7

PION-NUCLEON PHASE SHIFT ANALYSIS

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1

INTRODUCTION

IN the last few years tremendous strides forward have been made in our knowledge and understanding of the pion-nucleon system. Three years ago, the details of pion-nucleon scattering were known with some uncertainty up to 310 MeV. They are now known in considerable detail to beyond 1310 MeV. This information explosion, which it may fairly be called, had as its basis the development of a realistic dynamical theory of pion-nucleon scattering by partial wave dispersion relations, the advent of a series of experimental results of a quantity and quality hitherto lacking, and, growing out of these, detailed phase shift analysis. It is with this latter topic that these lectures will be concerned.

Beyond recognizing their existence, lack of time and space precludes any further mention of the experiments, although they deserve a better fate than to be mouldering in the references. The theory of partial wave dispersion relations will fare rather better since it plays a crucial role in the most extensive and complete phase shift analysis yet attempted.

Phase shift analysis is a lengthy, painstaking and at times, if the truth be told, a rather tedious business, involving a vast amount of computing. To my knowledge no one has ever dared to estimate the cost per phase, but it must be high. It can have its exciting moments, and this has been particularly true of pion-nucleon phase shift analysis with new resonances popping up, if not every other day, at least with sufficient regularity to make one wonder what is going to happen next. To the outside observer, this unexpected wealth of structure is undoubtedly the most striking and interesting result of the analyses.

Since the nature of the subject is largely a descriptive one, these lectures will follow suit. In preparing lecture notes, it is very tempting to re-derive every formula used, "for completeness". This has not been done, the necessary equations being written down with reference and without proof.

The exceptions to this are some odds and ends which have been gathered into two appendices and consigned to the end, out of harm's way.

2

KINEMATICS AND FORMULAE

The S -matrix for elastic pion-nucleon scattering can be written as

$$S_{fi} = \delta_{fi} - (2\pi)^4 i \delta^4(p_1 + q_1 - p_2 - q_2) \left\{ \frac{m^2}{4E_1 E_2 \omega_1 \omega_2} \right\}^{\frac{1}{2}} \bar{u}_2 T_{fi} u_1 \quad \dots [1]$$

where q_1 and q_2 are the four-momenta of the initial and final pions, p_1 and p_2 are the four-momenta of the initial and final nucleons. The nucleon energies E_i and the pion energies ω_i are given by

$$E_i = (m^2 + p_i^2)^{\frac{1}{2}}, \quad \omega_i = (\mu^2 + q_i^2)^{\frac{1}{2}}, \quad i = 1, 2 \quad \dots [2]$$

where m and μ are respectively the nucleon and pion masses, and u_1 and u_2 are the Dirac spinors for the initial and final nucleon states.

The scattering amplitude T is of the explicitly Lorentz invariant form¹

$$T_{fi} = -A_{fi} + i\gamma \cdot Q B_{fi} \quad \dots [3]$$

where $Q = \frac{1}{2}(q_1 + q_2)$, the γ^μ are the Dirac γ matrices and A and B are scalar invariant functions.

The isotopic spin decomposition of the amplitudes A and B is given by¹

$$A_{\beta\alpha} = \delta_{\beta\alpha} A^{(+)} + \frac{1}{2} [\tau_\beta, \tau_\alpha] A^{(-)} \quad \dots [4a]$$

$$B_{\beta\alpha} = \delta_{\beta\alpha} B^{(+)} + \frac{1}{2} [\tau_\beta, \tau_\alpha] B^{(-)} \quad \dots [4b]$$

where α and β are respectively the charge indices of the initial and final pions and the τ_α ($\alpha = 1, 2, 3$) are the usual isotopic spin operators.

The amplitudes $A^{(\pm)}$, $B^{(\pm)}$ are related to the amplitudes $A^{(T)}$, $B^{(T)}$ ($T = \frac{1}{2}, \frac{3}{2}$), the eigenstates of isotopic spin, by

$$A^{(+)} = \frac{1}{3} A^{(\frac{1}{2})} + \frac{2}{3} A^{(\frac{3}{2})}, \quad A^{(-)} = \frac{1}{3} A^{(\frac{1}{2})} - \frac{1}{3} A^{(\frac{3}{2})} \quad \dots [5a]$$

or conversely

$$A^{(\frac{1}{2})} = A^{(+)} + 2A^{(-)}, \quad A^{(\frac{3}{2})} = A^{(+)} - A^{(-)} \quad \dots [5b]$$

with identical relations holding for the B amplitudes.

The A 's and B 's are functions of the kinematic invariants s , t , u defined by

$$s = (p_1 + q_1)^2, \quad t = (q_1 - q_2)^2, \quad u = (p_1 - q_2)^2. \quad \dots [6]$$

These are not independent, being related by

$$s + t + u = 2m^2 + 2\mu^2. \quad \dots [7]$$

If we use the subscripts $+$ and $-$ to label the amplitudes for the elastic

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processes $\pi^+ + p \rightarrow \pi^+ + p$ and $\pi^- + p \rightarrow \pi^- + p$ respectively, then a convenient expression of crossing symmetry^{1,2} is the equality of the matrix elements

$$\langle p_2, q_2 | T_+ | p_1, q_1 \rangle = \langle p_2, -q_1 | T_- | p_1, -q_2 \rangle. \quad \dots[8]$$

Under the exchange $q_1 \rightleftharpoons -q_2$, $s \rightleftharpoons u$ and $t \rightarrow t$. Since the amplitudes A and B are independent (the term A is independent of nucleon spin, while the B term depends on it via the factor $\gamma \cdot Q$) crossing symmetry can be applied separately to A and $\gamma \cdot Q B$. Noting that $(\gamma \cdot Q) \rightarrow -(\gamma \cdot Q)$ under crossing, we obtain finally the crossing relation

$$A^{(+)}(s, t, u) = A^{(+)}(u, t, s), \quad A^{(-)}(s, t, u) = -A^{(-)}(u, t, s), \quad \dots[9a]$$

$$B^{(+)}(s, t, u) = -B^{(+)}(u, t, s), \quad B^{(-)}(s, t, u) = B^{(-)}(u, t, s). \quad \dots[9b]$$

We will work invariably in the c.m. system. In this system let W be the total energy, q the magnitude of the pion momentum and θ the scattering angle. Then

$$\left. \begin{aligned} s &= W^2 \\ t &= -2q^2(1 - \cos \theta) \\ u &= 2m^2 + 2\mu^2 + 2q^2(1 - \cos \theta) - W^2. \end{aligned} \right\} \quad \dots[10]$$

Since the ultimate intention is to discuss phase shifts, it is necessary to express the amplitudes A and B in terms of partial wave $\pi-N$ amplitudes. The first step is to relate the transition amplitude [3] to the 2×2 Pauli scattering matrix M .

From [1], the c.m. differential cross-section is

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{4\pi W} \right)^2 \sum |\bar{u}_2 T_{fi} u_1|^2 \quad \dots[11]$$

where Σ implies a sum over final spin states and an average over the initial spin states. In the M form, the c.m. differential cross-section is

$$\frac{d\sigma}{d\Omega} = \sum |\langle f | M | i \rangle|^2 \quad \dots[12]$$

where $|i\rangle$ and $|f\rangle$ are the Pauli spinors for the initial and final nucleon spin states and M is related to T by

$$\langle f | M | i \rangle = -\frac{m}{4\pi W} \bar{u}_2 T_{fi} u_1. \quad \dots[13]$$

In terms of the $\pi-N$ helicity amplitudes³ $f_1(\theta), f_2(\theta)$,

$$M = f_1(\theta) + \frac{\sigma \cdot q_2 \sigma \cdot q_1}{q^2} f_2(\theta) \quad \dots[14]$$

where σ is the nucleon spin, q_1 and q_2 the initial and final pion momenta and θ the scattering angle. The helicity amplitudes f_1 and f_2 can now be expressed in terms of A and B , using [3], [13] and [14], yielding

$$f_1 = \frac{(E+m)}{8\pi W} [A + (W-m)B], \quad \dots[15a]$$

$$f_2 = \frac{(E-m)}{8\pi W} [-A + (W+m)B]. \quad \dots[15b]$$

The inverse of [15] is

$$\frac{1}{4\pi} A = \frac{W+m}{E+m} f_1 - \frac{W-m}{E-m} f_2, \quad \dots[16a]$$

$$\frac{1}{4\pi} B = \frac{1}{E+m} f_1 + \frac{1}{E-m} f_2. \quad \dots[16b]$$

The helicity amplitudes are given in terms of the partial wave amplitudes $f_{l\pm}$, corresponding to total angular momentum $j = l \pm \frac{1}{2}$ by⁴

$$f_1 = \sum_{l=0}^{\infty} (f_{l+} P'_{l+1}(x) - f_{l-} P'_{l-1}(x)), \quad \dots[17a]$$

$$f_2 = \sum_{l=1}^{\infty} (f_{l-} - f_{l+}) P'_l(x) \quad \dots[17b]$$

where $x = \cos \theta$ and $P'_l(x)$ is the first derivative of the Legendre polynomial $P_l(x)$. The partial wave amplitudes $f_{l\pm}$ are expressed in turn in terms of phase shifts $\delta_{l\pm}$ by

$$f_{l\pm} = \frac{e^{2i\delta_{l\pm}} - 1}{2iq}. \quad \dots[18]$$

Writing $\delta_{l\pm} = \alpha_{l\pm} + i\beta_{l\pm}$ this becomes

$$f_{l\pm} = \frac{e^{-2\beta_{l\pm}} e^{2i\alpha_{l\pm}} - 1}{2iq} = \frac{\eta_{l\pm} e^{2i\alpha_{l\pm}} - 1}{2iq} \quad \dots[19]$$

$\alpha_{l\pm}$, $\eta_{l\pm}$ are real, and unitarity requires that $0 \leq \eta_l \leq 1$.

In terms of $\xi_J = \eta_J e^{2i\alpha_J}$ the partial wave total, elastic and inelastic cross-sections are given by

$$\sigma_{\text{Total}}^{(J)} = \frac{4\pi}{q} (J + \frac{1}{2}) \text{Im } f_J, \quad \dots[20a]$$

$$\sigma_{\text{Elastic}}^{(J)} = \frac{\pi}{q^2} (J + \frac{1}{2}) |1 - \xi_J|^2, \quad \dots[20b]$$

$$\sigma_{\text{Inelastic}}^{(J)} = \frac{\pi}{q^2} (J + \frac{1}{2}) \{1 - |\xi_J|^2\}. \quad \dots[20c]$$

By means of the orthogonality relation

$$\int_{-1}^1 dx P'_k(x) \{P_{l+1}(x) - P_{l-1}(x)\} = \delta_{kl} \quad \dots[21]$$

any partial wave amplitude can be extracted from the helicity amplitudes by the relation

$$f_{l\pm} = \frac{1}{2} \int_{-1}^1 dx \{P_l(x)f_1 + P_{l\pm 1}(x)f_2\}. \quad \dots[22]$$

An alternative form for the scattering matrix M is in terms of the non-spin-flip and spin-flip amplitudes $f(\theta)$ and $g(\theta)$ ⁵

$$M = f(\theta) + i \frac{\sigma \cdot (q_2 \times q_1)}{|q_2 \times q_1|} g(\theta). \quad \dots[23]$$

The expansion of $f(\theta)$ and $g(\theta)$ in terms of the scattering amplitudes $f_{l\pm}$ is

$$f(\theta) = \sum_{l=0}^{\infty} [(l+1)f_{l+} + lf_{l-}] P_l(\cos \theta) \quad \dots[24a]$$

$$g(\theta) = \sum_{l=1}^{\infty} [f_{l+} - f_{l-}] P_l^{(1)}(\cos \theta) \quad \dots[24b]$$

where $P_l^{(1)}(\cos \theta) = \sin \theta (d/d(\cos \theta)) P_l(\cos \theta)$.

In terms of $f(\theta)$ and $g(\theta)$ the elastic differential cross-section is given by

$$\frac{d\sigma(\theta)}{d\Omega} = |f(\theta)|^2 + |g(\theta)|^2 \quad \dots[25]$$

and the polarization $P(\theta)$ of the recoil nucleon by

$$P(\theta) = 2 \operatorname{Im} [f^*(\theta)g(\theta)] \left/ \frac{d\sigma(\theta)}{d\Omega} \right. \quad \dots[26]$$

The π^+p system, which has a z component of isotopic spin $T_z = \frac{3}{2}$, can exist only in the isotopic spin state $T = \frac{3}{2}$. However, the π^-p system, which has $T_z = -\frac{1}{2}$, is a linear combination of the $T = \frac{1}{2}$ and $T = \frac{3}{2}$ isotopic spin states. The appropriate combinations are

$$A(\pi^+ + p \rightarrow \pi^+ + p) = A(T = \frac{3}{2}), \quad \dots[27a]$$

$$A(\pi^- + p \rightarrow \pi^- + p) = \frac{1}{3}[A(T = \frac{3}{2}) + 2A(T = \frac{1}{2})], \quad \dots[27b]$$

$$A(\pi^- + p \rightarrow \pi^0 + n) = \frac{\sqrt{2}}{3}[A(T = \frac{3}{2}) - A(T = \frac{1}{2})]. \quad \dots[27c]$$

The scattering amplitudes of equations [24] must be extended to take into account the non-relativistic Coulomb effects as well as the first order relativistic Coulomb corrections.^{7,8,9} To do this, we introduce the following quantities:

$$\eta = \frac{e^2}{h\nu} \quad \dots[28]$$

where v is the velocity of the incident pion in the laboratory (η should not be confused with the inelasticity parameter $\eta_{l\pm}$ of [19])

$$B = \frac{\frac{1}{2}(\mu_p \beta_p \beta_\pi) + \frac{1}{4}(2\mu_p - 1)\beta_p^2}{1 + \beta_\pi \beta_p} \quad \dots[29]$$

where μ_p is the magnetic moment of the proton (in nuclear magnetons) and β_p, β_π are respectively the c.m. velocities of the proton and pion, divided by the velocity of light.

The non-relativistic Coulomb phase shift Φ_l of order l is zero for $l = 0$ and is given by

$$\Phi_l = \sum_{n=1}^l \tan^{-1}(\eta/n) \quad \dots[30]$$

for $l \geq 1$.

The *total* phase shifts are related to the nuclear phase shifts by

$$\alpha_{l\pm}(+) = \alpha_{l\pm} + \Phi_{l\pm}, \quad \alpha_{l\pm}(-) = \alpha_{l\pm} - \Phi_{l\pm} \quad \dots[31]$$

where $\Phi_{l\pm}$ is the complete Coulomb phase shift of order l , consisting of the non-relativistic Coulomb phase shift [30] plus a first order relativistic correction

$$\Phi_{l\pm} = \Phi_l + \Delta\Phi_{l\pm} \quad \dots[32]$$

where

$$\Delta\Phi_{0+} \approx \eta[\frac{1}{2}(\beta_p \beta_\pi) + \frac{1}{4}(2\mu_p - 1)\beta_p^2/(1 + \beta_\pi \beta_p)], \quad \dots[33a]$$

$$\Delta\Phi_{l+} \approx \eta B/(l+1), \quad l \geq 1, \quad \dots[33b]$$

$$\Delta\Phi_{l-} \approx -\eta B/l, \quad l \geq 1. \quad \dots[33c]$$

Using the superscripts ⁽¹⁾ and ⁽³⁾ to denote states of isotopic spin $\frac{1}{2}$ and $\frac{3}{2}$ respectively, the Coulomb corrected scattering amplitudes are

(i) $\pi^+ + p \rightarrow \pi^+ + p$

$$\begin{aligned} f(\theta) = & \frac{-\eta}{2q \sin^2(\theta/2)} \exp \{ -i\eta \ln [\sin^2(\theta/2)] \} \\ & + \sum_{l=0} \left[(l+1) \frac{\eta_{l+}^{(3)} \exp \{ 2i\alpha_{l+}^{(3)}(+) \} - \exp \{ 2i\Phi_l \}}{2iq} \right. \\ & \left. + l \frac{\eta_{l-}^{(3)} \exp \{ 2i\alpha_{l-}^{(3)}(+) \} - \exp \{ 2i\Phi_l \}}{2iq} \right] P_l(\cos \theta) \dots[34] \end{aligned}$$

$$\begin{aligned} g(\theta) = & \frac{\eta B \sin \theta}{2q \sin^2(\theta/2)} + \sum_{l=1} \left[\frac{\eta_{l+}^{(3)} \exp \{ 2i\alpha_{l+}^{(3)}(+) \} - \eta_{l-}^{(3)} \exp \{ 2i\alpha_{l-}^{(3)}(+) \}}{2iq} \right. \\ & \left. - \frac{\eta B}{q} \frac{2l+1}{l(l+1)} \right] P_l^{(1)}(\cos \theta) \quad \dots[35] \end{aligned}$$

(ii) $\pi^- + p \rightarrow \pi^- + p$

$$\begin{aligned}
f(\theta) = & \frac{\eta}{2q \sin^2(\theta/2)} \exp \{i\eta \ln [\sin^2(\theta/2)]\} \\
& + \sum_{l=0} \left[(l+1) \frac{\eta_{l+}^{(3)} \exp \{2i\alpha_{l+}^{(3)}(-)\} + 2\eta_{l+}^{(1)} \exp \{2i\alpha_{l+}^{(1)}(-)\}}{6iq} \right. \\
& \quad \left. + l \frac{\eta_{l-}^{(3)} \exp \{2i\alpha_{l-}^{(3)}(-)\} + 2\eta_{l-}^{(1)} \exp \{2i\alpha_{l-}^{(1)}(-)\}}{6iq} \right. \\
& \quad \left. - (2l+1) \frac{\exp \{-2i\Phi_l\}}{2iq} \right] P_l(\cos \theta) \quad \dots[36]
\end{aligned}$$

$$\begin{aligned}
g(\theta) = & \frac{\eta B \sin \theta}{2q \sin^2(\theta/2)} + \sum_{l=1} \left[\frac{\eta_{l+}^{(3)} \exp \{2i\alpha_{l+}^{(3)}(-)\} + 2\eta_{l+}^{(1)} \exp \{2i\alpha_{l+}^{(1)}(-)\}}{6iq} \right. \\
& \quad \left. - \frac{\eta_{l-}^{(3)} \exp \{2i\alpha_{l-}^{(3)}(-)\} + 2\eta_{l-}^{(1)} \exp \{2i\alpha_{l-}^{(1)}(-)\}}{6iq} \right. \\
& \quad \left. + \frac{\eta B}{q} \frac{2l+1}{l(l+1)} \right] P_l^{(1)}(\cos \theta). \quad \dots[37]
\end{aligned}$$

Equations [34], [35], [36] and [37], together with [25] and [26], form the basis of the phase shift analysis. The general notation we will use for the partial waves is $L_{2T, 2J}$ where L is the angular momentum (in the usual spectroscopic notation $S, P, D, F, \dots$ etc.), T is the isotopic spin and J the total angular momentum.

