sample of three track events or to a sample of apparently peripheral collisions or to one single event, even if of large multiplicity).
4. COMPUTING TECHNIQUE FOR THE PREDICTIONS PRESENTED HERE

At the time of writing this report we are working along the lines of formula (7a). The numerical predictions of 300 GeV data and the actual fits to available experiments as presented here (Figs. 6 to 19), have been found in another way starting from Eq. (7):

- we assume the \( \mathcal{Q} \) integration to be carried out and write, with suitable averaged functions \( \overline{\eta}_i(x), \overline{\gamma}(x), \overline{T}(x) \) (on which we omit the bar in the following):

\[
W_i(\vec{p},m_i) = \int dx \, \eta_i(x) \mathcal{L}[\gamma(x)] \frac{1}{\exp \left[ \frac{1}{\overline{T}(x)} \sqrt{\vec{p}^2 + m_i^2} \right] + 1}
\]

- in order to reduce the number of functions, we absorb \( \eta_i(x) \) into the other part of the integrand, where we instead allow \( \gamma_i(x) \) and \( T_i(x) \) to be different for different kinds of particles; we define these functions by

\[
\eta_i(x) \mathcal{L}[\gamma(x)] \frac{1}{\exp \left[ \frac{1}{\overline{T}(x)} \sqrt{\vec{p}^2 + m_i^2} \right] + 1} \equiv \]

\[
\equiv \mathcal{L}[\gamma_i(x)] \frac{1}{\exp \left[ \frac{1}{\overline{T}(x)} \sqrt{\vec{p}^2 + m_i^2} \right] + 1}
\]

but we deny here explicitly once and forever that there is any physical significance in different distributions \( T_i(x) \) and \( \gamma_i(x) \) for different kinds of particles; it is only a computational technique; consequently \( x \) should no longer be considered to represent a geometrical distance but should merely be thought of as an integration
variable serving to superpose spectra of different $\gamma$ and different $T$. In order to emphasize this, we redefine $x$ to be a dimensionless variable ranging from $-1$ to $+1$.

- Our final formula (as used for the numerical fits presented here) is then

$$W_i(\tilde{p}, m_i) = \int_{-1}^{1} dx \frac{1}{L[\gamma(x)]} \frac{1}{\exp \left[ \frac{1}{T_i(x)} \sqrt{p^2 + m_i^2} \right] + 1}$$

This formula has been programmed for the CERN CDC 6600 computer, leaving the functions $\gamma_1(x)$ and $T_1(x)$ open to be fitted to experimental spectra. One important improvement of the above formula turned out to be necessary in the course of the fitting procedure:

- Kinematical limits: although the spectra decrease exponentially towards larger momenta, the formula (8) does not know of any kinematical limit due to energy-momentum conservation, because its basis is still a thermodynamical formula derived from a Gibbs ensemble in which the temperature is given and the energy fluctuates. It turned out that formula (8), with functions $\gamma_1(x)$ and $T_1(x)$ fitted as well as possible by a least square fit to experimental spectra, not only gave — of course — non-vanishing contributions above the kinematical limits, but also — and that was not expected — was unable to reproduce the rather well visible deviation from a straight line (i.e., purely exponential decrease) of the experimental CM spectra in their high-energy part (see Fig. 9). We found that this stronger-than-exponential decrease can be well explained in the framework of our model and is indeed of purely kinematic origin (see Fig. 5): take the pions as an example and consider the volume element $\Gamma(x)$; it has a velocity corresponding to $\gamma(x)$. Now, for given $\gamma$, this volume element can contain at most a maximal mass $m(\gamma)$, which is given by energy-momentum conservation: $m(\gamma)$ is
attained when in the opposite direction only a proton is emitted (the contribution when the other particle is a pion, can be neglected as it is much less probable) and is then

$$m(\chi) = \sqrt{\gamma^2 E^2 - E^2 + m^2}$$

(9)

$$E = E_{CM}; \quad m = m(\text{proton}).$$

Now let this fireball of mass $m(\chi)$ decay into a pion and a proton with the pion going in the forward direction $\tilde{\gamma}$, the same as $m(\chi)_{\tilde{\gamma}}$ and the proton backwards. This pion has then the maximum energy a pion coming from this volume element could possibly have. This momentum has been built into the program as a limiting momentum belonging to that $\tilde{\gamma}$; the spectrum is calculated with Eq. (8) for all momenta up to this limit and put equal to zero above it; for non-zero angles we calculate the limit by assuming $m(\chi)$ going straight forward but then the pion being emitted under the angle in question. At the extreme high energy end of the spectrum there is another cut-off (belonging to central collisions: small $x$) in which the absolute maximum of the pion momentum is obtained when both nucleons go in one direction and the pion in the other. Here we simply cut off the pion spectrum. Corresponding kinematical cut-offs are obtained for $k$ mesons, antinucleons, etc. For nucleons (apart from those coming from pair production) only one kinematical cut-off is used, namely that their momentum cannot exceed the initial one.

