Entrainment parameters in cold superfluid neutron star core

Nicolas Chamel  
*Copernicus Astronomical Center, Polish Academy of Science, ul. Bartycka 18, 00-716 Warszawa, Poland*  
*LUTH, Paris Observatory, 5 place Jules Janssen, 92195 Meudon, France*

Pawel Haensel  
*Copernicus Astronomical Center, Polish Academy of Science, ul. Bartycka 18, 00-716 Warsaw, Poland*  
(Dated: September 14, 2006)

Hydrodynamical simulations of neutron star cores, based on a two fluid description in terms of a neutron-proton superfluid mixture, require the knowledge of the Andreev-Bashkin entrainment matrix which relates the momentum of one constituent to the currents of both constituents. This matrix is derived for arbitrary nuclear asymmetry at zero temperature and in the limits of small relative currents in the framework of the energy density functional theory. The Skyrme energy density functional is considered as a particular case. General analytic formulae for the entrainment parameters and various corresponding effective masses are obtained. These formulae are applied to the liquid core of a neutron star, composed of an homogeneous plasma of nucleons, electrons and possibly muons in beta equilibrium.

PACS numbers: 26.60.+c, 97.10.Sj, 97.60.Jd, 47.37.+q.  
Keywords: neutron star - two fluid model - superfluid densities - entrainment - effective mass

I. INTRODUCTION

In a standard model of neutron star core, matter is a uniform plasma consisting of neutrons, of number density $n_n$, and a small admixture of protons and electrons, of number densities $n_p$ and $n_e$, respectively. Electrons ensure the overall stability of the star by the condition of electroneutrality, $n_e = n_p$, but play a negligible role in the mass transport because their mass is very small compared to the nucleon mass. If the electron Fermi energy exceeds the muon rest mass, muons are present in matter, even within this simple mean field model take a rather

$$\rho_q = \rho_q^q q_q \pi_1, \quad \rho_q = n_q m$$  \hspace{1cm} (2)

in which it is recalled that the momenta $\pi_q$ are defined by the partial derivative with respect to the nucleon current $n_q q_q$ of the Lagrangian density $\Lambda(q_n, q_p)$ of the system. The mass current of some given nucleon species $q$

$$\rho_q = \sum q' \rho_{qq'} V_{qq'}$$  \hspace{1cm} (3)

in which $\rho_{qq'}$ is the (symmetric) entrainment or mass density matrix. Only one of these matrix elements has to be specified since the other elements can be obtained from the identities due to Galilean invariance

$$\rho_{nn} + \rho_{np} = \rho_n, \quad \rho_{pp} + \rho_{pn} = \rho_p.$$  \hspace{1cm} (4)

This matrix is a necessary ingredient in dynamical simulations of neutron star cores, such as for instance the study of oscillation modes. The (static) equation of state and the entrainment matrix are usually obtained using different microscopic models. In earlier calculations and even recently, the mass density matrix is postulated to have some density dependence whose parameters are determined from rough estimates.

Comer et al. \cite{Comer:2005} have built a self-consistent equation of state in the framework of a minimal relativistic $\sigma-\omega$ mean field model, ignoring non linear couplings between the meson fields which are however essential in order to reproduce nuclear properties such as the incompressibility of nuclear matter. They have obtained semi analytical formulae for the entrainment parameters in the limit of small fluid velocities (compared to that of light), which even within this simple mean field model take a rather

$$V_q = \pi_q / m$$  \hspace{1cm} (1)
It is not clear that analytical formulae could still be obtained with realistic relativistic mean field models, taking into account self meson couplings and including as well the ρ meson which is required for a correct treatment of the symmetry energy. Despite the fact that non relativistic mean field models have been widely applied in the study of terrestrial nuclei and in neutron stars, there has been no attempt so far to apply these models for the calculation of the mass density matrix. The purpose of the present work is therefore to fill this gap and to further investigate the density dependence of the various entrainment parameters and effective masses that have been introduced in the literature.

II. ENTRAINMENT IN A MIXTURE OF FERMI LIQUIDS

At zero temperature the entrainment effects have been shown to be independent of the nucleon pairing, giving rise to superfluidity \[3\]. Even at finite temperatures well below the critical temperatures for the onset of superfluidity, pairing as well as thermal effects are very small \[3\]. We can therefore ignore pairing interactions and restrict ourself to the limit of zero temperature.