3

PHASE SHIFT ANALYSIS I

The complexity of the phase shift expansions ([34], [35], [36] and [37]) of the scattering amplitude means that there is no simple way of deriving values for the phase shifts and inelasticities from experimental data. The procedure to be followed is to minimize the function

$$M = \sum_i \left\{ \frac{X_i^{(e)} - X_i^{(c)}}{\Delta X_i^{(e)}} \right\}^2 \quad \dots[38]$$

where $X_i^{(e)}$ is the quantity X_i as obtained from experiment, $\Delta X_i^{(e)}$ is the experimental error (standard deviation) in $X_i^{(e)}$, $X_i^{(c)}$ is the quantity X_i as calculated for a given set of phase parameters $\alpha_{l\pm}^{(T)}$, $\eta_{l\pm}^{(T)}$ and M is considered as a function of these phase parameters. The quantity M is also called "chi-squared" (χ^2). Having obtained a minimum, errors can be assigned to the phase parameters using the standard error matrix approach. The shape of the M hypersurface near the minimum is examined by evaluating the second

partial derivatives of M with respect to each of the phase parameters (p_i , say) used. These partial derivatives form a matrix G defined by

$$G_{ij} = \frac{1}{2}(\partial^2 M / \partial p_i \partial p_j). \quad \dots[39]$$

The inverse of this matrix yields the error matrix G^{-1} with the properties

$$[(G^{-1})_{ii}]^{\frac{1}{2}} = (\Delta p_i)_{\text{R.M.S.}} \quad \dots[40a]$$

and

$$(G^{-1})_{ij} = C_{ij}(\Delta p_i)_{\text{R.M.S.}}(\Delta p_j)_{\text{R.M.S.}} \quad \dots[40b]$$

where C_{ij} is the correlation coefficient.

The available experimental data consist of π^+p and π^-p total and elastic cross-sections, elastic differential cross-sections and recoil proton polarizations and π^-p charge exchange differential cross-sections, and are available at a wide variety of angles and energies. This list does not exhaust all the possible experiments, there also being the possibility of measuring charge exchange polarization (i.e. the polarization of the recoil neutron in π^-p charge exchange scattering) and of performing triple scattering experiments. However, these experiments are all ones of considerable difficulty and as yet have not been investigated seriously.

Ideally, it should be possible to obtain a unique set of phase parameters at a given energy by including in the function M all data available at (or very close to) that energy. The number of energies at which there exist sufficient measurements of all the above data to make this theoretically possible is small, and even at such energies the ideal is very seldom realized and the word "unique" must be used with caution. Several things conspire to thwart the hopeful phase shift analyst.

1. Systematic errors can exist in any given measurement. These errors are not included in the standard deviation $X_i^{(e)}$ and frequently it is not possible to modify the latter in a consistent way to take account of these extra errors, even assuming one is aware of their existence. The most common errors of this type arise in absolute normalization, which is a comparatively tractable problem, although occasionally there are discrepancies in the shape of an angular distribution coming from different measurements. This is a much more serious situation.
2. A measured angular distribution can be incomplete in the sense that only a part of the complete angular range has been covered. A common result of this situation is that two (or more) possible phase solutions are obtained, which give very similar predictions in that part of the angular range in which the measurements lie, but which differ considerably in the region where there are no measurements.
3. The errors on an experiment can be sufficiently large to allow different solutions, giving different predictions, to fit the experimental data

equally well in the sense that the chi-squared contribution from the data is approximately the same for each solution.

4. The partial wave expansion must be truncated at some value of l , $l = l_{\max}$ say. An interesting anomaly of phase shift analysis is that while it is necessary to include all partial waves up to and including $l = l_{\max}$ to obtain a good fit to the data, the highest partial waves are not well determined by the data and the possible experimental errors on them are large. Although in themselves small in magnitude, they can interfere strongly with the lower partial waves causing considerable ambiguity there.

All this makes a conventional energy-by-energy phase shift analysis extremely difficult, posing serious problems of uniqueness and of continuity. This does not mean that such an analysis cannot be performed¹⁰ and although it gives a realistic picture of the large (and consequently dominant) partial waves, the smaller are, at best, only qualitatively correct.

One approach which overcomes many of the above difficulties (although at the expense of introducing others) is that of the energy dependent analysis.^{11,12,13} In this approach an energy dependent parametric form is assumed for each partial wave and *all* data at *all* energies are fitted simultaneously with all partial waves up to and including $l = l_{\max}$. The advantages of this method are that continuity is built in *ab initio*, systematic errors tend to be ironed out, and the lack of data at some energies and the incompleteness or poor quality of data at others are no longer serious obstacles. The problem of the higher partial waves remains and new problems arise. To make such an analysis numerically tractable, the number of parameters allowed for each partial wave is necessarily small and the choice of parametric form is arbitrary. Consequently, there is the danger of biasing the solution by an unfortunate choice of parametric form, and of ironing out genuine structure along with the systematic singularities. Because of the limited number of parameters, energy dependent fits are invariably statistically poor fits (typically $\chi^2/N_F > 2$, where N_F is the number of degrees of freedom) and it is very difficult to assign realistic errors to the solution.

The parametrization chosen by Roper *et al.*¹¹ for the scattering amplitudes $A_{l\pm} = qf_{l\pm}$ was a combination of specifically resonant and non-resonant forms

$$A_{l\pm} = \varepsilon_{l\pm} A_{l\pm}(\text{resonant}) + A_{l\pm}(\text{non-resonant}) \quad \dots[41]$$

where

$$A_{l\pm}(\text{resonant}) = \frac{\frac{1}{2}\Gamma_{l\pm}^{(e)}}{(\omega_{l\pm}^R - \omega) - \frac{1}{2}i\Gamma_{l\pm}^{(o)}} \quad \dots[42]$$

is a resonance amplitude of the Layson form.¹⁴

$$\Gamma_{l\pm}^{(el)} = \frac{4m}{\omega_{l\pm}^R + \omega} (qr_{l\pm}) \gamma_{l\pm}^2 V_l(qr_{l\pm}) \quad \dots[43]$$

is the resonance elastic width and

$$V_l(qr_{l\pm}) = [(qr_{l\pm})^2 \{j_l^2(qr_{l\pm}) + n_l^2(qr_{l\pm})\}]^{-1} \quad \dots[44]$$

is the barrier penetration factor.

$$\Gamma_{l\pm}^{(t)} = \Gamma_{l\pm}^{(el)} + \Gamma_{l\pm}^{(in)} \quad \dots[45]$$

is the resonance total width and

$$\Gamma_{l\pm}^{(in)} = \tilde{\Gamma}_{l\pm}^{(in)} (q - q_I)^{2l+1}. \quad \dots[46]$$

q_I is the threshold pion c.m. momentum for one-pion production, $\varepsilon_{l\pm} = 1$ for states in which it was chosen to put a resonance and $\varepsilon_{l\pm} = 0$ for states without a resonance. The non-resonant amplitude

$$A_{l\pm}(\text{non-resonant}) = (\eta_{l\pm} e^{2i\alpha_{l\pm}} - 1)/2i \quad \dots[47]$$

is the background for resonant states ($\varepsilon_{l\pm} = 1$) or the amplitude for states in which there are no resonances and was parametrized either by

$$\tan \alpha_{l\pm} = q^{2l+1} \sum_{n=0}^{l_{\max}-l} (a_{l\pm})_n q^n \quad \dots[48a]$$

or

$$\left. \begin{aligned} \alpha_{l\pm} &= q^{2l+1} \sum_{n=0}^{l_{\max}-l} (a_{l\pm})_n q^n \\ \eta_{l\pm} &= e^{-2v_{l\pm}} \end{aligned} \right\} \quad \dots[48b]$$

with

$$\left. \begin{aligned} v_{l\pm} &= 0 \quad \text{for } q < q_I, \\ v_{l\pm} &= (q - q_I)^{2l+1} \sum_{n=0}^{l_{\max}-l} (b_{l\pm})_n (q - q_I)^n \quad \text{for } q \geq q_I. \end{aligned} \right\} \quad \dots[48c]$$

With this parametrization the correct threshold behaviour is guaranteed for the phase shifts and approximated for the absorption parameters.

This parametrization has the merit of comparative simplicity but, since the phase shifts $\alpha_{l\pm}$ and the inelasticities $\eta_{l\pm}$ are parametrized independently, there is no possibility of allowing for correlations between the two. That $\alpha_{l\pm}$ and $\eta_{l\pm}$ are correlated as regards their energy dependence is well known, and any changes in $\eta_{l\pm}$ must be reflected in a specific way in $\alpha_{l\pm}$.¹⁵ The parametrization of Bransden *et al.*¹² was chosen to allow for this explicitly.

Their parametrization is based on the dispersion relation for the inverse amplitude $f_{l\pm}^{-1}$.¹⁶

$$\begin{aligned} \operatorname{Re} f_{l\pm}^{-1}(q) = & -\frac{q^2 - q_0^2}{\pi} \int_0^\infty dq'^2 \frac{q' R_{l\pm}(q')}{(q'^2 - q^2)(q'^2 - q_0^2)} \\ & + \frac{q^2 - q_0^2}{\pi} \int_{-\infty}^{-a} dq'^2 \frac{\Delta f_{l\pm}^{-1}(q')}{(q'^2 - q^2)(q'^2 - q_0^2)} + \sum_{n=0}^l \lambda_n / q^{2n} \quad \dots [49] \end{aligned}$$

where

$$a = \mu^2 \frac{(1 - \mu^2/4m^2)}{(1 + 2\mu^2/m^2)} \quad \dots [50]$$

and the λ_n are (arbitrary) constants. $\Delta f_{l\pm}^{-1}$ is the discontinuity of $-2if_{l\pm}^{-1}$ across the left-hand cut. For those cases in which the amplitude has a zero for a physical value of the energy, an additional pole must be added to the right-hand side of [49]. The function $R_{l\pm}$ is given by $\sigma_{l\pm}^{(\text{total})}/\sigma_{l\pm}^{(\text{elastic})}$.

Delta functions for $\Delta f_{l\pm}^{-1}$ were considered to be a reasonable form of parametrization since most of the energies concerned in the analysis were well away from the unphysical region. Thus

$$\Delta f_{l\pm}^{-1} = \sum_{n=1}^K (r_{l\pm})_n \delta(q^2 + q_n^2) \quad \dots [51]$$

where $(r_{l\pm})_n$ and q_n^2 are the parametric constants. The value of K differs for different l and in general decreases with increasing l although there is no *a priori* way of choosing it.

$R_{l\pm}$ was parametrized by ratios of polynomials, chosen in such a way as to keep computing time to a minimum and yet remain flexible. In all searches, particular cases of the following general expression were used

$$\begin{aligned} R_{l\pm}(q) = & 1 + \theta(q - q_I) \left\{ a_{l\pm} \frac{q(q - q_I)}{(1 + q^2/A^2)} + b_{l\pm} \frac{q - q_I}{(1 + q^2/B^2)} \right\} \\ & + \theta(q - q_\eta) \left\{ c_{l\pm} \frac{q(q - q_\eta)}{(1 + q^2/C^2)} + \delta_{l0} d_{l\pm} \frac{q(q^2 - q_\eta^2)^{\frac{1}{2}}}{(1 + q^2/D^2)} \right\} \quad \dots [52] \end{aligned}$$

where $a_{l\pm}, b_{l\pm}, c_{l\pm}, d_{l\pm}, A, B, C, D$ are constants, q_I is the first inelastic threshold and q_η is the threshold for η production. This form allowed the integral in equation [49] to be evaluated analytically.

The numerical technique of finding the best fit in the energy dependent analysis is essentially the same as that of the energy independent analysis except that the summation in equation [38] is extended over all energies and the quantities varied to find a minimum of M are the parameters of the phase shifts and inelasticities, not the phase shifts and inelasticities themselves.

Roper *et al.*¹¹ performed the analysis over three energy ranges: 0–350 MeV, 0–700 MeV and 300–800 MeV with $l_{\max} = 3$ in the first case and $l_{\max} = 3$ or 4 in the two latter, and a series of different starting points were used (32 in the 0–350 MeV analysis, 17 in the 0–700 MeV analysis, 3 in the

300–800 MeV analysis). The best fits in the three cases had χ^2/N_F ratios of 1.86 ($\chi^2 = 1002$, $N_F = 539$), 2.59 ($\chi^2 = 2762$, $N_F = 1067$) and 3.19 ($\chi^2 = 2706$, $N_F = 848$) respectively. As a final check on the consistency of the 0–350 MeV analysis, 0–100 MeV and 0–50 MeV analyses were performed with $l_{\max} = 1$.

The different analyses are consistent in their overlap regions in the behaviour of the largest (and consequently dominant) amplitudes, but there are significant differences in the small amplitudes. From the discussion given above, this latter situation is not unexpected.

Bransden *et al.* performed the analysis in two energy ranges: 300–700 MeV¹² and 750–1100 MeV.¹³ In the 300–700 MeV region two different types of solution were found, both with $\chi^2/N_F \approx 2.5$ (the difference were not general, being confined to a few partial waves). The solution obtained in the 750–1100 MeV region is a reasonable extrapolation of the better of the two lower energy solutions, although not all amplitudes can be joined on smoothly (again this is a not unexpected result of this type of analysis).

An encouraging feature of the three analyses discussed so far is the consistency in their results for the large amplitudes despite the poor statistical fits of the energy dependent analyses^{11,12,13} and the difficulties inherent in the simple energy-by-energy analysis.¹⁰ Although there are certainly quantitative differences, a clear qualitative picture is obtained of the dominant amplitudes in pion-nucleon scattering below 1 GeV. These results will be discussed in more detail in section 7.

There is yet a third approach to phase shift analysis, namely to inject as much theory as possible into the analysis (this is a conspicuous omission from the approaches discussed so far). Although this makes the analysis considerably more complicated, it pays handsome dividends and as we shall see in section 5 it has all the advantages of *both* the energy-by-energy and energy independent analyses, with a few more advantages besides, and has none of the disadvantages. For the moment then we will turn to a brief discussion of the theory to be used and of its validity.