With these kinematical additions we have fitted (by a procedure described in Ref. 8) Eq. (8) to experimental spectra by assuming the following Gauss-type functional forms (after playing around with several possible trial functions, we found these convenient) for $T_1(x)$ and $\chi_1(x)$:

66/1255/5
\[ \bar{T}(x) = T_0' + (1 - T_0') \exp \left( -x^2/a^2 \right) \]

\[ \chi(x) = \chi_{cm} + (1 - \chi_{cm}) \exp \left( -x^2/b^2 \right) \]

where we consider \( T_0', \ T \) and \( a \) as free parameters for \( T \), and \( b \) as free parameter for \( \chi \). As explained above, we believe that these constants — in particular \( T_0' \) — are practically energy independent.

According to what we said about the different \( T_i \) and \( \chi_i \) we reserve one set of parameters to each kind of particles: protons, \( \pi^+, \pi^-, K^+, K^-, \pi, \) etc. We expect that nucleons are coming from the outer regions, pions and \( K \) mesons from everywhere, \( \bar{K} \) mesons and \( \bar{\pi} \) more from the inner regions. As Figs. 3 and 4 show, we attach large \( T \) with small \( \chi \) to the inner and small \( T \) with large \( \chi \) to the outer parts. Consequently we expect narrow distributions (type Fig. 4b) for protons and wide distributions (type Fig. 4a) for antiprotons, pions and kaons lying somewhere between; in terms of the width \( a \) and \( b \):

\[ (a, b)_N \lesssim (a, b)_{\pi, K} \lesssim (a, b)_{\bar{K}} \lesssim (a, b)_{\bar{\pi}} \]

whereas \( T_0' \) should be almost constant and of the order of 150 - 160 MeV. These expectations are indeed roughly verified by our numerical findings (see Table 1); we were, however, surprised to see \( b \) vary so little, and to find a \( T_0' \) for \( \bar{K} \) and \( \bar{\pi} \) which is distinctly smaller than that for the other particles.
The fit was performed using the following experimental data:

**Protons**: Anderson et al. 9) at $p_0 = 30$ GeV/c
Dekkers et al. 10) at $p_0 = 18.8$ and 23.1 GeV/c
Diddens et al. 11) at $p_0 = 19.2$ and 24 GeV/c

**Pions**: Anderson et al. 9) at $p_0 = 30$ GeV/c
The data by Dekkers et al. 10) at 18.8 and 23.1 GeV/c and Allaby et al. 12) at 19.4 GeV/c were not used for the fit.
Extrapolation of the resulting spectra to these primary proton momenta gave very good agreement with these data (see Fig. 8a).

**Kaons and antiprotons**: Not enough data on $H_2$ targets being available.
We used the Be target production data which are compiled in Ref. 13).

In Table 1, we give the parameters $a$, $b$, $C$, and $T_0'$ which were obtained by the fits. The fits obtained using a common $T_0'$ value for all kinds of particles are not much worse. That $T_0'$ for $K^-$ and $p$ is distinctly smaller than for the other particles need not have any deeper meaning in our present technique of calculation; by the by we are not sure that there might not exist other, equally good, solutions to the fit as those of Table 1.

<table>
<thead>
<tr>
<th>particle</th>
<th>$a$</th>
<th>$b$</th>
<th>$C$</th>
<th>$T_0'$ (GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>0.232</td>
<td>0.697</td>
<td>0.686</td>
<td>0.145</td>
</tr>
<tr>
<td>$\pi^+$</td>
<td>0.665</td>
<td>0.962</td>
<td>0.546</td>
<td>0.168</td>
</tr>
<tr>
<td>$\pi^-$</td>
<td>0.918</td>
<td>0.869</td>
<td>0.160</td>
<td>0.159</td>
</tr>
<tr>
<td>$K^+$</td>
<td>0.681</td>
<td>0.871</td>
<td>0.753</td>
<td>0.151</td>
</tr>
<tr>
<td>$K^-$</td>
<td>2.16</td>
<td>0.795</td>
<td>0.646</td>
<td>0.119</td>
</tr>
<tr>
<td>$\bar{p}$</td>
<td>3.93</td>
<td>1.47</td>
<td>0.609</td>
<td>0.121</td>
</tr>
</tbody>
</table>

- Table 1 -

The parameters $a$, $b$, $C$, and $T_0'$ for $p$, $\pi^+$, $\pi^-$, $K^+$, $K^-$, and $\bar{p}$ production
The parameters for $K^\pm$ and $\bar{p}$ production will certainly change when better measurements become available. The experimental values for $K$ and $\bar{p}$ production used in the fits are not very accurate.

A normalization factor of the form
\[ N_i(E_{CM}) = A \cdot E_{CM}^B \] (12)

has been kept open for the fits in which we used experimental spectra belonging to different CME. In Table 2 we give the parameters $A$ and $B$ obtained (primary proton momenta $p_o$ between 18.8 and 30 GeV/c). Note that we do not use formula (12) for any extrapolation; it serves only to give the fitting procedure the necessary freedom in those cases where scarce experimental material forced us to fit at several energies simultaneously. Our normalization procedure for the predicted spectra will be discussed in Section 5.