Borumand et al. \[7\] have shown how to obtain the entrainment matrix of a neutron-proton mixture in the framework of the Landau Fermi liquid theory. In what follows, we will limit ourselves to spin-unpolarized nuclear matter. Therefore, spin indices will not appear in our formulae, and all quantities are to be understood as spin-averages. Under our assumptions, the change in the total energy density of the system due to a small current is expressed as

\[
\mathcal{E} = 2 \sum_q \int \frac{d^3k}{(2\pi)^3} e^{(q)}(\{k\}) \delta n^{(q)}(\{k\})
+ 2 \sum_{q,q'} \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} f^{qq'}(\{k, k'\}) \delta n^{(q)}(\{k\}) \delta n^{(q')}(\{k'\}),
\]

in which \(e^{(q)}(\{k\})\) is the energy of a quasiparticle \((q = n, p\) for neutron and proton respectively) of wave vector \(k\), \(f^{qq'}(\{k, k'\})\) is the (spin averaged) interaction between the quasiparticles. Moreover, \(\delta n^{(q')}(\{k'\})\) denotes the change in the distribution function of quasiparticle states from that of the static (zero current) ground state characterized by the Heaviside functions \(\Theta(\{k'_F - k\})\), where \(k_F^{(q)}\) is the Fermi momentum (in the units of \(\hbar\)) \(k_F^{(n)} = (3\pi^2 n_q)^{1/3}\). In the presence of neutron and proton currents, the corresponding Fermi surfaces are displaced by a vector \(Q_q\). In the limit of small currents \(Q_q \ll k_F^{(q)}\), writing the “superfluid velocities” from \[f\] as

\[
V_q = \hbar Q_q / m
\]

it can be shown that the mass current \(\rho_q = \rho_p v_q\) of each nucleon species is linearly related to both the neutron and proton superfluid velocities

\[
\rho_q = \sum_{q'} \rho_{qq'} V_{q'},
\]

where the (symmetric) entrainment matrix \(\rho_{qq'}\) is given by

\[
\rho_{qq'} = \sqrt{\rho_q \rho_{q'}} \frac{m}{\sqrt{m_q m_{q'}}} \left(\delta_{qq'} + \mathcal{F}_1^{qq'} / 3\right).
\]

The (Landau) effective mass \(m_n^{(q)}\) and the dimensionless Landau parameters \(\mathcal{F}_1^{qq'}\) are defined respectively by

\[
\frac{1}{m_n^{(q)}} = \frac{1}{\hbar^2 k_F^{(q)}} \frac{d e}{d k} \bigg|_{k = k_F^{(q)}},
\]

\[
\mathcal{F}_1^{qq'} = \sqrt{N_q N_{q'}} \mathcal{F}_1^{qq'},
\]

in which \(N_q\) is the density of quasiparticle states at the Fermi surface,

\[
N_q = \frac{m_n^{(q)} k_F^{(q)}}{\hbar^2 \pi^2},
\]

and the parameters \(\mathcal{F}_1^{qq'}\) are obtained from the Legendre expansion of the spin averaged quasiparticle interaction

\[
f^{qq'}(\{k, k'\}) = \sum_{\ell} \mathcal{F}_1^{qq'}(\ell) P_{\ell}(\cos \theta)
\]

where \(\theta\) is the angle between the wave vectors \(k\) and \(k'\) lying on the corresponding Fermi surface.

Alternative formulae of the entrainment matrix \[8\] have been used in the literature, based on the decomposition of the Landau effective masses in the form

\[
m_n^{(q)} = m + \delta m_n^{(q)},
\]

\[
m_p^{(q)} = m + \delta m_p^{(q)} + \delta m_p^{(p)}
\]

where the various contributions to the effective masses are related to the Landau parameters by the simple formula \[9\]

\[
\delta m_{qq'} = \frac{1}{3} \mathcal{F}_1^{qq'} m \sqrt{\frac{n_q m_n^{(q)}}{n_p m_p^{(p)}}}.
\]

The mass density matrix can then be equivalently written explicitly as

\[
\rho_{nn} = \rho_n m + \delta m_{nn} / m_n^{(n)}
\]

\[
\rho_{pp}^* = \rho_p m + \delta m_{pp} / m_p^{(p)}
\]
\begin{align}
\rho_{pp} &= \rho_p \frac{m + \delta m_{pp}}{m_p} \tag{17} \\
\rho_{np} &= \rho_{pn} = \rho_n \frac{\delta m_{np}}{m_n} = \rho_p \frac{\delta m_{pp}}{m_p}. \tag{18}
\end{align}

It should be remarked that in the formulae provided by Sauls (see \[3\]), the terms proportional to \(\delta n_{nn}^\oplus\) and \(\delta m_{np}^\oplus\) in the expressions for \(\rho_{nn}\) and \(\rho_{pp}\) are omitted and therefore those formulae violate Galilean invariance.

The quasiparticle energies \(e^{(q)}\{k\}\) and the quasiparticle interaction \(f^{qq}\{k,k'\}\) can be deduced from a microscopic approach. The solution of the many body problem, starting from the bare nucleon-nucleon interactions is very difficult. We shall here adopt a simpler approach based on self-consistent mean field models with phenomenological effective interactions (for a review, see for instance \[10\]), which have been very successful in describing the nuclear properties of terrestrial nuclei. Such mean field models have also been widely applied in the context of neutron stars.

### III. LANDAU PARAMETERS IN THE ENERGY DENSITY FUNCTIONAL THEORY

We shall calculate in this section the Landau parameters for asymmetric nuclear matter in the framework of the Hohenberg-Kohn-Sham energy density functional theory \[11,12\].