4

PARTIAL WAVE DISPERSION RELATIONS

The extraction from the Mandelstam representation¹⁷ of the singularities in the complex s -plane of the partial wave amplitudes $f_{l\pm}(s)$, [18], is well documented¹⁸ and the resulting singularities are shown in Fig. 1. The dispersion relation for $f_{l\pm}(s)$ is

$$\operatorname{Re} f_{l\pm}(s) = \frac{P}{\pi} \int_{(m+\mu)^2}^{\infty} ds' \frac{\operatorname{Im} f_{l\pm}(s')}{s' - s} + \frac{1}{2\pi i} \int_{(\text{unphysical cuts})} ds' \frac{\Delta f_{l\pm}(s')}{s' - s} \quad \dots[53]$$

where $\Delta f_{l\pm}(s')$ is the discontinuity in $f_{l\pm}(s')$ across the cut at s' . The contribution to [53] from the integral along the physical cut $(m+\mu)^2 \leq s \leq \infty$

gives the rescattering, and the various unphysical cuts give contributions to [53] which can be regarded as the forces producing the $\pi-N$ scattering. The leading diagrams giving rise to these singularities are shown in Fig. 2 and the different unphysical cuts may be classified as follows.

- (a) *The short Born cut:* $(m-\mu^2/m)^2 \leq s \leq m^2+2\mu^2$.

The discontinuity across this cut is determined completely by the $\pi-N$ coupling constant $f^2 = 0.0822 \pm 0.0018$.^{19, 20, 21}

- (b) *The crossed physical cut:* $0 \leq s \leq (m-\mu)^2$.

The discontinuity across this cut is given in terms of the imaginary part of the physical $\pi-N$ scattering amplitudes. It is dominated by the contributions from the N_{33}^* resonance (the main contribution) and from the low energy s wave $\pi-N$ amplitudes. Detailed calculation verifies that the contributions from the higher mass $\pi-N$ resonances can be ignored safely. Thus the only information required to evaluate the contributions from this cut is the position and width of the N_{33}^* resonance, and the s wave $\pi-N$ amplitudes up to 100 MeV or so, all of which are well known.^{19, 20}

- (c) *The circle cut:* $|s| = m^2 - \mu^2$.

The discontinuity across the circle is given in terms of the absorptive parts $\text{Im} f_{\pm}^J(t)$ of the helicity amplitudes for the process $\pi+\pi \rightarrow N+\bar{N}$. J is the angular momentum and t the square of the total energy in this channel. Extended unitarity²² relates $\text{Im} f_{\pm}^J(t)$ directly to the absorptive part of $\pi-\pi$ scattering at energy $\sqrt{t}$.

Using the notation $s = (m^2 - \mu^2) \exp(i\phi)$ for points on the circle, the values of t which can occur at an angle ϕ are

$$4\mu^2 \leq t \leq t_{\max}$$

FIG. 1. The singularities of the partial wave amplitudes in the s -plane.

where

$$t_{\max} = 2[(m^2 + \mu^2) - (m^2 - \mu^2) \cos \phi]$$

and it follows that only low energy $\pi-\pi$ scattering can contribute to $\Delta f_{l\pm}(s)$ for small values of $|\phi|$ and the higher the $\pi-\pi$ scattering energy, the higher is the least value of $|\phi|$ involved. Present methods do not allow us to determine $\Delta f_{l\pm}(s)$ for $|\phi| > 66^\circ$.²³ However, since $|\phi| = 66^\circ$ corresponds to $t_{\max} = 56\mu^2$ and we have no real knowledge of the helicity amplitudes above $t = 50\mu^2$, this is of little practical consequence and the discontinuity across the circle is evaluated only for $0 \leq |\phi| < 66^\circ$.

There are two distinct contributions to the front of the circle. The first comes from the isospin $T = 0$ helicity amplitudes for $\pi + \pi \rightarrow N + \bar{N}$, which occur for even values of J . The $T = 0, J = 2$ state is known to give no significant contribution,²⁴ so that only the $T = 0, J = 0$ (s wave $\pi-\pi$ scattering) need be considered. The parameters of the s wave $\pi-\pi$ contribution to the $\pi-N$ partial wave amplitudes have been

FIG. 2. Leading diagrams of pion-nucleon scattering.

studied in great detail,²⁵ so for our present purposes we shall simply assume that this contribution is accurately known.

The second contribution comes from the $T = 1, J = 1$ helicity amplitude and can be explained completely by the $T = 1, J = 1$ $\pi - \pi$ resonance, the ρ . Using a narrow resonance approximation, the relevant helicity amplitudes $\text{Im } f_{\pm}^1(t)$ are given by δ -functions, and the absorptive parts of the invariant $\pi - N$ amplitudes $A^{(-)}$ and $B^{(-)}$ by

$$\text{Im } A^{(-)}(s, t) = 12\pi^2 C_2 (s + \frac{1}{2}t_R - m^2 - \mu^2) \delta(t - t_R), \quad \dots[54a]$$

$$\text{Im } B^{(-)}(s, t) = -12\pi^2 (C_1 + 2mC_2) \delta(t - t_R). \quad \dots[54b]$$

Here $t_R = m_\rho^2$, where m_ρ is the mass of the ρ , and C_1, C_2 are the $\rho - N$ coupling constants. The gyromagnetic ratio of the nucleon gives²⁶ $C_2/C_1 = 0.27$, leaving one unknown coupling constant. This is readily determined from low energy s wave $\pi - N$ scattering.^{25, 27}

(d) *The left-hand cut:* $-\infty \leq s \leq 0$.

All processes so far discussed contribute to this cut also, but in addition there are many other contributions from, for example, the more massive intermediate states in the t channel ($4\pi, 6\pi, K\bar{K}$, etc.). Some of these contribute also to the back of the circle and since they are at present quite incalculable, the total contribution from the left-hand cut and the back of the circle is unknown. As we shall see below, they can be classified as "distant singularities" giving rise to "short range forces", in contrast to the other "nearby singularities" which give rise to "long range forces". It will be assumed that the contribution from the distant singularities does not give rise to any structure in the physical region, i.e. it varies slowly with energy and can be adequately represented by a simple parametric form, e.g. one or two simple poles. Note that this assumption says nothing about the *magnitude* of the short range contribution, it merely specifies its *form*. To date all experimental evidence appears to justify this assumption.

Rather than study the amplitude $f_{l\pm}(s)$ it is convenient to introduce $F_{l\pm}(s)$, defined by

$$F_{l\pm}(s) = \frac{f_{l\pm}(s)}{q^{2l}}. \quad \dots[55]$$

The relation between s and q^2 is

$$s = \{(m^2 + q^2)^{\frac{1}{2}} + (\mu^2 + q^2)^{\frac{1}{2}}\}^2 \quad \dots[56]$$

and to each value of s there is a unique value of q^2 given by

$$q^2 = \{s - (m - \mu)^2\}\{s - (m + \mu)^2\}/4s. \quad \dots[57]$$

Note that q^2 is real on *all* the cuts shown in Fig. 1. On the physical cut

q^2 increases from 0 at $s = (m + \mu)^2$ to ∞ as $s \rightarrow \infty$. On the crossed physical cut $q^2 = 0$ at $s = (m - \mu)^2$ and increases to $+\infty$ as $s \rightarrow 0$ along this cut, and on the short Born cut $q^2 \simeq -\mu^2$. At the point $s = (m^2 - \mu^2) \exp(i\phi)$ on the circle $|s| = m^2 - \mu^2$,

$$q^2 = -\{m^2 \sin^2(\phi/2) + \mu^2 \cos^2(\phi/2)\}. \quad \dots[58]$$

Thus $q^2 = -\mu^2$ at $s = m^2 - \mu^2$ and as s moves round the circle to $-(m^2 - \mu^2)$, q^2 decreases to $-m^2$. Finally, on the cut $-\infty \leq s \leq 0$, q^2 has a maximum value of $-m^2$ at $s = -(m^2 - \mu^2)$ and then decreases to $-\infty$ both as $s \rightarrow 0$ and as $s \rightarrow -\infty$.

The well-known threshold behaviour of the phase shift δ_l of the l -th partial wave

$$\delta_l = a_l q^{2l+1}, \quad \dots[59]$$

with a_l a constant as $q \rightarrow 0$, implies that $f_{l\pm}(s) = q^{2l}$ near the threshold. It follows from [58] that q^{-2} is a regular function of s except for simple poles at $s = (m + \mu)^2$ and $s = (m - \mu)^2$. Thus, the function $F_{l\pm}(s)$ defined by [55] is bounded at the physical threshold and has a simple branch point there. At the crossed threshold $s = (m - \mu)^2$ the function $F_{l\pm}(s)$ is not bounded in general, the type of singularity there being different from that of $f_{l\pm}(s)$. However, as we shall see, this causes no difficulty.

Thus in place of [53] we shall use the dispersion relation

$$\operatorname{Re} F_{l\pm}(s) = \frac{P}{\pi} \int_{(m+\mu)^2}^{\infty} ds' \frac{\operatorname{Im} F_{l\pm}(s')}{s' - s} + \frac{1}{2\pi i} \int_{(\text{unphysical cuts})} ds' \frac{\Delta F_{l\pm}(s')}{s' - s} \quad \dots[60]$$

where

$$\Delta F_{l\pm}(s') = (\Delta f_{l\pm}(s')) / (q^2(s'))^l \quad \dots[61]$$

and special care is taken at $s' = (m - \mu)^2$. [60] has considerable advantages over [53] on account of its improved convergence for all $l \geq 1$. The presence of the factor q^{-2l} in $\Delta F_{l\pm}(s)$ suppresses the contribution to the second integral in [60] from the unknown distant singularities (thereby giving added strength to the assumption about the lack of structure in the contribution from these singularities) and in addition, since q^{-2l} is small at high physical energies, any uncertainties arising from the high-energy portion of the re-scattering integral are very much reduced. Of course, the higher the value of l the greater is the suppression and the smaller one would expect the contribution from the distant singularities to be. This is physically very sensible in that it would not be expected that high partial waves would have any great dependence on the very short range interaction.

One can go further than this. There are two constraints which can be applied which, within the limits of the "no structure" assumption, determine the magnitude and sign of the short range interaction.

(1) *Crossing*

The crossing relations, [9], can be used not only to obtain the discontinuity across the crossed cut in terms of the imaginary part of the physical $\pi-N$ scattering amplitudes, but also to obtain the *real part* of the scattering amplitude on the crossed cut in terms of the real parts of the physical $\pi-N$ scattering amplitudes. This crossing relation is of the general form

$$\operatorname{Re} F_J(s_C) = \sum_{J'} \int_{\text{some physical } s'} ds' K_{JJ'}(s_C, s') \operatorname{Re} f_{J'}(s') \quad \dots[62]$$

where $0 \leq s_C \leq (m-\mu)^2$. The summation is over all J' and the integral over some (finite) range of physical energies, the exact limit depending on the particular value of s_C . In practice, of course, the summation over J' must be truncated but, as in evaluating the discontinuity across the cut, the right-hand side of [62] is dominated by the N_{33}^* and low energy s waves for values of s_C close to the crossed cut threshold. Further, for $s_C = (m-\mu)^2$, i.e. at the crossed cut threshold, the summation is truncated *exactly* at $l' = l$ (due to the threshold behaviour of the scattering amplitudes the contribution from all partial waves with $l' > l$ vanishes there). The integration also disappears, the right-hand side of [62] being evaluated at the physical threshold only. Consequently, at the threshold of the crossed cut the value of the real part of the l -th partial wave is given *exactly* in terms of the physical threshold parameters of the partial waves $l' \leq l$.

The importance of this is that the real part of the scattering amplitude can be obtained on the crossed cut by a second method, namely evaluation of the dispersion relation [60] for $s = s_C$, and the two real parts so obtained should be equal. This crossing constraint is an extremely powerful one to apply in obtaining solutions to partial wave dispersion relations.

(2) *The unitary sum rules*

Suppose, for the convenience of writing (there is no loss of generality in the argument), that we represent the unphysical cuts by the line $-\infty \leq s \leq s_1$, so that the amplitude $F_{l\pm}(s)$ satisfies the dispersion relation

$$\operatorname{Re} F_{l\pm}(s) = \frac{P}{\pi} \int_{s_0}^{\infty} ds' \frac{\operatorname{Im} F_{l\pm}(s')}{s' - s} + \frac{1}{\pi} \int_{-\infty}^{s_1} ds' \frac{\operatorname{Im} F_{l\pm}(s')}{s' - s} \quad \dots[63]$$

where $s_0 = (m+\mu)^2$ is the physical threshold.

We want to consider the limiting form of [63] for infinite physical energies ($s \rightarrow +\infty$).

For physical s

$$\operatorname{Re} F_{l\pm}(s) = \frac{\eta_{l\pm}(s) \sin(2\alpha_{l\pm}(s))}{2[q(s)]^{2l+1}}$$

so that

$$|\operatorname{Re} F_{l\pm}(s)| \leq \frac{1}{2}[q(s)]^{-(2l+1)}. \quad \dots[64]$$

From the relation [57] between q^2 and s it follows immediately that the magnitude of the left-hand side of [63] is bounded by $2^{2l}/s^{l+\frac{1}{2}}$ for large physical s .

Now look at the right-hand side of [63] as $s \rightarrow +\infty$. We require to make the assumption that for $l \geq 1$, $\operatorname{Im} F_{l\pm}(s)$ decreases faster than $(-s)^{-1-\varepsilon}$, $\varepsilon > 0$, as $s \rightarrow -\infty$. This assumption is reasonable, since without it the high-energy behaviour of the dispersion relation must be very pathological. With this assumption,

$$\frac{1}{\pi} \int_{-\infty}^{s_1} ds' \frac{\operatorname{Im} F_{l\pm}(s')}{s' - s} = -\frac{1}{\pi s} \int_{-\infty}^{s_1} ds' \operatorname{Im} F_{l\pm}(s') + O\left(\frac{1}{s^2}\right) \quad \dots[65]$$

as $s \rightarrow +\infty$.

Since the first term on the right of equation [63] is a principal value integral, it is not so easy to find its asymptotic form. Sufficient conditions¹⁹ for

$$\frac{P}{\pi} \int_{s_0}^{\infty} ds' \frac{\operatorname{Im} F_{l\pm}(s')}{s' - s}$$

to behave like

$$-\frac{1}{\pi s} \int_{s_0}^{\infty} ds' \operatorname{Im} F_{l\pm}(s')$$

as $s \rightarrow +\infty$ are that

$$(i) \quad \int_{s_0}^{\infty} ds' \operatorname{Im} F_{l\pm}(s') \text{ exists} \quad \dots[66a]$$

and

(ii) given any $\varepsilon > 0$, there exists an S such that

$$\left| \frac{s' \operatorname{Im} F_{l\pm}(s') - s \operatorname{Im} F_{l\pm}(s)}{s' - s} \right| < \frac{\varepsilon}{(ss')^{\frac{1}{2}}} \text{ for all } s' \geq s > S. \quad \dots[66b]$$

Since

$$\operatorname{Im} F_{l\pm}(s) = \frac{1 - \eta_{l\pm}(s) \cos(2\alpha_{l\pm}(s))}{2[q(s)]^{2l+1}}$$

condition (i) is satisfied for $l \geq 1$. However, we can make no definite statement about condition (ii) without detailed knowledge of how $\alpha_{l\pm}$ and $\eta_{l\pm}$ behave for large s , although it should be noted that for simple models of high-energy scattering it is certainly true. What we can say by using [63], [64] and [65] is that

$$\lim_{s \rightarrow +\infty} \left[\frac{s}{\pi} P \int_{s_0}^{\infty} ds' \frac{\text{Im } F_{l\pm}(s')}{s' - s} \right] = \frac{1}{\pi} \int_{-\infty}^{s_1} ds' \text{Im } F_{l\pm}(s') \quad \dots[67]$$

for $l \geq 1$, so that the limit on the left-hand side certainly exists, and it is a not unreasonable conjecture that this limit is indeed

$$-\frac{1}{\pi} \int_{s_0}^{\infty} ds' \text{Im } F_{l\pm}(s').$$

Equation [67] then gives the unitary sum rule for $l \geq 1$,

$$\int_{s_0}^{\infty} ds' \text{Im } F_{l\pm}(s') + \int_{-\infty}^{s_1} ds' \text{Im } F_{l\pm}(s') = 0. \quad \dots[68]$$

In a similar fashion one can obtain extra sum rules for higher values, e.g. for $l \geq 2$ one obtains the unitary sum rule

$$\int_{s_0}^{\infty} ds' s' \text{Im } F_{l\pm}(s') + \int_{-\infty}^{s_1} ds' s' \text{Im } F_{l\pm}(s') = 0 \quad \dots[69]$$

and so on.

The consequence of this is that if we are given a solution to the dispersion relation [63], then from the values of $\text{Im } F_{l\pm}(s)$ on the physical cut $s_0 \leq s \leq \infty$ the unitary sum rule provides a relation governing the values which $\text{Im } F_{l\pm}(s)$ can have on the unphysical cut. As we have seen already, the long and fairly long range parts of the $\pi-N$ interaction are known reasonably well, so that the unitary sum rule gives us information on the short range part of the $\pi-N$ interaction.