<table>
<thead>
<tr>
<th>particle</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>6410.4</td>
<td>0</td>
</tr>
<tr>
<td>$\bar{p}$</td>
<td>22.95</td>
<td>0.5</td>
</tr>
<tr>
<td>$\bar{\pi}^+$</td>
<td>18.78</td>
<td>0.5</td>
</tr>
<tr>
<td>$\pi^-$</td>
<td>2.38</td>
<td>1.352</td>
</tr>
<tr>
<td>$K^+$</td>
<td>6.794</td>
<td>1.011</td>
</tr>
<tr>
<td>$K^-$</td>
<td>1.945</td>
<td>1.378</td>
</tr>
</tbody>
</table>

- Table 2 -

Normalization parameters $A$ and $B$
We can use the fitted spectra at $p_o = 18.8$ and 30 GeV/c to calculate for every kind of particles the mean transverse momenta, the multiplicities and the inelasticities (fraction of CM energy carried away by the corresponding kind of secondary particles). The results are given in Table 3.

<table>
<thead>
<tr>
<th>particle</th>
<th>18.8 GeV/c</th>
<th>30 GeV/c</th>
<th>(energy-independent)*)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$n_i$</td>
<td>$K_i$</td>
<td>$n_i$</td>
</tr>
<tr>
<td>$p$</td>
<td>0.67</td>
<td>0.27</td>
<td>0.98</td>
</tr>
<tr>
<td>$\pi^+$</td>
<td>1.01</td>
<td>0.101</td>
<td>1.17</td>
</tr>
<tr>
<td>$\pi^-$</td>
<td>0.62</td>
<td>0.056</td>
<td>0.70</td>
</tr>
<tr>
<td>$K^+$</td>
<td>0.066</td>
<td>0.010</td>
<td>0.098</td>
</tr>
<tr>
<td>$K^-$</td>
<td>0.037</td>
<td>0.0046</td>
<td>0.048</td>
</tr>
<tr>
<td>$\bar{p}$</td>
<td>0.0016</td>
<td>0.0003</td>
<td>0.0022</td>
</tr>
</tbody>
</table>

- Table 3 -

Multiplicities $n_i$, inelasticities $K_i$ and mean transverse momenta $p_{ti}$ at $p_o = 18.8$ and 30 GeV/c

The total charged multiplicity at 18.8 GeV/c is $n_{ch} = 2.61$ which is to be compared with experimental findings $^{14)} n_{ch}(18.8) = 3.2 - 4.5$. At 30 GeV/c we get $n_{ch} = 2.90$ which is to be compared with $n_{ch}(30) = 3.5 - 5$. As our production curves fit the measured spectra very well we conclude that this difference between the multiplicities is presumably due to a disagreement between two different sets of measurements.

*) because $T$ is energy independent.
We assume the inelasticities for neutron production $K_n = K_p$, for $K^0$ production $K K^0 = K K^+$, for $K_0$ and $\bar{K}_0$ productions $K K_0 = K K^+$ and $K K_0 = K K^-$ and for hyperon production $K_y = 0.07$ (determined from the missing number of baryons).

Having all the inelasticities we can test the energy conservation and get at $16.8$ GeV/c $\sum K_1 = 0.95$ at $p_0 = 30$ GeV/c $\sum K_1 = 0.92$. Considering the errors of the measured spectra and the assumptions made, a better agreement could not be expected. In Figs. 6 to 12 we compare the calculated spectra of $p$, $\pi^+$, $\pi^-$, $K^+$, $K^-$ and $\bar{p}$ with the measurements. $p$, $\pi^+$ and $\pi^-$ spectra are given at $p_0 = 30$ GeV/c. $K^+$, $K^-$ and $\bar{p}$ spectra are given at $p_0 = 16.8$ GeV/c. All spectra are plotted in the laboratory system, the $\pi^-$ spectrum is also given in the CM system. The agreement between curves and measured values is satisfactory in view of the fact that for each kind of particle (i.e., each set of curves) we have only three parameters. If one calculates with these parameters the pion spectra at other energies where measured values are available, then these predicted curves fit the experimental data as well as at $30$ GeV/c; an example is shown in Fig. 8a.