A good check on the validity of the sum rule is provided by the N_{33}^* resonance, since the integrand in the physical integral in [68] is dominated by the resonance region. The long range forces are, of course, known and representing the short range interaction by a single pole, determined by the sum rule, the resulting P_{33} scattering length is 0.219 ± 0.011 (in units $\hbar = \mu = c = 1$) which is to be compared with the experimental value of 0.215 ± 0.005 .¹⁹

Using this theoretical structure, explicit prediction can be made of the pion-nucleon scattering amplitudes,^{28, 29} so long as there is no appreciable inelasticity (the theory is essentially an elastic one). The energy range over which the results are meaningful varies from partial wave to partial wave since usually the higher the partial wave the later the onset of inelasticity, although there are of course exceptions. Simple impact parameter consider-

ations²⁸ indicate that p waves should be realistic to ~ 300 MeV, d waves to ~ 500 MeV, f waves to ~ 700 MeV and g waves and higher beyond 1 GeV. We will justify these statements *a posteriori*. What justification for the theory can we give *a priori*?

1. The predictions of the theory are in good quantitative agreement with single energy phase shift analyses up to 310 MeV²⁸ and beyond.²⁹
2. The best results of Feld *et al.*¹¹ in the 0-350 MeV analysis were obtained when the theoretical predictions were used as a starting point, and the experimental values did not differ appreciably from the theoretical ones.
3. General statements can be made about the relative magnitude and signs of the partial wave amplitudes at energies up to 1 GeV or more. These are consistent with experimental evidence.³⁰
4. The partial wave amplitudes can be classified into "resonant" and "non-resonant" and upper limits put on the resonance energies of the resonant ones.³¹ The resulting quantum numbers are in agreement with and the energies consistent with the known N_{33}^* (1238), N_{13}^* (1520), N_{15}^* (1670) and N_{37}^* (1920). [As we shall see, the forces producing these "old" resonances come from the $\pi-N$ channel itself, although appreciably modified by inelastic effects. In the case of the "new" resonances, the forces come almost completely from channels other than the $\pi-N$ one, and the theory can say nothing about them.]

5

PHASE SHIFT ANALYSIS II

In order to apply the partial wave dispersion relations to a phase shift analysis of the data, a semi-phenomenological technique has been developed whereby the experimental phase shifts and inelasticities in a given partial wave are fitted simultaneously by a parametrization which satisfies the dispersion relation exactly and has the correct long range forces, the short range forces being treated as variables (in effect, they form part of the parametrization) and the whole subjected to the unitary sum rule. The higher angular momentum states, which cannot be expected to be determined by the data but which nevertheless are important (see section 3), are given solely in terms of the theory. Although this does not amount to a complete *a priori* prediction, it imposes strong consistency conditions on the data, enabling decisions to be made between different sets of phase shifts and allowing interpolation from energies with complete experiments to those where they are incomplete. Since errors can be placed on the theoretical predictions, they can be used as "theoretical data" to supplement the inadequate experimental data at these latter energies, and a second (more extensive) fit to the experiments made. The results are then used to re-evaluate the partial wave dispersion relations, and so on until consistency is achieved. By this means,

it is possible to obtain all the advantages of the energy dependent fits with none of their attendant disadvantages. Continuity is ensured, systematic errors are smoothed out, the correct energy dependent correlations between the phase shifts $\alpha_{l\pm}$ and the inelasticities $\eta_{l\pm}$ are obtained and a realistic assessment of the errors on $\alpha_{l\pm}$ and $\eta_{l\pm}$ can be made. Further, by fitting one partial wave at a time, it is possible to use a large number of parameters, ensuring that the results come from the dispersion relation itself and not from the particular parametrization, avoiding the danger of bias arising from having too few parameters.

By its very nature, this is a technique which allows the analysis to be extended to ever increasing energy domains and this is what has happened in practice. Starting from a $\pi^\pm - p$ analysis between 300–700 MeV³² the $\pi^+ - p$ analysis was extended to 1100 MeV³³ and this in turn extended to a $\pi^\pm - p$ analysis from 0–1400 MeV.³⁴ Let us now look at the procedure in some detail.

The starting point is with those energies where $\pi^+ - p$ and $\pi^- - p$ polarization measurements and $\pi^- - p$ charge exchange measurements are available, as well as $\pi^+ - p$ and $\pi^- - p$ differential cross-section measurements. The highest partial waves (usually g and h waves) are fixed at the theoretical values and a conventional energy independent phase shift analysis is performed. If necessary, some of the smaller, elastic, amplitudes may be constrained to the theoretical values and used as “theoretical data” to supplement the experimental data. Having found a minimum in M a series of “random searches” is performed. These consist of taking an arbitrary step in parameter space (the arbitrariness being obtained by using a pseudo-random number generator) and carrying out the minimization once again from this new starting point. This procedure, which is repeated many times, is a very necessary one.

There is no guarantee that M has only one minimum. It may have several. Hopefully it has only one deep one, with a statistically significant χ^2 , the others being local shallow minima with no real statistical significance. This case presents no serious problem. If the minimization has found a shallow minimum, then the random search procedure takes the parameters away from this region and it is unlikely that this particular “solution” will reappear. On the other hand, if the minimum is a genuinely deep one, then the same solution will be obtained repeatedly. If after a large number of random searches only one such deep minimum has been found, there is some justification for claiming a unique solution. Frequently, however, there exist two (and sometimes more) deep minima, each with approximately the same statistical significance, and further action must be taken. This, and other configurations of parameter space which cause difficulties, will be discussed below.

Having obtained solutions at a number of energies E_i ($i = 1, \dots, N$) (with corresponding c.m. s values s_i) these are then fitted to the dispersion

relations. Denoting the known contribution from the long-range forces by $\mathcal{F}_{l\pm}(s)$ and approximating the short-range contribution by a number of simple poles, the dispersion relation can be written as

$$\operatorname{Re} F_{l\pm}(s) = \mathcal{F}_{l\pm}(s) + \frac{P}{\pi} \int_{s_0}^{\infty} ds' \frac{\operatorname{Im} F_{l\pm}(s')}{s' - s} + \sum_{j=1}^n \frac{r_j}{s - a_j} \quad \dots[70]$$

and we know (experimentally) $F_{l\pm}(s_i)$, ($i = 1, \dots, N$).

To evaluate the dispersion relations, we use a conformal mapping technique, details of which are given in appendix 2. For the moment it is sufficient to know that there exist two sets of functions $g_k(s)$ and $h_k(s)$, ($k = 1, \dots, \infty$), with the following properties:

$$(i) \quad g_k(s) = \frac{P}{\pi} \int_{s_0}^{\infty} ds' \frac{h_k(s')}{s' - s} \quad \dots[71]$$

$$(ii) \quad \operatorname{Im} F_{l\pm}(s) = \sum_{k=1}^{\infty} (a_{l\pm})_k h_k(s), \quad \dots[72]$$

and this series converges under very general conditions.

(iii) $g_k(s)$ and $h_k(s)$ have the correct threshold and asymptotic behaviour.

The dispersion relation [70] then takes the form

$$\operatorname{Re} F_{l\pm}(s) = \mathcal{F}_{l\pm}(s) + \sum_{k=1}^{\infty} (a_{l\pm})_k g_k(s) + \sum_{j=1}^n \frac{r_j}{s - a_j} \quad \dots[73]$$

Truncating the series at some value $k = K$ and evaluating $\operatorname{Re} F_{l\pm}(s)$ and $\operatorname{Im} F_{l\pm}(s)$, and their errors, at the points $s = s_i$ ($i = 1, \dots, N$) we obtain the equations

$$\operatorname{Im} F_{l\pm}(s_i) = \sum_{k=1}^K (a_{l\pm})_k h_k(s_i), \quad \dots[74a]$$

$$[\operatorname{Re} F_{l\pm}(s_i) - \mathcal{F}_{l\pm}(s_i)] = \sum_{k=1}^K (a_{l\pm})_k g_k(s_i) + \sum_{j=1}^n \frac{r_j}{s_i - a_j}, \quad \dots[74b]$$

for $i = 1, \dots, N$.

The unknowns $(a_{l\pm})_k$, ($k = 1, \dots, K$), and r_j , ($j = 1, \dots, n$), can then be varied to obtain a best fit to equations [74]. [The pole positions a_j , ($j = 1, \dots, n$), are kept fixed.] The values of K and n cannot be fixed *a priori* and they are automatically adjustable up to specified maximum values $K_{\max}$ and $n_{\max}$. In accordance with the "no-structure" assumption $n_{\max}$ is kept small (≤ 3) but $K_{\max}$ can be larger—in some cases 20 or more. Another variable in the fit is the conformal zero, which gives a considerable measure of control over the behaviour of the fit.

Once a reasonable fit is achieved, we then have a dispersion relation prediction of the amplitudes at *all* energies. This technique is a very powerful

one, as the example in Figs. 3a and 3b shows. This is the S_{31} amplitude which is one of the more difficult ones to fit because of the large and rapid changes in α and η above 600 MeV. Even so the dispersion relation *prediction* (the dotted line), based on experiments up to 600 MeV and at the two energies 870 and 990 MeV and the final fit (the solid line) are in remarkably good agreement. In amplitudes which show less structure, the final fits rarely depart significantly from the dispersion relation predictions.

Although the use of the dispersion relations allows the rejection of *most* alternative solutions, it cannot reject them all and above 700 MeV there is at present no unique solution to the analysis. There are three types of non-uniqueness.

1. Two (or more) genuinely different solutions exist. In parameter space this corresponds to having two (or more) distinct deep minima, with statistically meaningful χ^2 .
2. Several minor variations exist of the one type of solution. In parameter space this corresponds to there being a number of small local minima grouped close together at the bottom of the deep minimum.
3. Different solutions exist which appear superficially to be type 1, but are in fact a variant of type 2. In this case major differences are confined to only one or two parameters and in parameter space corresponds to there being a flat valley in the direction of each of these parameters, but otherwise the minimum is steep. This type of situation is easily discernible and the extent of the valley can be found readily by tracking across all values of the parameter concerned. For example $\eta(S_{11})$ is one of the frequent culprits and in this case a typical tracking search would be to fix $\eta(S_{11})$ successively at 0, 0.1, 0.2, ..., 1.0 and minimize with respect to the other parameters for each of those fixed values of $\eta(S_{11})$.

Nothing can be done at present about the type 1 ambiguity. Clarification of this ambiguity will need to wait until the more difficult experiments of charge exchange polarization or of triple scattering are performed.

Types 2 and 3 are however a different matter. Their significance is that the statistical errors obtained in the analysis are not a realistic measure of the true experimental error. The final step is to take all variations of the same solution and combine them, weighted appropriately according to their relative probabilities, into a single composite solution having a realistic experimental error. This is the final output of the phase shift analysis.

6

RESONANCES

As we shall see later, one of the most interesting results of pion-nucleon phase shift analysis has been the existence of hitherto unsuspected resonances in the pion-nucleon system and in clarifying the structure of the ones which

FIG. 3a. The S_{31} phase shift plotted as a function of energy.
(—— result: ---- dispersion relation prediction.)

FIG. 3b. The S_{31} inelasticity parameter plotted as a function of energy.
(—— result: ---- dispersion relation prediction.)

were known to exist. These new resonances are very different in character from the conventional idea of a resonance, typified for example by the N_{33}^* (1238) resonance, and some explanation of possible resonance structures is necessary.

For all pion-nucleon resonances, apart from the N_{33}^* , the presence of inelastic channels is of considerable importance. In the presence of such channels, the scattering and reaction processes can be described best by the reaction matrix $K^{35,36}$ which is related to the scattering matrix T by

$$T = K(1 - iQK)^{-1} \quad \dots[75]$$

where Q is the matrix of available channel momenta.

A resonant state corresponds to the situation when one eigenphase of K increases rapidly through 90° , and in the neighbourhood of the resonance energy the eigenvalue K_R and the eigenphase δ_R are given approximately by

$$K_R = \tan \delta_R = \frac{\gamma}{W_R - W} + \tan \delta_B \quad \dots[76]$$

where δ_B denotes the non-resonant background phase of this eigenstate. This may be described in channel space by specifying the coefficients $C_{R\beta}$ appropriate to channel β , the coefficients being normalized such that $\sum_\beta |C_{R\beta}|^2 = 1$.

As emphasized by Adair,³⁷ in the neighbourhood of the resonance the scattering amplitude has then the form

$$f_J = \frac{1}{2iq} \exp \{2i\delta_\beta\} \frac{\Gamma_1}{\Gamma} \left\{ \frac{W^* - W + \frac{1}{2}i\Gamma}{W^* - W - \frac{1}{2}i\Gamma} \right\} + A_J. \quad \dots[77]$$

The Breit-Wigner resonance energy W^* and total width Γ are given by

$$W^* = W_R - \frac{1}{2}\gamma \sin 2\delta_B \quad \dots[78]$$

and

$$\Gamma = 2\gamma \cos^2 \delta_B \quad \dots[79]$$

and the partial width Γ_1 for the elastic channel by

$$\Gamma_1 = |C_{R1}|^2 \Gamma. \quad \dots[80]$$

The complex amplitude A_J consists of contributions from all the non-resonant eigenphases and from δ_B , and (hopefully) is slowly varying over the resonance region.

A convenient parameter to introduce is

$$R_J = \left\{ \frac{\sigma_{\text{Total}}^{(J)}}{\sigma_{\text{Elastic}}^{(J)}} \right\} \geq 1 \quad \dots[81]$$

which is given in terms of the phase shift α_J and the inelasticity parameter η_J by

$$R_J = 1 + \frac{1 - \eta_J^2}{1 - 2\eta_J \cos 2\alpha_J + \eta_J^2}. \quad \dots[82]$$

The behaviour of R_J as a function of η_J is shown in Fig. 4. For $-1 \leq \cos 2\alpha_J \leq 0$, $1 \leq R_J \leq 2$, the maximum and minimum values of R_J being attained at the minimum and maximum values of η_J . For $1 \geq \eta_J \geq \cos 2\alpha_J (> 0)$, $1 \leq R_J \leq 2$ and for $\cos 2\alpha_J \geq \eta_J \geq 0$, $R_J \geq 2$.

FIG. 4. R as a function of η .

To obtain a prescription for a resonant situation, we now discuss the Adair plot³⁷ (Argand diagram).

Let $z = 2qf_J(s)$. From the definition, [19], of f_J , it follows immediately that

$$|z - i| = \eta_J \quad \dots[83]$$

so that z must lie inside or on the circle with centre i and unit radius, Fig. 5a, the scattering phase being given by $2\alpha_J = \widehat{AOZ}$ and inelasticity by $\eta_J = OZ$. In terms of R_J the unitarity condition is

$$\{\operatorname{Re}(2qf_J)\}^2 + \{\operatorname{Im}(2qf_J)\}^2 = 2 \operatorname{Im}(2qf_J)/R_J \quad \dots[84]$$

or,

$$|z - i/R_J| = 1/R_J \quad \dots[85]$$

so that at any given moment z is describing the arc of a circle centre i/R_J and radius $1/R_J$, Fig. 5a.

Equation [77] tells us that in the region of a resonance the scattering amplitude is dominated by a pair of Breit-Wigner poles, modified by some background term. Let us consider first the case when the background is zero, when we have the usual multichannel Breit-Wigner description of the scattering amplitude with the various cross-sections given by

$$\sigma_{\text{Total}}^{(J)} = \frac{\pi}{q^2} (J + \frac{1}{2}) \frac{\Gamma^2}{(W_R - W)^2 + \Gamma^2/4} \quad \dots[86a]$$

$$\sigma_{\text{Elastic}}^{(J)} = \frac{\pi}{q^2} (J + \frac{1}{2}) \frac{\Gamma\Gamma_1}{(W_R - W)^2 + \Gamma^2/4} \quad \dots[86b]$$

$$\sigma_{\text{Inelastic}}^{(J)} = \frac{\pi}{q^2} (J + \frac{1}{2}) \sum_{\beta} \frac{\Gamma\Gamma_{\beta}}{(W_R - W)^2 + \Gamma^2/4} \quad \dots[86c]$$

with $\Gamma = \Gamma_1 + \sum_{\beta} \Gamma_{\beta}$. In this case $\delta_B = 0$ and so from [78] and [79], $W^* = W_R$ and $\Gamma = 2\gamma$.

In the neighbourhood of the resonance, Γ_1 and Γ_{β} ($\beta = 2, 3, \dots$) are approximately constant and so $R_J = \Gamma/\Gamma_1$ will vary slowly with energy with the consequence that the function z will trace out a considerable part of the arc of a circle as the energy W is increased through and beyond the resonance energy W_R . In the purely elastic case, Fig. 5b, the amplitude stays on the circle, starting at the bottom at zero energy and traversing the circle in an

FIG. 5. Resonance configurations in the Adair plot.

anti-clockwise direction. The point $W = W_R$ occurs at the top of the circle ($\alpha_J = 90^\circ$) and the width $\Gamma = W_2 - W_1$, where W_2 and W_1 are the energies at which $\alpha_J = 135^\circ$ and 45° respectively.

Two different situations occur in the case of there being some inelasticity.

(i) $1 \leq R_J \leq 2$.

In this case, Fig. 5c, which of course includes the purely elastic case ($R_J = 1$), the phase shift α_J passes through 90° at the top of the circle and the resonance width can be defined as above, so long as the departure of R_J from a constant value is small between the upper and lower half widths. Appreciable deviation from a constant causes severe distortion of the arc from the simple circular form, although the resonance position remains at the top of the circle. Outside the resonance region departure of R_J from a constant is certainly to be expected and a complete circle will not be obtained experimentally.

Since $R_J \leq 2$, $\Gamma_1 > \frac{1}{2}\Gamma$ and this type of resonance can be classified as quasi-elastic.

(ii) $R_J > 2$.

In this class of highly absorptive resonances, the scattering phase shift α_J does not go through 90° at the resonance position, but through 0° , Fig. 5d. The definition of the resonance width is as before, with the same reservations about distortion due to R_J departing from a constant value. In this case $\Gamma_1 < \frac{1}{2}\Gamma$ and so, in the case when there is only one important inelastic channel open, $\Gamma_2 > \frac{1}{2}\Gamma$ and inelastic resonances in one channel are quasi-elastic ones in another channel.

Let us now consider the effect of background, which we have so far ignored. We may rewrite [77] as

$$2qf_J = -i \frac{\Gamma_1}{\Gamma} \exp \{2i\delta_B\} \exp \{i\phi(W)\} + \tilde{A}_J(W) \quad \dots[87]$$

where

$$\phi(W) = 2 \tan^{-1}(\Gamma/2(W_R - W)). \quad \dots[88]$$

If $\tilde{A}_J(W)$ is slowly varying with energy in the resonance region, then [87] gives a circle, centre $\tilde{A}_J(W_R)$ and radius $\Gamma_1 \exp \{2i\delta_B\}/\Gamma$, with the phase $\phi(W)$ going from ~ 0 to $\sim 2\pi$ through the resonance. The top of the circle gives the energy W^* but *not* the true resonance energy W_R since δ_B of [76] is no longer zero. Unlike the "no-background" cases, which were essentially symmetrical about the vertical diameter of the Adair circle, the effect of the background can move the resonance circle to any part of the diagram. Two typical situations are shown in Figs. 5e and 5f, which are cases of a strong attractive background and a strong repulsive background respectively. Determination of the resonance width is also a hazardous procedure in this case, since not only can there be distortion of the circle due to energy dependence

of the radius term, as in the "no-background" cases, but also there can be further distortion due to energy dependence of the background term $\tilde{A}_J(W)$ itself. It is in fact virtually impossible to determine accurately the true parameters of such a resonance from a study of only one of its possible channels.

However, that such resonance circles do occur, and quite noticeably too, can be seen from the Argand plots of the forward pion-nucleon scattering amplitude made by Höhler and Ebel,³⁸ there being markedly circular paths in the neighbourhood of the $N^*(1238)$ and $N^*(1690)$ resonances, despite the presence of strongly energy-dependent background contributions.

7

RESULTS

The present upper limit in pion-nucleon phase shift analysis is 1.4 GeV, this limit being dictated largely by the lack of sufficient data, although the need to take into account more and more partial waves is beginning to create a barrier which will be difficult to pass. Unfortunately 1.4 GeV is still a long way from the asymptotic region ($\gtrsim 3.5$ GeV, say) where the scattering can be described reliably in terms of high-energy models, and there is evidence of considerable resonance structure in the intervening energy range. This is clearly of considerable interest, but detailed investigation of this region will be very difficult and almost certainly will be model dependent and not done "exactly" in the sense of the phase shift analyses described here.

To avoid littering the remaining few pages with an overabundance of reference numbers, we will refer to each of the phase shift analyses by the initials of the authors, namely BBSV,¹⁰ RWF,¹¹ BOM^{12,13} and DKLL.³⁴

Since the analysis of DKLL is the most extensive yet completed, we quote their results as the basis for discussion (Tables I, II, III and IV) pointing out the similarities and differences in the solutions obtained by BBSV, RWF and BOM. The DKLL results are "semi-final" in the sense that although they represent the final fit to the experimental data they have still to undergo their final theoretical smoothing. Also DKLL have two possible $T = \frac{1}{2}$ solutions between 700 and 1000 MeV. The ambiguity is confined almost entirely to the S_{11} and P_{11} amplitudes. It is hoped that the final application of theory will resolve this ambiguity. [The power of this approach can be gauged from the fact that DKLL found it impossible to obtain a unique $T = \frac{3}{2}$ solution without the aid of the theory and at one point in the analysis they had seven different $T = \frac{1}{2}$ solutions of which the theory has rejected five successfully.]

DKLL considered all partial waves up to, and including, h waves, and the presence of the g and h waves was found to be important (in particular G_{17}). Neither BOM nor BBSV took account of these higher partial waves, truncating the expansion at f waves. Since the g and h waves give a non-negligible contribution, this undoubtedly produces errors, since the other amplitudes distort to try to compensate for these missing contributions.

Further the parametrization of BOM requires that zeros of the amplitudes be inserted at the beginning as poles in the inverse amplitude; they cannot be found in the course of the fit. Since many of the small amplitudes do appear to have zeros (and some more than one) their results for these amplitudes are doubtful. Also their analysis was done in two parts, 300–700 MeV and 750–1050 MeV and there is a distinct discontinuity between the two sets, making it difficult to evaluate the results quantitatively.

$T = \frac{3}{2}$ scattering

S_{31} . This is the most interesting of the $T = \frac{3}{2}$ amplitudes. The results show an inelastic peak at 800 MeV, accompanied by a very rapid drop in the phase shift. When plotted in the Argand diagram, it appears that in the 800 MeV region S_{31} describes part of an anti-clockwise circle, displaced far to the left by the presence of a strongly repulsive background, and it can be interpreted in terms of a resonance of type 5f. BBSV also observe this phenomenon, although the inelastic peak is not so marked as in DKLL ($\eta_{\min} \simeq 0.4$ compared with $\eta_{\min} \simeq 0.2$) nor is the drop in the phase shift so steep. BOM failed to find this effect, their phase shift and elasticity decreasing slowly together with no marked structure. The RWF analysis does not go to a high enough energy to observe the resonance. At the lower energies (< 700 MeV) all solutions are compatible.

P_{31} . In the DKLL solution the phase shift is negative, reaching a minimum of $\sim -25^\circ$ at 1 GeV and then rising, this being accompanied by a steady decrease in η . The BOM and BBSV solutions are roughly similar, the phase shift in the former being less negative ($\delta_{\min} \sim -15^\circ$) and in the latter more negative ($\delta_{\min} \sim -35^\circ$) than that of DKLL, and the BBSV solution is almost elastic. The RWF solution is incompatible with these three, becoming positive at 660 MeV and already being strongly absorptive there ($\eta < 0.5$).

P_{33} . This is the well-known N_{33}^* resonance, occurring at an energy of about 200 MeV. The main differences in the solutions are that in BBSV and DKLL the phase shift levels off near 180° , while in RWF and BOM it levels off near 160° . In all solutions there is no appreciable inelasticity below 700 MeV, and thereafter η falls slowly with increasing energy.

D_{33} . The DKLL, BBSV and RWF solutions all find the phase shift to be initially small and positive, then becoming negative in the vicinity of 500 MeV. In the BOM solution the phase shift is small and negative throughout. The BOM solution is essentially elastic, in the DKLL and the BBSV solutions η decreases steadily above 400 MeV.

D_{35} . All solutions find this phase shift to be negative with some inelasticity at higher energies ($\eta \sim 0.8$). The BOM and RWF phase shift is more negative ($\sim -10^\circ$) than the DKLL and BBSV phase shift ($\sim -5^\circ$).

TABLE I $T = \frac{3}{2}$ parameters

E_{π} (MeV)	S_{31}		P_{31}		P_{33}		D_{33}		D_{35}		F_{35}		F_{37}	
	δ	η	δ	η	δ	η	δ	η	δ	η	δ	η	δ	η
24.8	- 3.08 ± 0.14		- 0.37 ± 0.09		1.91 ± 0.08		0.005		- 0.007					
31.0	- 3.86 ± 0.15		- 0.45 ± 0.12		2.39 ± 0.13		0.010		- 0.010					
37.0	- 3.68 ± 0.34		- 0.57 ± 0.17		3.30 ± 0.21		0.013		- 0.016					
41.5	- 4.77 ± 0.33		- 0.79 ± 0.17		3.76 ± 0.25		0.016		- 0.021					
61.0	- 6.34 ± 0.40		- 1.71 ± 0.24		8.71 ± 0.23		0.037		- 0.049		0.002		0.006	
83.0	- 9.81 ± 0.82		- 1.81 ± 0.50		14.35 ± 0.36		0.067		- 0.093		0.005		0.016	
98.0	- 8.49 ± 0.29		- 2.66 ± 0.36		21.12 ± 0.43		0.090		- 0.130 ± 0.011		0.008		0.026	
120.0	- 10.61 ± 0.85		- 2.93 ± 0.44		31.19 ± 0.38		0.125		- 0.19 ± 0.01		0.013		0.045	
150.0	- 12.13 ± 2.00		- 4.71 ± 0.57		53.69 ± 0.57		0.17 ± 0.01		- 0.29 ± 0.01		0.021		0.080	
165.0	- 19.12 ± 3.86		- 4.60 ± 1.27		62.94 ± 2.21		0.20 ± 0.02		- 0.35 ± 0.02		0.024		0.104	
170.0	- 18.03 ± 3.54		- 5.22 ± 1.41		64.79 ± 1.61		0.21 ± 0.02		- 0.36 ± 0.02		0.025		0.112	
176.0	- 17.16 ± 6.59		- 5.27 ± 1.52		71.16 ± 3.86		0.22 ± 0.02		- 0.39 ± 0.02		0.026		0.124	

TABLE I (continued)

E_{π} (MeV)	S_{31}		P_{31}		P_{33}		D_{33}		D_{35}		P_{35}		P_{37}	
	δ	η	δ	η	δ	η	δ	η	δ	η	δ	η	δ	η
200.0	- 8.13 $\pm$ 3.11		- 4.89 $\pm$ 1.78		96.56 $\pm$ 2.23		0.25 $\pm$ 0.024		- 0.48 $\pm$ 0.02		0.028		0.18	
225.0	- 16.55 $\pm$ 0.69		- 4.73 $\pm$ 0.94		109.12 $\pm$ 0.86		0.29 $\pm$ 0.03		- 0.59 $\pm$ 0.025		0.028		0.24	
247.5	- 18.05 $\pm$ 0.20		- 7.66 $\pm$ 0.30		121.02 $\pm$ 0.18		0.32 $\pm$ 0.04		- 0.69 $\pm$ 0.03		0.028		0.30	
270.0	- 18.19 $\pm$ 0.84		- 5.56 $\pm$ 0.74		125.82 $\pm$ 0.74		0.36 $\pm$ 0.05	0.997 $\pm$ 0.003	- 0.79 $\pm$ 0.04		0.027		0.37	
310.0	- 21.59 $\pm$ 0.22		- 8.37 $\pm$ 0.29		135.64 $\pm$ 0.16		0.42 $\pm$ 0.05	0.996 $\pm$ 0.001	- 1.01 $\pm$ 0.04		0.02 $\pm$ 0.02		0.52 $\pm$ 0.01	
370.0	- 25.29 $\pm$ 0.35	0.996 $\pm$ 0.004	- 10.51 $\pm$ 0.37		144.16 $\pm$ 0.19		0.48 $\pm$ 0.09	0.983 $\pm$ 0.002	- 1.19 $\pm$ 0.06		- 0.01 $\pm$ 0.02		0.83 $\pm$ 0.01	
410.0	- 26.60 $\pm$ 0.76	0.982 $\pm$ 0.013	- 11.81 $\pm$ 0.89	0.971 $\pm$ 0.029	148.48 $\pm$ 0.51		0.41 $\pm$ 0.64	0.986 $\pm$ 0.014	- 1.83 $\pm$ 0.76		- 0.05 $\pm$ 0.03		1.07 $\pm$ 0.04	
450.0	- 28.38 $\pm$ 0.39	0.984 $\pm$ 0.007	- 14.33 $\pm$ 0.45	0.999 $\pm$ 0.001	153.24 $\pm$ 0.24		0.20 $\pm$ 0.12	0.959 $\pm$ 0.003	- 1.43 $\pm$ 0.10		- 0.08 $\pm$ 0.04		1.33 $\pm$ 0.03	
490.0	- 29.03 $\pm$ 0.86	0.940 $\pm$ 0.007	- 15.30 $\pm$ 2.04	0.985 $\pm$ 0.015	157.72 $\pm$ 1.00		1.21 $\pm$ 1.42	0.955 $\pm$ 0.015	- 2.13 $\pm$ 1.24		- 0.11 $\pm$ 0.03		1.65 $\pm$ 0.04	
533.0	- 29.27 $\pm$ 1.47	0.951 $\pm$ 0.042	- 14.86 $\pm$ 3.74	0.943 $\pm$ 0.057	161.03 $\pm$ 1.99		1.32 $\pm$ 2.23	0.939 $\pm$ 0.035	- 2.51 $\pm$ 1.91		- 0.15 $\pm$ 0.10		2.01 $\pm$ 0.09	
550.0	- 30.89 $\pm$ 0.53	0.905 $\pm$ 0.013	- 17.74 $\pm$ 0.91	0.999 $\pm$ 0.001	162.69 $\pm$ 0.69	0.999 $\pm$ 0.001	- 0.79 $\pm$ 0.12	0.916 $\pm$ 0.005	- 1.46 $\pm$ 0.14		- 0.12 $\pm$ 0.06		2.20 $\pm$ 0.05	
581.0	- 28.62 $\pm$ 0.48	0.911 $\pm$ 0.016	- 20.29 $\pm$ 0.63	0.999 $\pm$ 0.001	165.41 $\pm$ 0.40	0.984 $\pm$ 0.012	- 1.24 $\pm$ 0.13	0.902 $\pm$ 0.006	- 0.87 $\pm$ 0.22		- 0.17 $\pm$ 0.07		4.14 $\pm$ 0.24	

TABLE I (concluded)