5. **EXTRAPOLATION OF THE SPECTRA TO HIGHER ENERGY ; ABSOLUTE NORMALIZATION (MULTIPLECTILITIES)**

Using the parameters $a, b, c$ and $t_0$ given in Table 1, we can no. calculate the spectra at higher energies. However only the shape of the spectra is determined by these parameters. We do not yet know the over-all energy dependent normalization factors or, if we assume the spectra to be normalized to one, the multiplicities at higher energies. The normalization factors given in Table 2 are valid only between $p_0 = 20$ to $30$ GeV/c and we cannot assume them to be valid still at much higher primary proton momenta.
We proceed as follows to determine the energy dependence of the multiplicities. We combine this time thermodynamics with energy conservation. In principle thermodynamics alone should be sufficient, but we then would have to solve a rather complicated problem: as already discussed earlier, we learn from thermodynamics that the initial fireball is an (imperfect) equilibrium of fireballs, which in turn are... and so on; when the initial fireball decays, it will not decay into pions, kaons, nucleons, etc., but into smaller fireballs (some of which are pions, kaons, nucleons, etc.), which in turn decay in a similar way. We thus may speak of a chain of decays, containing several generations. Fireballs in each generation have a momentum distribution (and mass distribution) following from applying thermodynamics [essentially Eq. (4)] and Lorentz transformations to the fireballs of the previous generation. By solving this recurrence problem, we would automatically arrive at properly normalized particle spectra without invoking energy conservation (energy would not exactly, but only on the average, be conserved for e, reasons already explained).

This problem seems to us too complicated to be pursued here. In fact, one can do something much simpler once one has the normalized particle spectra. The idea is:

- We know that in each generation pions, kaons, nucleons, etc., are produced, along with heavier fireballs, according to thermodynamics. They will contribute to the finally observed particles, whereas the heavier fireballs will decay further and contribute to later generations in the same way.

- We know (see Fig. 2) that with decreasing excitation energy the temperature remains a long time near to $T_0$ until it rather suddenly drops, when the energy becomes smaller than, say, 500 MeV. That means that in each generation the probabilities (per decaying fireball) for creation of pions, kaons, nucleons, etc., are roughly the same and independent of the CME; we only do not know, without solving the above-mentioned complicated problem, what the total number of contributing fireballs is.
We shall determine the "total number of contributing fireballs" - without assigning to this number any deep physical significance - by assuming constant thermodynamic probabilities for particle production and requiring that the energies of all particles produced taken together add up to the total CM energy; for this the non-normalized spectra are sufficient.

The calculation goes as follows:

We first calculate the average energy of each particle from the by now known spectra:

\[
\xi_i = \frac{\int W_i(p) \sqrt{p^2 + m_i^2} \, d^3p}{\int W_i(p) \, d^3p} \tag{13}
\]

Next we calculate the relative probabilities for creation of each kind of particle: integrating Eq. (4) (without the multiplicity function \( n_i \)) over all momenta yields the probability for unconditional production (i.e., when no conservation law restricts production):

\[
n_i = \frac{V_o}{2\pi^2} \, T \, m_i^2 \, K_2 \left( \frac{m_i}{T} \right) \left( 2J_i + 1 \right) \left( 2I_i + 1 \right) \tag{14}
\]

where \( V_o \approx (4\pi/3)m^{-3} \), \( K_2 \) = second modified Hankel function, \( J \) spin, I isospin, \( T \lesssim T_0 \) the temperature.

The particles observed are, however, partly created under restricting conditions. One can show \(^7\) that the numbers \( n_i \) given by Eq. (14) can be treated as independent probabilities as long as they are small (and they are small except for pions, where we need no combined probabilities); hence we write the production probabilities as

\[
W_i = n_i \quad \text{for single unconditional production}
\]

\[
W_{iK} = n_i n_K \quad \text{for associated production} \tag{15}
\]
Now we write down the probabilities for each kind of particle, taking into account properly their production mechanisms (e.g., kaons are produced in $K\bar{K}$ pairs and in associated production $p+p \rightarrow N+K+Y+\ldots$ or $\rightarrow 2K+2Y+\ldots$):

\[
\begin{align*}
\omega_{\pi} &= \eta_{\pi} \\
\omega_{K} &= \eta_{K}^2 + \frac{\eta_{K}\eta_{Y}}{\eta_{N}} + 2 \left( \frac{\eta_{K}\eta_{Y}}{\eta_{N}} \right)^2 \\
\omega_{\bar{K}} &= \eta_{K}^2 + \frac{\eta_{K}\eta_{Y}}{\eta_{N}} \\
\omega_{Y} &= \frac{\eta_{K}\eta_{Y}}{\eta_{N}} + 2 \left( \frac{\eta_{K}\eta_{Y}}{\eta_{N}} \right)^2 + \frac{\eta_{K}\eta_{N}\eta_{Y} + \eta_{Y}^2}{\eta_{N}} \\
\omega_{\bar{N}} &= \eta_{N}^2 + \frac{\eta_{K}\eta_{N}\eta_{Y}}{\eta_{N}} \\
\omega_{\bar{Y}} &= \eta_{Y}^2 + \frac{\eta_{K}\eta_{N}\eta_{Y}}{\eta_{N}} \\
\omega_{N} &= \eta_{N}^2 + \frac{\eta_{K}\eta_{N}\eta_{Y} - \eta_{K}\eta_{Y}}{\eta_{N}} + 2 \left( \frac{\eta_{K}\eta_{Y}}{\eta_{N}} \right)^2
\end{align*}
\]

(16)

Note that the probabilities (14) depend only on the mass (and $T$) and are equal for particle and antiparticle; with this remark (16) is self-explanatory; in the last line only newly created nucleons (or those absorbed into hyperons $Y$) are listed, to these add the two brought-in protons.