E_{π} (MeV)	S_{31}		P_{31}		P_{33}		D_{33}		D_{35}		F_{35}		F_{37}	
	δ	η	δ	η	δ	η	δ	η	δ	η	δ	η	δ	η
600.0	-28.58 $\pm$ 0.65	0.867 $\pm$ 0.016	-20.06 $\pm$ 1.11		166.76 $\pm$ 0.64	0.980 $\pm$ 0.011	-1.49 $\pm$ 0.12	0.896 $\pm$ 0.007	-0.96 $\pm$ 0.37		-0.19 $\pm$ 0.08		2.76 $\pm$ 0.06	
650.0	-26.15 $\pm$ 1.14	0.673 $\pm$ 0.021	-22.25 $\pm$ 0.98		169.69 $\pm$ 0.58	0.963 $\pm$ 0.019	-2.36 $\pm$ 0.16	0.875 $\pm$ 0.007	0.12 $\pm$ 0.57		-0.16 $\pm$ 0.10		3.41 $\pm$ 0.08	
698.0	-24.57 $\pm$ 4.00	0.500 $\pm$ 0.062	-21.94 $\pm$ 2.63	0.913 $\pm$ 0.087	171.89 $\pm$ 1.58	0.975 $\pm$ 0.025	-3.62 $\pm$ 1.67	0.809 $\pm$ 0.061	-1.53 $\pm$ 1.84	0.982 $\pm$ 0.018	-0.08 $\pm$ 0.29		4.14 $\pm$ 0.24	
746.5	-35.14 $\pm$ 5.93	0.315 $\pm$ 0.108	-15.95 $\pm$ 6.59	0.866 $\pm$ 0.097	170.22 $\pm$ 2.53	0.902 $\pm$ 0.098	-0.15 $\pm$ 4.40	0.915 $\pm$ 0.085	-3.29 $\pm$ 4.79	0.903 $\pm$ 0.043	0.39 $\pm$ 0.65	0.998 $\pm$ 0.002	3.62 $\pm$ 1.55	0.988 $\pm$ 0.012
795.9	-52.30 $\pm$ 3.24	0.373 $\pm$ 0.044	-17.63 $\pm$ 1.45	0.834 $\pm$ 0.106	169.73 $\pm$ 2.40	0.925 $\pm$ 0.057	-4.59 $\pm$ 1.33	0.923 $\pm$ 0.043	-3.01 $\pm$ 2.57	0.791 $\pm$ 0.019	0.23 $\pm$ 0.90	0.985 $\pm$ 0.015	6.48 $\pm$ 1.62	0.985 $\pm$ 0.015
845.4	-48.12 $\pm$ 6.32	0.142 $\pm$ 0.023	-25.94 $\pm$ 1.89	0.982 $\pm$ 0.018	188.19 $\pm$ 1.27	0.995 $\pm$ 0.005	-12.31 $\pm$ 1.52	0.544 $\pm$ 0.045	-3.23 $\pm$ 1.19	0.991 $\pm$ 0.009	0.28 $\pm$ 0.80	0.852 $\pm$ 0.019	6.86 $\pm$ 0.90	0.999 $\pm$ 0.001
870.0	-65.21 $\pm$ 1.57	0.458 $\pm$ 0.023	-24.51 $\pm$ 0.98	0.864 $\pm$ 0.021	178.26 $\pm$ 0.62	0.737 $\pm$ 0.016	-6.68 $\pm$ 0.27	0.788 $\pm$ 0.007	-3.04 $\pm$ 0.46	0.899 $\pm$ 0.012	0.51 $\pm$ 0.25	0.969 $\pm$ 0.005	9.10 $\pm$ 0.37	0.988 $\pm$ 0.004
899.8	-62.37 $\pm$ 1.13	0.588 $\pm$ 0.014	-22.91 $\pm$ 0.55	0.899 $\pm$ 0.021	176.03 $\pm$ 0.73	0.816 $\pm$ 0.019	-7.97 $\pm$ 0.25	0.794 $\pm$ 0.006	-1.02 $\pm$ 0.47	0.854 $\pm$ 0.009	1.05 $\pm$ 0.25	0.949 $\pm$ 0.006	8.55 $\pm$ 0.34	0.983 $\pm$ 0.005
949.4	-48.44 $\pm$ 1.04	0.657 $\pm$ 0.011	-18.63 $\pm$ 0.69	0.668 $\pm$ 0.015	169.26 $\pm$ 0.48	0.905 $\pm$ 0.010	-9.16 $\pm$ 0.34	0.569 $\pm$ 0.008	-4.81 $\pm$ 0.23	0.990 $\pm$ 0.009	-0.65 $\pm$ 0.24	0.833 $\pm$ 0.011	13.46 $\pm$ 0.26	0.999 $\pm$ 0.001
990.0	-60.27 $\pm$ 5.96	0.830 $\pm$ 0.106	-23.59 $\pm$ 2.92	0.713 $\pm$ 0.112	171.27 $\pm$ 2.88	0.857 $\pm$ 0.076	-9.16 $\pm$ 2.07	0.826 $\pm$ 0.064	-5.83 $\pm$ 2.47	0.764 $\pm$ 0.048	3.96 $\pm$ 2.08	0.850 $\pm$ 0.056	9.53 $\pm$ 1.96	0.991 $\pm$ 0.009
1048.6	-69.80 $\pm$ 1.38	0.734 $\pm$ 0.029	-21.86 $\pm$ 1.43	0.710 $\pm$ 0.056	180.00 $\pm$ 1.21	0.757 $\pm$ 0.038	-12.39 $\pm$ 0.91	0.772 $\pm$ 0.019	-3.73 $\pm$ 0.75	0.853 $\pm$ 0.022	1.94 $\pm$ 0.52	0.830 $\pm$ 0.003	12.37 $\pm$ 0.55	0.928 $\pm$ 0.019
1227.6	-63.46 $\pm$ 2.21	0.367 $\pm$ 0.021	-17.32 $\pm$ 2.09	0.609 $\pm$ 0.041	172.16 $\pm$ 1.38	0.767 $\pm$ 0.044	-16.59 $\pm$ 1.33	0.705 $\pm$ 0.030	-6.21 $\pm$ 1.97	0.783 $\pm$ 0.054	-6.13 $\pm$ 1.22	0.735 $\pm$ 0.031	32.38 $\pm$ 1.21	0.783 $\pm$ 0.040
1311.0	-60.80 $\pm$ 10.05	0.739 $\pm$ 0.255	-4.03 $\pm$ 16.63	0.444 $\pm$ 0.199	170.17 $\pm$ 5.31	0.882 $\pm$ 0.106	-11.88 $\pm$ 6.76	0.874 $\pm$ 0.126	-19.33 $\pm$ 9.24	0.496 $\pm$ 0.131	-3.92 $\pm$ 5.48	0.598 $\pm$ 0.085	28.56 $\pm$ 2.71	0.563 $\pm$ 0.192

F_{35} . This amplitude is not well determined. All solutions agree that the phase shift is very small, BBSV and RWF find it positive, and elastic, BOM find it negative (but tending to zero with increasing energy) and elastic and DKLL find it in turn positive (< 350 MeV), negative (< 700 MeV) and then positive again, with some evidence of increasing inelasticity at the higher energies.

F_{37} . All solutions agree that the phase shift is positive, and increasing with energy, with η decreasing steadily above 750 MeV. This amplitude is the candidate for the resonance observed in $\pi - N$ total cross-sections in the region of 1300 MeV (the old "fourth resonance"). The only solution which extends to this energy is that of DKLL and they find that there is indeed a resonance although it appears to be of the absorptive type 5d rather than the quasi-elastic type 5c, with the phase shift going through zero in the region of 1350 MeV. Thus the resonance energy is probably somewhat higher than that quoted from the total cross-section measurements.

$T = \frac{1}{2}$ scattering

S_{11} . DKLL have two possible solutions for this amplitude. In both, the phase shift rises initially to about 50° at 600 MeV and then drops to near zero at 700 MeV, associated with a steep maximum in the inelastic cross-section at about 650 MeV ($\eta < 0.2$). Above 700 MeV the solutions diverge; in one case the phase shift goes negative (solution 1) while η rises slowly and in the second case the phase shift rises sharply, passing through 90° at about 820 MeV while η rises steeply and falls again (solution 2). Both results can be interpreted in terms of a resonance, the former of type 5d/5e and the latter a rather deformed type 5c/5e. The BBSV solution is similar to solution 2, although the initial maximum is not reached until 700 MeV, the dip in the phase shift is much less marked and the subsequent rise much slower, the phase shift not reaching 90° until about 950 MeV. The RWF solution (up to 700 MeV) is very similar to the BBSV solution. Because of a large discontinuity, the BOM solution is difficult to assess. Below 700 MeV they have two solutions, one of which is very similar to the DKLL solution 1 and the second of which has a rapidly rising phase shift, reaching 150° by 700 MeV. Their solution between 750 and 1050 MeV is a compromise between DKLL solution 2 and BBSV. All solutions have a resonance interpretation, but not only are the mass and width uncertain, so also is the type. Also a "double resonance" interpretation is not completely excluded.

P_{11} . The ambiguities in S_{11} are reflected in this amplitude. In both DKLL solutions the phase shift has a broad maximum of about 120° between 550 and 600 MeV and then decreases, steeply in solution 1 and more slowly in solution 2, eventually becoming negative in both cases. In both

$T = \frac{1}{2}$ parameters 0-700 MeV

TABLE II

E_{π} (MeV)	S_{11}		P_{11}		P_{13}		D_{13}		D_{15}		P_{15}		F_{17}		G_{17}	
	δ	η	δ	η	δ	η	δ	η	δ	η	δ	η	δ	η	δ	η
31.0 ± 0.14	6.13		- 0.64 ± 0.02		- 0.14 ± 0.03		0.015		0.012							
41.5 ± 0.31	6.58		- 0.89 ± 0.04		- 0.16 ± 0.04		0.024		0.023							
61.0 ± 0.47	6.17		- 1.26 ± 0.07		- 0.25 ± 0.07		0.064		0.052		0.003					
98.0 ± 0.30	7.80		- 1.54 ± 0.13		- 0.40 ± 0.07		0.17 ± 0.05		0.15 ± 0.03		0.013		0.003			
120.0 ± 0.36	7.20		- 0.96 ± 0.19		- 0.05 ± 0.25		0.29 ± 0.05		0.23 ± 0.05		0.028		0.004			
150.0 ± 0.22	9.11		- 0.92 ± 0.27		- 1.13 ± 0.46		0.59 ± 0.06		0.31 ± 0.07		0.057		0.007			
165.0 ± 1.00	8.60		- 0.22 ± 0.36		- 1.09 ± 0.98		0.69 ± 0.08		0.36 ± 0.09		0.076		0.008			
225.0 ± 2.19	9.85		4.42 ± 0.68	0.999 ± 0.001	- 2.46 ± 0.34		1.60 ± 0.13		0.65 ± 0.15		0.21		0.008			
247.5 ± 1.95	12.94		7.49 ± 0.80	0.993 ± 0.004	- 2.72 ± 0.62		2.01 ± 0.15		0.76 ± 0.14		0.28		0.006			
270.0 ± 2.09	14.14		10.43 ± 0.88	0.986 ± 0.008	- 3.10 ± 0.51		2.52 ± 0.19		0.80 ± 0.15		0.36		0.002			
310.0 ± 0.59	16.61	0.955 ± 0.011	18.82 ± 0.60	0.965 ± 0.008	- 3.80 ± 0.15		3.76 ± 0.16		0.95 ± 0.18		0.72 ± 0.1		0.05 ± 0.06		0.06 ± 0.03	
370.0 ± 0.85	19.96	0.975 ± 0.016	27.12 ± 0.76	0.788 ± 0.020	- 4.24 ± 0.22		6.26 ± 0.19	0.965 ± 0.010	1.40 ± 0.23		1.11 ± 0.14		- 0.04 ± 0.08		0.23 ± 0.03	

TABLE II (concluded)

E_{π} (MeV)	S_{11}		P_{11}		P_{13}		D_{13}		D_{15}		F_{15}		F_{17}		G_{17}	
	δ	η	δ	η	δ	η	δ	η	δ	η	δ	η	δ	η	δ	η
410.0	20.42 ± 6.93	0.950 ± 0.050	36.74 ± 3.86	0.581 ± 0.194	- 4.62 ± 2.60	0.992 ± 0.008	8.24 ± 1.80	0.926 ± 0.074	0.97 ± 3.20	0.997 ± 0.003	1.57 ± 0.95		- 0.79 ± 1.28		0.57 ± 0.70	
450.0	27.71 ± 1.93	0.862 ± 0.020	49.42 ± 2.42	0.392 ± 0.038	- 5.25 ± 0.64	0.999 ± 0.001	12.85 ± 0.40	0.999 ± 0.001	2.16 ± 0.34	0.999 ± 0.001	2.12 ± 0.28		- 0.01 ± 0.12		0.40 ± 0.09	
490.0	25.56 ± 6.38	0.800 ± 0.200	79.56 ± 15.35	0.290 ± 0.192	- 2.51 ± 3.63	0.960 ± 0.040	20.63 ± 1.91	0.939 ± 0.061	1.77 ± 1.53	0.999 ± 0.001	2.19 ± 0.16		0.25 ± 0.83		1.46 ± 0.95	
533.0	27.47 ± 10.51	0.650 ± 0.350	112.59 ± 48.02	0.304 ± 0.163	- 4.63 ± 6.61	0.937 ± 0.063	32.77 ± 4.69	0.772 ± 0.133	3.18 ± 3.81	0.989 ± 0.011	6.41 ± 4.86		- 0.11 ± 2.57		2.66 ± 2.23	
550.0	41.41 ± 4.38	0.731 ± 0.049	91.56 ± 10.58	0.169 ± 0.055	- 5.79 ± 0.77	0.999 ± 0.001	32.42 ± 1.74	0.633 ± 0.040	4.08 ± 0.42	0.999 ± 0.001	5.65 ± 0.54		0.01 ± 0.19		0.72 ± 0.17	
581.0	44.22 ± 5.35	0.540 ± 0.140	117.45 ± 11.96	0.411 ± 0.154	- 3.88 ± 4.44	0.884 ± 0.116	47.52 ± 8.91	0.390 ± 0.113	5.84 ± 3.18	0.971 ± 0.029	5.05 ± 3.41		1.25 ± 1.83		1.82 ± 0.99	
600.0	28.82 ± 3.36	0.523 ± 0.036	120.0 ± 4.35	0.386 ± 0.050	- 4.92 ± 0.71	0.999 ± 0.001	94.75 ± 3.48	0.270 ± 0.037	8.19 ± 0.53	0.991 ± 0.009	6.04 ± 0.73		- 0.20 ± 0.23		0.91 ± 0.20	
650.0	10.08 ± 1.17	0.178 ± 0.029	90.49 ± 7.28	0.232 ± 0.028	- 5.93 ± 1.04	0.954 ± 0.024	135.82 ± 1.86	0.532 ± 0.040	10.47 ± 0.74	0.910 ± 0.018	8.59 ± 0.86		- 0.20 ± 0.32		1.50 ± 0.29	
698.0	- 1.90 ± 25.92	0.400 ± 0.400	56.86 ± 59.06	0.260 ± 0.260	- 5.71 ± 7.80	0.842 ± 0.158	144.21 ± 10.75	0.693 ± 0.236	11.21 ± 7.39	0.889 ± 0.111	14.58 ± 6.63		- 0.85 ± 3.43		5.75 ± 4.54	

TABLE III

$T = \frac{1}{2}$ parameters 700–1400 MeV (solution I)

E_{π} (MeV)	S_{11}		P_{11}		P_{13}		D_{13}		D_{15}		F_{15}		F_{17}		G_{17}	
	δ	η	δ	η	δ	η	δ	η	δ	η	δ	η	δ	η	δ	η
746.5	- 36.30 ± 49.28	0.300 ± 0.200	- 96.34 ± 15.26	0.439 ± 0.199	- 1.54 ± 3.12	0.889 ± 0.111	164.38 ± 7.67	0.892 ± 0.108	15.96 ± 3.42	0.691 ± 0.099	18.78 ± 2.12	0.828 ± 0.165	- 0.43 ± 2.09		0.96 ± 2.15	
795.9	0.21 ± 50.21	0.278 ± 0.278	- 94.31 ± 11.94	0.848 ± 0.152	7.33 ± 5.36	0.951 ± 0.049	175.14 ± 7.95	0.893 ± 0.107	14.98 ± 6.09	0.540 ± 0.087	28.19 ± 4.09	0.611 ± 0.196	- 0.77 ± 2.26	0.994 ± 0.006	2.09 ± 2.60	0.999 ± 0.001
845.4	- 92.70 ± 12.23	0.500 ± 0.215	- 31.64 ± 27.03	0.248 ± 0.239	3.66 ± 5.64	0.871 ± 0.105	175.69 ± 6.99	0.667 ± 0.214	- 4.08 ± 12.55	0.291 ± 0.106	71.31 ± 16.60	0.360 ± 0.084	3.86 ± 2.27	0.977 ± 0.023	7.41 ± 4.18	0.972 ± 0.028
870.0	- 83.93 ± 10.27	0.550 ± 0.263	- 42.94 ± 43.16	0.201 ± 0.169	5.47 ± 4.70	0.916 ± 0.071	169.70 ± 4.78	0.779 ± 0.187	- 18.27 ± 8.52	0.285 ± 0.085	73.55 ± 13.29	0.342 ± 0.075	0.35 ± 1.49	0.987 ± 0.009	8.52 ± 3.05	0.999 ± 0.001
899.8	- 58.95 ± 2.66	0.746 ± 0.069	- 23.57 ± 6.55	0.344 ± 0.031	9.35 ± 2.28	0.686 ± 0.030	177.84 ± 1.18	0.606 ± 0.050	- 18.24 ± 0.90	0.566 ± 0.037	120.17 ± 2.47	0.625 ± 0.033	2.17 ± 0.50	0.887 ± 0.022	11.20 ± 0.95	0.932 ± 0.024
949.4	- 63.80 ± 6.98	0.528 ± 0.147	25.40 ± 23.69	0.170 ± 0.090	14.71 ± 4.07	0.502 ± 0.094	186.05 ± 4.16	0.660 ± 0.153	- 14.58 ± 2.73	0.507 ± 0.081	136.30 ± 4.02	0.733 ± 0.123	- 3.25 ± 1.22	0.931 ± 0.008	6.64 ± 1.80	0.998 ± 0.001
990.0	- 63.44 ± 6.05	0.603 ± 0.213	- 33.84 ± 59.11	0.147 ± 0.147	2.19 ± 5.66	0.537 ± 0.088	174.01 ± 3.48	0.887 ± 0.113	- 19.99 ± 3.45	0.601 ± 0.156	146.18 ± 3.77	0.569 ± 0.142	3.37 ± 2.75	0.931 ± 0.069	3.12 ± 2.41	0.890 ± 0.110
1148.0	- 55.56 ± 1.13	0.645 ± 0.226	- 51.79 ± 11.51	0.127 ± 0.042	6.63 ± 1.19	0.656 ± 0.038	173.73 ± 1.30	0.703 ± 0.025	- 12.71 ± 0.95	0.795 ± 0.030	164.12 ± 0.95	0.832 ± 0.025	- 0.84 ± 0.55	0.901 ± 0.022	5.41 ± 0.53	0.808 ± 0.022
1227.6	- 84.86 ± 4.99	0.193 ± 0.019	- 53.68 ± 4.27	0.539 ± 0.075	- 14.66 ± 2.89	0.564 ± 0.047	181.22 ± 1.28	0.760 ± 0.073	- 11.34 ± 1.67	0.767 ± 0.068	167.86 ± 1.59	0.771 ± 0.040	1.28 ± 0.86	0.789 ± 0.044	4.48 ± 0.78	0.954 ± 0.031
1311.0	- 92.70 ± 92.39	0.308 ± 0.308	11.83 ± 24.21	0.392 ± 0.249	- 16.34 ± 7.38	0.524 ± 0.162	175.53 ± 5.52	0.823 ± 0.091	- 13.41 ± 8.03	0.657 ± 0.179	173.65 ± 10.60	0.516 ± 0.079	3.45 ± 2.89	0.937 ± 0.063	5.57 ± 2.87	0.886 ± 0.104

the BBSV and RWF solutions, the phase shift passes through 90° at 600 MeV, and continues rising. In the RWF solution it flattens off at about 120° , but in the BBSV it continues up through 180° . In the BOM solution the phase shift rises more slowly, passing through 90° at about 800 MeV. All solutions agree that this amplitude is highly absorptive with $\eta \sim 0.15$. A resonance interpretation is again possible, but the details are again obscure.

P_{13} . BBSV, RWF and BOM all find this phase shift small and negative without much inelasticity. In the DKLL solution there is evidence for some structure in the region of 1 GeV, although errors are large. The phase shift, initially negative, becomes positive, rising to $\sim 15^\circ$ before turning and becoming negative again, this being associated with a fairly rapid decrease in η . In the Argand diagram the amplitude describes part of a rather small, squashed anti-clockwise circle (a miniature type 5d). A resonance interpretation is almost certainly too dramatic here, a "woolly cusp" interpretation being more likely.

D_{13} . All solutions have a resonance of type 5c. In the DKLL solution the phase shift passes through 90° close to 600 MeV, in the BOM and BBSV solutions close to 620 MeV and in the RWF solution close to 650 MeV. In all cases η has a deep minimum near this energy ($\eta \sim 0.2$). This amplitude is the old "second resonance", and is much narrower than was thought previously.

D_{15} . The DKLL, BBSV and BOM solutions all agree that there is a resonance in this amplitude. The DKLL and BBSV solutions are of type 5d, the BOM of type 5c. Since at the resonance position ($\alpha \simeq 0^\circ$ in BBSV and DKLL, $\alpha \simeq 90^\circ$ in BOM), η is very small ($\simeq 0.1$) this difference in interpretation is difficult to make meaningful. The zeros in the real parts of the amplitude occur at different energies: about 850 MeV in BOM and DKLL and about 900 MeV in BBSV.

F_{15} . The DKLL, BBSV and BOM solutions all have a resonance of type 5c at about 900 MeV, or just below. It is fairly absorptive, with $\eta \sim 0.3$. This is the old "third resonance", and like D_{13} is much narrower than thought previously.

F_{17} . Like F_{35} this amplitude is badly determined. All that can be said is that it is essentially elastic and the phase shift is very small.

G_{17} . Only DKLL have a solution for this amplitude. They find that the phase shift is surprisingly large for a g wave, rising steadily and reaching $\sim 10^\circ$.

We have discussed in some detail four independent phase shift analyses of pion-nucleon scattering using different techniques. Despite the quantitative differences, a remarkably clear qualitative picture emerges of pion-

nucleon scattering below 1.4 GeV. The only possible exceptions to this are the S_{11} and P_{11} amplitudes, and even there the qualitative interpretation of the situation is the same in each case despite the apparently very different results. This difference is in fact due more to the "phase shift and inelasticity" description of the amplitudes than to any fundamental difference, as can be seen if the different solutions are plotted on the Argand diagram, when they all look much more alike. Also it has been shown recently³⁹ that if the scattering amplitudes are described in terms of the eigenphases and mixing angles of the K matrix then, in certain circumstances, very small changes in these parameters produce very large changes in the phase shift and inelasticities in the elastic scattering channel, i.e., the "phase shift and inelasticity" description is not necessarily the best one to use.

Finally, in Table V we give a composite picture of the resonance structure in pion-nucleon scattering, with possible masses and widths. Further clarification awaits new experiments.

TABLE V

The probable resonance structure of pion-nucleon scattering below 1.4 GeV

Amplitude	Resonance energy (MeV)	Resonance width (MeV)
S_{31}	1690 ± 20	220 ± 20
P_{33}	1240 ± 10	110 ± 10
F_{37}	1920 ± 20	120 ± 30
S_{11} i)	1510 ± 10	100 ± 10
[ii)	1660 ± 60	230 ± 30 (Dubious!)]
P_{11}	1420 ± 20	$200 \pm ?$
D_{13}	1520 ± 10	80 ± 25
D_{15}	1650 ± 20	65 ± 20
F_{15}	1680 ± 10	70 ± 15

APPENDIX 1

THE EVALUATION OF PARTIAL WAVE DISPERSION RELATIONS

The Mandelstam representations for the amplitudes $A^{(T)}$, $B^{(T)}$ are

$$\begin{aligned}
 B^{(T)}(s, t, u) = & \frac{\zeta_1^{(T)} G_r^2}{s - m^2} + \frac{\zeta_2^{(T)} G_r^2}{u - m^2} \\
 & + \frac{1}{\pi^2} \int_{(m+\mu)^2}^{\infty} ds' \int_{4\mu^2}^{\infty} dt' \frac{B_{12}^{(T)}(s', t')}{(s' - s)(t' - t)} \\
 & + \frac{1}{\pi^2} \int_{4\mu^2}^{\infty} dt' \int_{(m+\mu)^2}^{\infty} du' \frac{B_{23}^{(T)}(t', u')}{(t' - t)(u' - u)} \\
 & + \frac{1}{\pi^2} \int_{(m+\mu)^2}^{\infty} ds' \int_{(m+\mu)^2}^{\infty} du' \frac{B_{13}^{(T)}(s', u')}{(s' - s)(u' - u)} \quad \dots [A.1]
 \end{aligned}$$

with $\zeta_1^{(\frac{1}{2})} = -3$, $\zeta_2^{(\frac{1}{2})} = -1$; $\zeta_1^{(\frac{3}{2})} = 0$, $\zeta_2^{(\frac{3}{2})} = 2$, and an identical relation for $A^{(T)}$ with $\zeta_1^{(T)} = \zeta_2^{(T)} = 0$, $T = \frac{1}{2}, \frac{3}{2}$.

By [22] and [15] the projection of $F_{l\pm}^{(T)}(s)$ from the amplitudes $A^{(T)}$ and $B^{(T)}$ is given by

$$F_{l\pm}^{(T)}(s) = \frac{1}{8\pi q^{2l}} \int_{-1}^1 dx \left\{ P_l(x) \frac{(W+m)^2 - \mu^2}{4W^2} [A^{(T)} + (W-m)B^{(T)}] \right. \\ \left. + P_{l\pm 1}(x) \frac{(W-m)^2 - \mu^2}{4W^2} [-A^{(T)} + (W+m)B^{(T)}] \right\} \dots [A.2]$$

The Born pole

This comes from the term in $(s-m^2)^{-1}$ in the amplitude $B^{(\frac{1}{2})}$. Since there is no $T = \frac{3}{2}$ contribution and since s is independent of x , the projection [A.2] is valid only for F_{1-} , and this term contributes only to the amplitude P_{11} . Evaluating the kinematical factor at $W = m$ and carrying out the integration over x , one obtains the contribution

$$-\frac{6m}{(4m^2 - \mu^2)} \cdot \frac{g_r^2}{(s - m^2)} \dots [A.3]$$

where $g_r^2 = G_r^2/4\pi$.

The Born cuts

These come from the terms in $(u-m^2)^{-1}$ in the amplitudes $B^{(T)}$ ($T = \frac{1}{2}, \frac{3}{2}$) and are the cuts $-\infty \leq s \leq 0$ (the left-hand cut) and $(m - \mu^2/m)^2 \leq s \leq (m^2 + 2\mu^2)$ (the short Born cut). They give the only contribution to this latter cut and it is this contribution we want to evaluate, i.e. we want to find

$$\frac{1}{\pi} \int_{(m - \mu^2/m)^2}^{(m^2 + 2\mu^2)} ds' \frac{\text{Im } F_l(s')}{s' - s} \dots [A.4]$$

Substituting the pole terms from [A.1] into [A.2] we obtain

$$F_{l\pm}^{(T)} = \frac{\zeta_2^{(T)}}{2q^{2l}} g_r^2 \int_{-1}^1 dx \left\{ P_l(x) \frac{[(W+m)^2 - \mu^2][W-m]}{4W^2} \right. \\ \left. + P_{l\pm 1}(x) \frac{[(W-m)^2 - \mu^2][W+m]}{4W^2} \right\} \frac{1}{u - m^2} \dots [A.5]$$

In terms of s and x

$$u = -\{[s^2 - 2s(m^2 + \mu^2) - (m^2 - \mu^2)^2] + x[s^2 - 2s(m^2 + \mu^2) + (m^2 - \mu^2)^2]\}/2s \dots [A.6]$$

so that

$$\text{Im } F_{l\pm}^{(T)} = -\frac{s\zeta_2^{(T)}}{q^{2l}} g_r^2 \left\{ \frac{[(W+m)^2 - \mu^2][W-m]}{4W^2} \text{Im } I_l + \frac{[(W-m)^2 - \mu^2][W+m]}{4W^2} \text{Im } I_{l\pm 1} \right\} \quad \dots[A.7]$$

where

$$I_l = \int_{-1}^1 dx \frac{P_l(x)}{a+bx} \quad \dots[A.8]$$

with

$$\left. \begin{aligned} a &= s^2 - 2s\mu^2 - (m^2 - \mu^2)^2, \\ b &= s^2 - 2s(m^2 + \mu^2) + (m^2 - \mu^2)^2. \end{aligned} \right\} \quad \dots[A.9]$$

In the region under consideration $|b| \geq |a|$, so $(a+bx)^{-1}$ has a pole for some x in $-1 \leq x \leq 1$ for each value of s in $(m - \mu^2/m)^2 \leq s \leq (m^2 + 2\mu^2)$. Since I_l is a singular integral, we can write

$$I_l = \int_{-1}^1 dx \frac{P_l(x)}{a+bx} = P \int_{-1}^1 dx \frac{P_l(x)}{a+bx} \pm i \frac{\pi}{b} P_l(-a/b) \quad \dots[A.10]$$

where the $-$ or $+$ sign is to be taken according as to whether the pole is passed from above or below in the x -plane. Letting $s = s + i\epsilon$ and evaluating x to the first order in ϵ , we obtain

$$x(s+i\epsilon) = -a[ab + 2i\epsilon\{(s-\mu^2)b - (s-m^2-\mu^2)a\}]/b^2$$

so that for $\epsilon > 0$ and for all $s \in (m - \mu^2/m)^2 \leq s \leq (m^2 + 2\mu^2)$, $\text{Im } x < 0$ and we must pass below the pole in the x -plane. Hence

$$\text{Im } I_l = \frac{\pi}{b} P_l(-a/b)$$

and the contribution from the short Born cut is given by

$$-\frac{\zeta_2^{(T)} \cdot g_r^2}{4} \int_{(m-\mu^2/m)^2}^{(m^2+2\mu^2)} ds' \frac{1}{(s'-s)b'q'^{2l}} \{ [W'-m][(W'+m)^2 - \mu^2] P_l(-a'/b') + [W'+m][(W'-m)^2 - \mu^2] P_{l\pm 1}(-a'/b') \} \quad \dots[A.11]$$

with

$$\zeta_2^{(\frac{1}{2})} = -1, \quad \zeta_2^{(\frac{3}{2})} = 2$$

and

$$\begin{aligned} a' &= s'^2 - 2s'\mu^2 - (m^2 - \mu^2)^2, \\ b' &= s'^2 - 2s'(m^2 + \mu^2) + (m^2 - \mu^2)^2. \end{aligned}$$

The crossed cut

We require to evaluate the contribution

$$\frac{1}{\pi} \int_0^{(m-\mu)^2} ds' \frac{\text{Im } F_{l\pm}^{(T')}(s')}{s' - s} \quad \dots[A.12]$$

with $\text{Im } F_{l\pm}^{(T')}(s')$ given by the projection

$$\begin{aligned} \text{Im } F_{l\pm}^{(T')}(s') &= \frac{(-1)}{8\pi q'^{2l}} \int_{-1}^1 dx' \left\{ P_l(x') \frac{[(W' + m)^2 - \mu^2]}{4W'^2} [\text{Im } A^{(T')} + (W' - m) \text{Im } B^{(T')}] \right. \\ &\quad \left. + P_{l\pm 1}(x') \frac{[(W' - m)^2 - \mu^2]}{4W'^2} [-\text{Im } A^{(T')} + (W' + m) \text{Im } B^{(T')}] \right\} \quad \dots[A.13] \end{aligned}$$

and all quantities are evaluated on the crossed cut.

For the moment, let us denote amplitudes evaluated on the crossed cut by a subscript c , and recalling [9a], we have

$$A_c^{(\pm)} = A_c^{(+)} + 2A_c^{(-)} = A^{(+)} - 2A^{(-)} = -\frac{1}{3}A^{(\pm)} + \frac{4}{3}A^{(\pm)}$$

and

$$A_c^{(\pm)} = A_c^{(+)} - A_c^{(-)} = A^{(+)} + A^{(-)} = \frac{2}{3}A^{(\pm)} + \frac{1}{3}A^{(\pm)}.$$

We may combine these results as

$$\begin{bmatrix} A_c^{(+)} \\ A_c^{(\pm)} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} A^{(+)} \\ A^{(\pm)} \end{bmatrix}$$

or

$$A_c^{(T)} = C_{TT'} A^{(T')} \quad \dots[A.14]$$

where

$$C_{TT'} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \quad \dots[A.15]$$

is the isospin crossing matrix. By [9b] the corresponding matrix for the B amplitudes is $-C_{TT'}$. Inserting this into [A.13], we obtain

$$\begin{aligned} \text{Im } F_{l\pm}^{(T')}(s') &= \frac{-C_{TT'}}{8\pi q'^{2l}} \int_{-1}^1 dx' \left\{ P_l(x') \frac{[(W' + m)^2 - \mu^2]}{4W'^2} [\text{Im } A^{(T'')} - (W' - m) \text{Im } B^{(T'')}] \right. \\ &\quad \left. + P_{l\pm 1}(x') \frac{[(W' - m)^2 - \mu^2]}{4W'^2} [-\text{Im } A^{(T'')} - (W' + m) \text{Im } B^{(T'')}] \right\}. \end{aligned}$$

Re-expressing the amplitudes $A^{(T'')}, B^{(T'')}$ in terms of the amplitudes $F_{l''}(s'')$ via [16] and [17] yields

$$\text{Im } F_{l\pm}^{(T')}(s') = \frac{-C_{T'T''}}{8W'^2 q'^{2l}} \int_{-1}^1 dx' (\mathcal{A} - \mathcal{B}) \quad \dots [A.16]$$

where

$$\begin{aligned} \mathcal{A} = & \{ [W' + m]^2 - \mu^2 \} P_l(x') - \{ [W' - m]^2 - \mu^2 \} P_{l\pm 1}(x') \} \\ & \left(\frac{W'' + m}{E'' + m} \sum_{l''=0}^{\infty} [\text{Im } F_{l''+}^{(T'')}(s'') P'_{l''+1}(x'') - \text{Im } F_{l''-}^{(T'')}(s'') P'_{l''-1}(x'')] (q'')^{2l''} \right. \\ & \left. + \frac{W'' - m}{E'' - m} \sum_{l''=1}^{\infty} [\text{Im } F_{l''+}^{(T'')}(s'') - \text{Im } F_{l''-}^{(T'')}(s'')] (q'')^{2l''} P'_{l''}(x'') \right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{B} = & \{ [W' - m] [(W' + m)^2 - \mu^2] P_l(x') + [W' + m] [(W' - m)^2 - \mu^2] P_{l\pm 1}(x') \} \\ & \left(\frac{1}{E'' + m} \sum_{l''=0}^{\infty} [\text{Im } F_{l''+}^{(T'')}(s'') P'_{l''+1}(x'') - \text{Im } F_{l''-}^{(T'')}(s'') P'_{l''-1}(x'')] (q'')^{2l''} \right. \\ & \left. - \frac{1}{E'' - m} \sum_{l''=1}^{\infty} [\text{Im } F_{l''+}^{(T'')}(s'') - \text{Im } F_{l''-}^{(T'')}(s'')] P'_{l''}(x'') (q'')^{2l''} \right). \end{aligned}$$

The energies s'' and angles x'' which appear on the right-hand side of equation [A.16] are real physical values of energy ($s'' \geq (m + \mu)^2$) and angle ($-1 \leq x'' \leq 1$), and they are functions of s' and x' . Since the variable t is unchanged under crossing, the equality

$$t' = t''$$

gives immediately that

$$q'^2(1 - x') = q''^2(1 - x''). \quad \dots [A.17]$$

The crossing relation $s'' = u'$ can be expanded, using [A.6], to give

$$s'' = (m^2 + \mu^2 - \frac{1}{2}s')(1 + x') + (m^2 - \mu^2)^2(1 - x')/2s'. \quad \dots [A.18]$$

[A.17] and [A.18] mean of course that the right-hand side of equation [A.16] can be expressed entirely as a function of s' and x' and the integration over x' performed.