Suppose now the number $N(E)$ of contributing fireballs to be known, then we would have the total multiplicities for particles of kind $i$

\[
\begin{align*}
N_{\pi}(E) &= N(E) \omega_{\pi} \quad \text{(except nucleons)} \\
N_{N}(E) &= 2 + N(E) \omega_{N} \quad \text{(for nucleons)}
\end{align*}
\]

(17)

In fact we do not know the function $N(E)$ but we can now require all-over energy conservation and determine $N(E)$ from that; the energy balance yields
\[ E = N(E) \times \sum_{\text{all particles}} \frac{\omega_i \epsilon_i}{\sum \omega_j \epsilon_j} + 2 \epsilon_N \] (18)

where in the sum the term \( \epsilon_N \omega_N \) is included. Inserting the \( \epsilon_i \) and \( \omega_i \) from Eqs. (14) to (16) yields then the "total number of contributing fireballs" \( N(E) \) and then by (17) the absolute multiplicities

\[ N_L(E) = \frac{E - 2 \epsilon_N \omega_L}{\sum \omega_j \epsilon_j} \] (except nucleons)

\[ N_N(E) = 2 + \frac{E - 2 \epsilon_N \omega_N}{\sum \omega_j \epsilon_j} \] (nucleons)

and finally the absolute spectra

\[ N_L(\vec{p}) = N_L(E) \frac{W_i(\vec{p})}{\int W_i(\vec{p}) \, d^3p} \] (20)

It is this normalization which we use in our predictions.

We have, however, not simply applied Eq. (14) as it stands, because we know neither the exact production volume \( V_0 \) nor the effective temperature \( T \). We therefore considered Eq. (14) and all what follows from it, as a set of equations containing three free parameters \( N(E), V_0 \) and \( T \) to be fitted to experimental values. (This is a test of the whole approach: it could turn out that no fit is obtainable, or that the parameters have impossible values.)

\( N(E) \) is common to all particles. Therefore we can compare the actual particle numbers, e.g., at \( p_0 = 30 \) GeV/c with those of (17). Supposing that we can get a fit at all, we would expect the parameters \( V_0 \) and \( T \) to lie near to
\[ V_0 \approx \left( \frac{1}{4} \text{ to } 1 \right) \times \frac{4\pi}{3} m^{-3}_\pi \]

\[ T \approx (0.8 \text{ to } 1) \times T_0 \]

The volume \( V_0 \) should be reduced owing to the fact that mostly the inner parts of the collision contribute to the production. The average temperature might be lower than \( T_0 \); but not too low, again because the inner regions have a temperature near to \( T_0 \).

The parameters \( V_0, T \) and \( N(p_0 = 30) \) were determined by a fit using the particle numbers at 30 GeV/c given in Table 3. The following values are obtained (all energies in GeV):

\[ V_0 = 0.4 \times \frac{4\pi}{3} m^{-3}_\pi \]

\[ T = 0.1586 \]

\[ N(p_0 = 30) = 5.05 \]

In Table 4 we list the average CM energies \( \bar{E} \), used in (18) and in Table 5 we compare the numbers of \( p, \pi, K, \bar{K} \) and \( \bar{N} \) observed at 30 GeV/c (derived from Table 3, \( n_{\pi} = 2n_p \), \( n_{\pi^+} = 2n_{\pi^-} \) and \( n_{\bar{K}} = 2n_{K^+} \), \( n_{\bar{K}^-} = 2n_{K^0} \), \( n_{\bar{N}} = 2n_p \)) with the results of formula (17) with the fitted parameters.
\[ p_0 \ (GeV/c) \nabla E_{CM} \ (GeV) \nabla p \nabla \Pi^+ \nabla \Pi^- \nabla K^+ \nabla K^- \nabla \bar{p} \]

- Table 4 -

Average CM energies \( \xi \) as function of \( p_0 \ (GeV) \)

\[
\begin{array}{cccccccc}
\text{particle} & N_{\text{exp}} & N_{\text{theor}} & T & V_0 \\
N & 1.3 & 1.867 & \frac{156.6 \text{ MeV}}{5} \nabla 0.4 \times \left( \frac{4 \xi}{5 - \xi} \right)^{-1} \\
\Pi & 3.04 & 3.202 & \\
K & 0.196 & 0.083 & \\
\bar{K} & 0.096 & 0.055 & \\
\bar{N} & 0.0044 & & \\
\end{array}
\]

- Table 5 -

Comparison of observed particle numbers at \( p_0 = 30 \text{ GeV/c} \) with \( N_{\text{theor}} \) from formula (17)
Having determined $V_o$ and $T$ at 30 GeV/c, we see that these parameters have the expected a priori values. This fact and the good agreement with the experiment revealed in Table 5 makes us believe that this thermodynamical approach is reliable. From now on the thermodynamic model requires the $v_1$ to be energy independent and always valid for p-p collisions.

$N(E)$ turns out to vary approximately like $N(E) \approx E^{\frac{1}{2}}$.

6. PREDICTIONS FOR 300GeV AND DISCUSSION

In Table 6 we give the multiplicities $n_i$ and the inelasticities $K_i$ for $p_o = 300$ GeV/c. The mean transverse moments at 300 GeV/c are the same as given in Table 3 for 10.8' to 30 GeV/c (they are energy independent in our model since the temperature is constant).

<table>
<thead>
<tr>
<th>particle</th>
<th>$n_i$</th>
<th>$K_i$</th>
<th>particle</th>
<th>$n_i$</th>
<th>$K_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>1.67</td>
<td>0.41</td>
<td>$K^-$</td>
<td>0.21</td>
<td>0.009</td>
</tr>
<tr>
<td>$p$</td>
<td>0.83</td>
<td>0.21</td>
<td>$K^+$</td>
<td>0.105</td>
<td>0.0045</td>
</tr>
<tr>
<td>$\pi^-$</td>
<td>0.04</td>
<td>0.45</td>
<td>$\bar{K}^-$</td>
<td>0.014</td>
<td>0.0009</td>
</tr>
<tr>
<td>$\pi^+$</td>
<td>0.02</td>
<td>0.18</td>
<td>$\bar{K}$</td>
<td>0.007</td>
<td>0.00045</td>
</tr>
<tr>
<td>$K$</td>
<td>0.56</td>
<td>0.045</td>
<td>$\bar{p}$</td>
<td>0.35</td>
<td>0.085</td>
</tr>
<tr>
<td>$K^+$</td>
<td>0.28</td>
<td>0.023</td>
<td>$\bar{K}$</td>
<td>0.008</td>
<td>0.0009</td>
</tr>
</tbody>
</table>

- Table 6 -

Multiplicities $n_i$ and inelasticities $K_i$ at $p_o = 300$ GeV/c
From Table 6 we get at 300 GeV/c a total charged multiplicity of 6.27. Experimental values from cosmic ray measurements \(^{14}\) lie between 8 and 10. The total charged multiplicity predicted by our formula is lower than the cosmic ray results. We observed the same effect at 18.8 and 30 GeV/c. The disagreement with experimental multiplicities at 300 GeV is a consequence of the corresponding disagreement at 18.8 to 30 GeV/c \(^{9}\).

The fact that the inelasticities in Table 6 add to 1 is trivial, because we used the energy conservation to derive the particle numbers at 300 GeV/c.

Cosmic ray results \(^{15}\) indicate that the nucleon inelasticity \(K_N\) is independent of \(p_o\) and lies between 0.50 and 0.60. At 18.8 to 30 GeV/c our formula gives \(K_N\) values within these limits. The nucleon multiplicity \(K_N = 0.41\) at 300 GeV/c is smaller than obtained in cosmic ray measurements. Correspondingly the pion inelasticity \(K_{\pi^-} = 0.45\) is too high. The \(\pi^-\) inelasticity \(K_{\pi^-} = 0.087\) is reasonable but the \(\pi^+\) (and \(\pi^0\)) inelasticity \(K_{\pi^+} = 0.19\) is too large. What is the reason for this behaviour? In the fit at 30 GeV/c we determined the functions \(\chi(x)\) and \(T(x)\). The mean CM energies of the particles and their inelasticities at all energies are mainly determined by these two functions. The fit of the \(\pi^-\) spectrum at 30 GeV/c was very good, much better than for the proton and especially the \(\pi^+\) spectra as can be seen from Figs. 6 to 9. The values for the inelasticities at 300 GeV/c indicate that our functions \(\chi(x)\) and \(T(x)\) may be considered to be reliable for \(\pi^-\) but that these functions for \(\pi^+\) and protons are not yet good enough. In our work for the further improvement of the model we shall try to improve this situation. It might be mentioned, however, that a small change of the parameters would probably be sufficient to bring about the right inelasticities at larger energies; such a small change indeed, that the curves will not much differ from those presented here.

\(^{9}\) As already pointed out, we think that at 18.8 and 30 GeV there is a contradiction between measurements.
In Figs. 13 to 18 we present the proton, pion, kaon and antiproton spectra at 300 GeV/c. In Fig. 19 we give the momentum spectra dn/dp for all kinds of particles. The momentum spectra for pions are compared with two other extrapolations. This is the extrapolation due to Trilling \(^2\) and the extrapolation of the empirical spectra (represented by a 10 parameter formula \(^3\)) using the method of Cocconi \(^1\). Our curves agree below 200 GeV/c qualitatively with the Trilling curves but they do not show the two maxima. Above 200 GeV/c our spectra are more realistic than the Trilling curves, they decrease steeply and vanish at the kinematical limit.

The Cocconi extrapolation predicts much less pions at high momenta.

We think that the Cocconi extrapolation underestimates the pions at high momenta. These curves may be regarded as a pessimistic lower limit of the actual production spectra.