Alternatively, [A.17] and [A.18] may be used to express x' as a function of s'' and the x' integration re-cast into an integration over s'' over some (finite) range of energies (the upper and lower limits are controlled by the value of s' occurring in [A.18]). In this form [A.16] may be written symbolically as

$$\begin{aligned} \text{Im } F_{l\pm}^{(T')}(s') = & -C_{T'T''} \sum_{l''} \int_{\text{Some Physical Energies}} ds'' \{ K_{ll''}^{+}(s', s'') \text{Im } F_{l''+}^{(T'')}(s'') \\ & + K_{ll''}^{-}(s', s'') \text{Im } F_{l''-}^{(T'')}(s'') \} \quad \dots [A.19] \end{aligned}$$

where the $K^\pm(s', s'')$ are (calculable) kernels, and the contribution from the crossed cut is

$$-\frac{C_{T'T''}}{\pi} \int_0^{(m-\mu)^2} \frac{ds'}{s'-s} \sum_{T''} \int ds'' \{K_{ll''}^+(s', s'') \operatorname{Im} F_{l''+}^{(T'')}(s'') + K_{ll''}^-(s', s'') \operatorname{Im} F_{l''-}^{(T'')}(s'')\}. \quad \dots[A.20]$$

The only difficulty which occurs in evaluating [A.20] is at the crossed threshold $s_1 = (m-\mu)^2$, where there is a divergence for d waves and higher. Evaluating [A.19] explicitly in the p wave case²⁵ shows that

$$\operatorname{Im} F_{1\pm}(s'+i0) = \frac{a_\pm}{(s_1-s')^{\frac{1}{2}}} + O((s_1-s')^2) \quad \dots[A.21]$$

as s' approaches s_1 along the cut, the constants $a_\pm$ depending on the $\pi-N$

FIG. 6. Contour of integration on the crossed cut.

s wave scattering lengths. Since this is only a square root singularity it gives no trouble.

In the d wave case it is easy to see that

$$\operatorname{Im} F_{2\pm}(s'+i0) = \frac{a'_\pm}{(s_1-s')^{\frac{3}{2}}} + \frac{b'_\pm}{(s_1-s')^{\frac{1}{2}}} + O((s_1-s')^{\frac{1}{2}}) \quad \dots[A.22]$$

as s' approaches s_1 along the cut. As before the constants $a'_\pm$ are related to the $\pi-N$ s wave scattering lengths, while $b'_\pm$ depend on the behaviour of the $\pi-N$ s waves near the physical threshold. The term in [A.22] which contains $(s_1-s')^{-\frac{3}{2}}$ gives rise to a divergence in the integral

$$\frac{1}{\pi} \int_0^{(m-\mu)^2} ds' \frac{\operatorname{Im} F_{2\pm}(s')}{s'-s}.$$

This divergence is easily removed. In Fig. 6 we show the path of integration around the cut $0 < s' \leq s_1$. The contribution arising therefrom is

$$\frac{1}{\pi} \int_0^{(s_1-r)} ds' \frac{\operatorname{Im} F_{2\pm}(s')}{s'-s} + \frac{1}{2\pi i} \int_{C_r} ds' \frac{F_{2\pm}(s')}{s'-s} \quad \dots[A.23]$$

where r is a small real number and C_r is the small circle of radius r centred on s_1 . Substituting [A.22] into the first integral in [A.23] gives

$$\frac{1}{\pi} \int_0^{(s_1-r)} ds' \frac{\operatorname{Im} F_{2\pm}(s')}{s'-s} = (\text{finite term}) + \frac{2}{\pi} \frac{a'_\pm}{r^{\frac{1}{2}}} \frac{1}{(s_1-s)} + O(r^{\frac{1}{2}}) \quad \dots[A.24]$$

where the "finite terms" are independent of r .

Substituting [A.22] into the second integral in [A.23] gives

$$\frac{1}{2\pi i} \int_{C_r} ds' \frac{F_{2\pm}(s')}{s'-s} = -\frac{2}{\pi} \frac{a'_{\pm}}{r^{\frac{1}{2}}} \frac{1}{s_1-s} + O(r^{\frac{1}{2}}). \quad \dots[A.25]$$

Adding [A.24] and [A.25] and letting $r \rightarrow 0$ gives a finite result.

In practice the integration is done analytically around C_r and along the cut to some value $\bar{s} < s_1 - r$, the remainder being done numerically.

The method can easily be extended to handle the higher order divergences which occur in the higher partial waves.

The circle contribution

The discontinuity in $F_l^{(\pm)}(s)$ across the circle is given by

$$\begin{aligned} \Delta F_{l\pm}^{(\pm)}(s) &= \frac{1}{8\pi q^{2l}} \int_{-1}^1 dx \left\{ P_l(x) \frac{(W+m)^2 - \mu^2}{4W^2} [\Delta A^{(\pm)} + (W-m)\Delta B^{(\pm)}] \right. \\ &\quad \left. + P_{l\pm 1}(x) \frac{(W-m)^2 - \mu^2}{4W^2} [-\Delta A^{(\pm)} + (W+m)\Delta B^{(\pm)}] \right\}. \quad \dots[A.26] \end{aligned}$$

The discontinuity of $A^{(\pm)}$, $B^{(\pm)}$ across the circle (i.e. the difference between the value of $A^{(\pm)}$, $B^{(\pm)}$ just inside the circle and the value just outside) is simply $2i \operatorname{Im} A^{(\pm)}$, $2i \operatorname{Im} B^{(\pm)}$ respectively, where

$$\begin{aligned} \operatorname{Im} A^{(\pm)} &= \sum_{J=0}^{\infty} (J+\frac{1}{2}) \frac{8\pi}{p_-^2} (iq_3 p_-)^J [P_J(\cos \theta_3) \operatorname{Im} f_+^J(t) \\ &\quad - \frac{m \cos \theta_3}{[J(J+1)]^{\frac{1}{2}}} P_J'(\cos \theta_3) \operatorname{Im} f_-^J(t)] \quad \dots[A.27a] \end{aligned}$$

$$\operatorname{Im} B^{(\pm)} = 8\pi \sum_{J=1}^{\infty} \frac{(J+\frac{1}{2})}{[J(J+1)]^{\frac{1}{2}}} (iq_3 p_-)^{J-1} P_J'(\cos \theta_3) \operatorname{Im} f_-^J(t) \quad \dots[A.27b]$$

where the $f_{\pm}^J(t)$ are the helicity amplitudes for the process $\pi + \pi \rightarrow N + \bar{N}$. J is even for the (+) case and odd for the (-) case. The momentum of the pion in the process $\pi + \pi \rightarrow N + \bar{N}$ is given by

$$q_3^2 = t/4 - \mu^2 \quad \dots[A.28]$$

and the scattering angle in $\pi + \pi \rightarrow N + \bar{N}$ is given by

$$\cos \theta_3 = \frac{s - p_-^2 + q_3^2}{2ip_- q_3} \quad \dots[A.29]$$

where

$$p_-^2 = (m^2 - t/4) \quad \dots[A.30]$$

[t , of course, is a function of s and x and consequently so is the right-hand

side of [A.26]. As in the case of the crossed cut, the integral over x may be converted to an energy integral—this time over t .]

As discussed in section 4, the only important contributions to [A.27] come from the $T = 0, J = 0$ (s wave $\pi-\pi$) and $T = 1, J = 1$ (ρ meson) states, which contribute respectively to the (+) and (−) amplitudes. Information on the former can be obtained from a study of low energy s wave pion-nucleon scattering²⁵ or from pion-nucleon backward dispersion relations,^{40,41} and information on the latter from the same sources and from the isovector form factor of the nucleon.²⁹ Once these are known, substitution into [A.27], and that in turn into [A.26], gives the circle contribution

$$\frac{1}{2\pi i} \int ds' \frac{\Delta F_{l\pm}(s')}{(s'-s)}$$

APPENDIX 2

CONFORMAL MAPPING METHODS IN EVALUATING DISPERSION INTEGRALS†

Suppose we require to evaluate the right-hand (principle value) integral in a dispersion relation. For simplicity, let the variable be T , the kinetic energy, so that the integral is from 0 to ∞ .

Now make the change of variable to

$$x = \cos \zeta = \frac{T_0 - T}{T_0 + T} \quad \dots[A.31]$$

where T_0 is an arbitrary constant. For obvious reasons $-T_0$ is called the “conformal zero”. The conformal mapping [A.31] maps the cut 0 to ∞ in the T plane on to the cut -1 to 1 in the x -plane. From [A.31] we have that

$$\begin{aligned} (1-x^2)^{\frac{1}{2}} &= \sin \zeta = \frac{2(TT_0)^{\frac{1}{2}}}{T+T_0} \\ &\sim q \quad \text{as } T \rightarrow 0 \\ &\sim q^{-1} \quad \text{as } T \rightarrow \infty \end{aligned} \quad \dots[A.32]$$

since

$$\begin{aligned} q^2 &= [s-(m+\mu)^2][s-(m-\mu)^2]/4s \\ &= \frac{m^2 T(T+4m\mu)}{2mT+(m+\mu)^2} \end{aligned} \quad \dots[A.33]$$

By unitarity, $\text{Im } F_l$ has the behaviour

$$\begin{aligned} \text{Im } F_l &\sim q^{2l+1} \quad \text{as } T \rightarrow 0 \\ &\sim q^{-2l-1} \quad \text{as } T \rightarrow \infty. \end{aligned}$$

† We are indebted to Dr. C. Lovelace for permission to quote freely from his unpublished notes on this subject.

Hence the approximation

$$\text{Im } F_l = \sum_{n=1}^N a_n (1-x^2)^{l+\frac{1}{2}} C_{n-1}^{l+1}(x) \quad \dots [A.34]$$

where $C_{n-1}^{l+1}(x)$ is a Gegenbauer polynomial,⁴² will have the correct behaviour both at threshold and infinity.

The rescattering contribution required is of the form

$$\int_0^\infty dT' \frac{\phi(T')}{T'-T} = - \int_{-1}^1 dx' \left[\frac{1}{x'-x} - \frac{1}{1+x'} \right] \phi(x'). \quad \dots [A.35]$$

Therefore the rescattering contribution to the real part will be given by

$$[\text{Re } F_l]_{\text{Rescatt.}} = \sum_{n=1}^N a_n [g_n^l(x) - g_n^l(-1)] \quad \dots [A.36]$$

where

$$g_n^l(x) = -\frac{P}{\pi} \int_{-1}^1 dx' \frac{(1-x'^2)^{l+\frac{1}{2}} C_{n-1}^{l+1}(x')}{(x'-x)} \quad \dots [A.37]$$

and the subtraction at -1 ensures that the real part vanishes at infinity.

It can be shown⁴³ that

$$\begin{aligned} C_{n-1}^{l+1}(x) &= \frac{(2l+2)_{n-1}}{(l+\frac{3}{2})_{n-1}} P_{n-1}^{(l+\frac{1}{2}, l+\frac{1}{2})}(x) \\ &= \frac{\Gamma(2l+n+1)\Gamma(l+\frac{3}{2})}{\Gamma(2l+2)\Gamma(l+n+\frac{1}{2})} P_{n-1}^{(l+\frac{1}{2}, l+\frac{1}{2})}(x). \end{aligned} \quad \dots [A.38]$$

Therefore⁴⁴

$$2(x^2-1)^{l+\frac{1}{2}} Q_{n-1}^{(l+\frac{1}{2}, l+\frac{1}{2})}(x) = - \int_{-1}^1 dx' \frac{(1-x'^2)^{l+\frac{1}{2}} P_{n-1}^{(l+\frac{1}{2}, l+\frac{1}{2})}(x')}{x'-x-i\epsilon} \quad \dots [A.39]$$

and we have

$$g_n^l(x) = \frac{1}{\pi} \frac{\Gamma(2l+n+1)\Gamma(l+\frac{3}{2})}{\Gamma(l+n+\frac{1}{2})\Gamma(2l+2)} 2(x^2-1)^{l+\frac{1}{2}} \text{Re } Q_{n-1}^{(l+\frac{1}{2}, l+\frac{1}{2})}(x). \quad \dots [A.40]$$

This may be expressed in terms of hypergeometric functions for $-1 \leq x \leq 1$ ⁴⁴

$$\begin{aligned} 2(x^2-1)^{l+\frac{1}{2}} Q_{n-1}^{(l+\frac{1}{2}, l+\frac{1}{2})}(x) &= \frac{2^{2l+1} \Gamma(l+\frac{1}{2}) \Gamma(l+n+\frac{1}{2})}{\Gamma(n+2l+1)} F(n, -n-2l; \frac{1}{2}-l; \frac{1}{2}-\frac{1}{2}x) \\ &\quad - \frac{\pi(x^2-1)^{l+\frac{1}{2}}}{\sin(\pi[l+\frac{1}{2}])} P_{n-1}^{(l+\frac{1}{2}, l+\frac{1}{2})}(x). \end{aligned} \quad \dots [A.41]$$

The second term is purely imaginary for $x^2 \leq 1$ and so in this region (corresponding to $T \geq 0$) we have

$$\begin{aligned}
 g_n^l(x) &= \frac{1}{\pi} \frac{\Gamma(l+\frac{1}{2})\Gamma(l+\frac{3}{2})}{\Gamma(2l+2)} 2^{2l+1} F(n, -n-2l; \frac{1}{2}-l; \frac{1}{2}-\frac{1}{2}x) \\
 &= \frac{1}{\sqrt{\pi}} \frac{\Gamma(l+\frac{1}{2})}{l!} F(n, -n-2l; \frac{1}{2}-l; \frac{1}{2}-\frac{1}{2}x). \quad \dots[A.42]
 \end{aligned}$$

The hypergeometric function reduces to a polynomial since the second argument is a negative integer. Convenient transformations to make are⁴⁵

$$F(n, -n-2l; \frac{1}{2}-l; \frac{1}{2}-\frac{1}{2}x) = F\left(\frac{n}{2}, -\frac{n}{2}-l; \frac{1}{2}-l; 1-x^2\right) \quad \dots[A.43]$$

and

$$F(n, -n-2l; \frac{1}{2}-l; \frac{1}{2}-\frac{1}{2}x) = xF\left(\frac{n+1}{2}, -\left(\frac{n-1}{2}\right)-l; \frac{1}{2}-l; 1-x^2\right). \quad \dots[A.44]$$

The hypergeometric function on the right-hand side of [A.43] terminates for n even, and that in [A.44] for n odd. Thus we have

$$g_{2m}^l(x) = \binom{l-\frac{1}{2}}{l} F(m, -m-l; \frac{1}{2}-l; 1-x^2), \quad \dots[A.45]$$

$$g_{2m+1}^l(x) = \binom{l-\frac{1}{2}}{l} x F(m+1, -m-l; \frac{1}{2}-l; 1-x^2) \quad \dots[A.46]$$

and these series terminate more quickly than that in [A.42]. It follows from [A.45] and [A.46] that

$$g_n^l(-1) = (-1)^n \binom{l-\frac{1}{2}}{l}. \quad \dots[A.47]$$

Thus given $\text{Im } F_l$ by [A.34], then [A.36], [A.45], [A.46] and [A.47] together determine the rescattering.

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