The inelasticity for \(\pi^+\) production which is predicted by our formula is higher than indicated by cosmic ray results. From this fact we conclude that the high momentum part of our \(\pi^+\) spectrum is still overestimated. The curve plotted should be regarded as an upper limit. The improvement of our model will result certainly in a \(\pi^+\) spectrum lying nearer to the \(\pi^-\) spectrum than presently. The \(\pi^-\) spectrum, however, is not expected to change much.

The inelasticity for nucleon production \(K_N\) being too small, indicates that the high momentum tail of the proton spectrum is underestimated. We expect that both the \(p\) and \(\pi^+\) spectra will undergo only rather small changes, at most perhaps a factor 2 in some parts of the curves.

The curves for kaon and antiproton production are much more uncertain than the pion and proton curves. The reason for this is the low accuracy of the measured kaon and antiproton spectra below 30 GeV/c.
It is possible that the kaon and antiproton production at 18.8 to 30 GeV/c is not yet far enough from threshold; this could be indicated by the empirically fitted normalization factors (Table 2) which grow for kaons and antiprotons much faster with $E_{CM}$ than for pions. This can, however, also be due to the errors of the measured spectra and/or to the fact that the experiments were done on beryllium instead of hydrogen. To get better predictions for kaon and antiproton production, better measurements of the spectra between 20 and 30 GeV/c are necessary.

Computer programmes for the CDC 6600 are available to calculate the spectra according to our or any other formula. The programmes have options to calculate the following data:

1) tables of spectra in the lab. or CM system;
2) graphs of $d^2N/dp\,d\Omega$ in the lab. or CM system;
3) graphs of $dN/dp$ in the lab. system;
4) graphs of the momentum spectra of decay products (2 or 3 particle decay);
5) multiplicities, inelasticities, mean momenta and energies, mean transverse momenta.
REFERENCES


2) G. Trilling - UCID 10 148 (1966).


7) R. Hagedorn - (to be published).


15) B. Peters - CERN 66-22 (1966);

16) The experimental mass spectrum has been brought up to date and smoothed out for comparison with the asymptotic prediction by Dr. M.J. Vandermeulen, University of Liège, Belgium (private communication).
Figure 1 The hadronic mass spectrum; dotted lines$^1$): smoothed experimental distribution (1964 and 1966); full line: asymptotic form predicted by the thermodynamic model; $T_0$ is not a free parameter here.

Figure 2 The relation between temperature and energy density in the thermodynamic model according to Eq. (2) with $\alpha = \frac{a}{b}$ and measured in GeV per nucleon volume: $V_0 \approx (4\pi/3)m^{-\frac{1}{2}}$.

Figure 3 Proton-proton collision seen from the CM frame. $z$ is the collision axis, $y$ the impact parameter.
Region (1): overlap;
Region (2): spread of interaction;
Region (3): nearly unaffected.

Figure 4 Qualitative picture of the expected temperature and velocity distribution; velocity expressed by $\gamma = (1 - v^2)^{-\frac{1}{2}}$.
(a) a rather central collision: $y \ll r$;
(b) a rather peripheral collision: $y \approx 2r$.

Figure 5 The maximal pion momentum coming from a fireball with given $\gamma$: the total CM energy decays into $p + m(\gamma)$ and then $m(\gamma) \rightarrow p + \pi$.

Figure 6 Secondary proton spectrum at $p_0 = 30$ GeV/c in the lab. system compared with experiment$^9$.

Figure 7 $\pi^+$ laboratory spectrum at $p_0 = 30$ GeV/c compared with experiment$^9$.

Figure 8 $\pi^-$ laboratory spectrum at $p_0 = 30$ GeV/c compared with experiment$^9$.
Figure 8a Predicted \( \pi^- \) laboratory spectrum at \( p_0 = 18.8 \) GeV/c compared with experiment \(^{10}\); no free parameters.

Figure 9 \( \pi^- \) CM spectrum at \( p_0 = 30 \) GeV/c compared with experiment \(^{9}\).

Figure 10 \( K^+ \) laboratory spectrum at \( p_0 = 18.8 \) GeV/c compared with experiment \(^{10}\).

Figure 11 \( K^- \) laboratory spectrum at \( p_0 = 18.8 \) GeV/c compared with experiment \(^{10}\).

Figure 12 \( \bar{p} \) laboratory spectrum at \( p_0 = 18.8 \) GeV/c compared with experiment \(^{10}\).

Figure 13 Secondary proton spectrum in the laboratory system at \( p_0 = 300 \) GeV/c.

Figure 14 \( \pi^+ \) laboratory spectrum at \( p_0 = 300 \) GeV/c.

Figure 15 \( \pi^- \) laboratory spectrum at \( p_0 = 300 \) GeV/c.

Figure 16 \( K^+ \) laboratory spectrum at \( p_0 = 300 \) GeV/c.

Figure 17 \( K^- \) laboratory spectrum at \( p_0 = 300 \) GeV/c.

Figure 18 \( \bar{p} \) laboratory spectrum at \( p_0 = 300 \) GeV/c.

Figure 19 Momentum spectra \( dN/dp \) of protons, \( \pi^{\pm} \), \( K^\pm \) and antiprotons at \( p_0 = 300 \) GeV/c. The pion momentum spectra are compared with the extrapolations due to Cocconi \(^{1}\) and Trilling \(^{2}\).

(our predictions: full curves; Cocconi and Trilling: dotted)
$\log_{10} g(m)$

Asymptotic formula:

$g(m) \approx \frac{a}{m^{10}} \exp\left(-\frac{m}{M_0}\right)$

All resonances
April 1966
971 states

All resonances
October 1964
609 states

FIG. 1
FIG. 3

FIG. 4a

FIG. 4b
all CME concentrated in these two

FIG. 5
$\frac{d^2N}{dp d\Omega}$

$P_0 = 30 \text{ GeV}/c$

$H_2$ Target

Lab. System

$0^\circ$

$1^\circ$

$2^\circ$

$3^\circ$

$4^\circ$

$5^\circ$

$6^\circ$

$7^\circ$

$8^\circ$

$9^\circ$

$10^\circ$

$11^\circ$

$12^\circ$

$13^\circ$

$14^\circ$

$15^\circ$

$16^\circ$

$17^\circ$

$18^\circ$

$19^\circ$

$20^\circ$

$21^\circ$

$22^\circ$

$23^\circ$

$24^\circ$

$25^\circ$

$26^\circ$

$27^\circ$

$28^\circ$

$29^\circ$

$30^\circ$

Number of protons (GeV/c, sr, interacting proton)

Secondary Proton Momentum $p$ (GeV/c)

NOT ALL MEASURED POINTS ARE PLOTTED

• 10.3° Anderson et al.

△ 2.35° “

○ 3.44° “

■ 5.73° “

□ 9.16° “

FIG. 6
$P_0 = 30 \text{ GeV/c}$

H$_2$ TARGET

LAB. SYSTEM

- $1^\circ$ Anderson et al.
- $5.7^\circ$ "
- $9.16^\circ$ "

NOT ALL EXPERIMENTAL POINTS ARE PLOTTED

Number of pions (GeV/c, sr, interacting proton)

Pion Momentum $p$ (GeV/c)
Figure 8: The plot shows the pion momentum spectrum for different angles with respect to the proton momentum. The legend indicates measurements by Anderson et al. at various angles: 1°, 5.7°, and 9.16°. The data points are not all measured, as indicated in the legend.
$\frac{d^2N}{dp\,d\Omega}$

**K$^+$ SPECTRUM**

$P_0 = 18.8 \text{ GeV/c}$

**Be TARGET**

**LAB. SYSTEM**

- $0^\circ$ Dekkers et al.
- $5.7^\circ$ "

**Number of kaons / (GeV/c, sr, interacting proton)**

**Kaon Momentum $p$ (GeV/c)**

$\log_{10}$ scale

$10^{-5}$ to $10^1$

$0$ to $18.80$

$24^\circ$ to $0^\circ$

FIG. 10
FIG. 11

$P_0 = 18.8 \text{ GeV/c}$

$K^-$ SPECTRUM

BeTARGET LAB. SYSTEM

- $0^\circ$ Dekkers et al.
- $5.7^\circ$ "

Number of kaons/(GeV/c, sr, interacting proton)

Kaon Momentum $p$ (GeV/c)
$P_0 = 18.8$ GeV/c  
Be TARGET  
LAB. SYSTEM  

$\frac{d^2N}{dp \, d\Omega}$  

ANTIPROTON SPECTRUM  

$\bullet 0^\circ$ Dekkers et al.  
$\circ 5.7^\circ$  

Number of antiprotons/(GeV/c, sr, interacting proton)  

Antiproton Momentum $p$ (GeV/c)  

$10^{-5}$  
$10^{-4}$  
$10^{-3}$  
$10^{-2}$  
$10^{-1}$  
$1$  

$14^\circ$  
$24^\circ$  
$20^\circ$  
$12^\circ$  
$10^\circ$  
$8^\circ$  
$6^\circ$  
$4^\circ$  
$0^\circ$  
$16^\circ$  
$2^\circ$  

FIG. 12
$P_0 = 300 \text{ GeV/c} \quad \text{H}_2 \text{ TARGET LAB. SYSTEM}$

Proton Spectrum

$\frac{d^2 N}{dp \, d\Omega}$

Secondary protons (GeV/c, sr, interacting proton)

Secondary Proton Momentum $p$ (GeV/c)

Fig. 13
$d^2 N / dp d\Omega$

Pions (GeV/c, sr, interacting proton)

Pion Momentum $p$ (GeV/c)

Fig. 15

$P_0 = 300$ GeV/c $\text{H}_2 \text{ TARGET LAB. SYSTEM}$

$\pi^-$ Spectrum
$P_0 = 300 \text{ GeV/c}$

Number of particles / (GeV/c, interacting proton)

Secondary Particle Momentum $p$ (GeV/c)

FIG. 19