The energy density functional for spin-unpolarized homogeneous nuclear matter is written as a sum of the isoscalar \((T = 0)\) and isovector \((T = 1)\) terms \[10\]

\[
E = \sum_{T=0,1} \delta \tau_T \frac{\hbar^2}{2m} \tau_T + C_T^\sigma n_b^2 + C_T^\tau n_T \tau_T + C_T^J j_T^2. \tag{19}
\]

The isoscalar and isovector parts of some quantity for a nucleon system is given respectively by the sum and the difference between the neutron and proton contributions. For example, the isoscalar and isovector densities are given by \(n_0 = n_n + n_p = n_b\) and \(n_1 = n_n - n_p\) respectively.

The nucleon density \(n_q\), kinetic energy density \(\tau_q\) (in the units of \(\hbar^2/2m\)), and nucleon current \(j_q\) are expressible in terms of the nucleon distribution function \(\tilde{n}^{(q)}\{k\}\) by

\[
n_q = \int \frac{d^3k}{(2\pi)^3} \tilde{n}^{(q)}\{k\} \tag{20}
\]

\[
\tau_q = \int \frac{d^3k}{(2\pi)^3} k^2 \tilde{n}^{(q)}\{k\} \tag{21}
\]

\[
\dot{\tilde{\rho}}\{k\} = \int \frac{d^3k}{(2\pi)^3} k^2 \tilde{n}^{(q)}\{k\} \tag{22}
\]

Energy density functionals of the form \[19\] can be obtained in the Hartree-Fock approximation with effective contact nucleon-nucleon interactions \(\hat{v}\{r_1, r_2\}\) of the Skyrme type, whose standard parametrisations (ignoring spin-orbit terms which are irrelevant in the present case) are

\[
\hat{v}\{r_1, r_2\} = t_0(1 + x_1 \hat{P}_\sigma) \delta\{r_1 - r_2\} + \frac{1}{2} t_1(1 + x_1 \hat{P}_\sigma) \delta\{r_1 - r_2\} + \delta\{r_1 - r_2\} \hat{k}^2 + t_2(1 + x_2 \hat{P}_\sigma) k^2 \cdot \delta\{r_1 - r_2\} \hat{k} + \frac{1}{6} t_3(1 + x_3 \hat{P}_\sigma) \delta\{r_1 - r_2\} n_b \left(\frac{r_1 + r_2}{2}\right)^2 \gamma, \tag{23}
\]

where \(\hat{P}_\sigma = (1 + \sigma_1 - \sigma_2)/2\) is the spin exchange operator and \(\hat{k} = -i (\nabla_1 - \nabla_2)/2\). The density dependent term proportional to \(t_3\) represents the effects of three body interactions. The coefficients \(C_T^\sigma\{n_b\}, C_T^\tau\) and \(C_T^J\) can then be expressed in terms of the parameters of the Skyrme interaction (see Appendix). It should be stressed however that the functional \[19\] is more general than the Skyrme functional. In particular, the coefficients \(C_T^\nu\{n_b\}\) can be any function of the baryon density \(n_b\).

The single particle energies are obtained from the functional derivative of the energy density

\[
e^{(q)}\{k\} = \frac{\delta E}{\delta \tilde{n}^{(q)}\{k\}} \bigg|_0 \tag{24}
\]

where the zero subscript indicates that the functional derivative is evaluated in the static ground state (in which the currents \(j_q\) vanish), characterized by the distribution function

\[
\tilde{n}_0^{(q)}\{k\} = \Theta\{t_0^{(q)} - k\}. \tag{25}
\]

Substituting the functional \[19\] yields

\[
e_q\{k\} = \frac{\hbar^2 k^2}{2m_q} + U_q, \tag{26}
\]

in which the effective mass \(m_q^\oplus\) (using the same symbol as for the Landau effective mass defined by \[19\] since both definitions coincide) and the single-quasiparticle potential \(U_q\) are given by

\[
\frac{\hbar^2}{2m_q^\oplus} = \frac{\delta E}{\delta \tau_q} = \frac{\hbar^2}{2m} + (C_0^\tau - C_1^\tau) n_b + 2C_1^J n_q \tag{27}
\]

\[
U_q = \frac{\delta E}{\delta n_q} = 4C_0^n n_q - 2n_b (C_0^\nu - C_1^\nu) + (C_0^\tau - C_1^\tau) n_b + 2C_1^\tau \tau_q + \frac{\delta C_0^n}{\delta n_b} n_b^2 + \frac{\delta C_1^n}{\delta n_b} (2n_q - n_b)^2, \tag{28}
\]
where \( \tau_b = \tau_n + \tau_p \).

The quasiparticle interaction, calculated as a second functional derivative of the energy functional \( E \), is

\[
E^{\text{q'p'}} = \frac{\delta^2 E}{\delta n^{(q')}\{k\} \delta n^{(p')}\{k'\}}
\]

and contains only \( \ell = 0 \) and \( \ell = 1 \) components.

The non-vanishing Landau parameters are found to be expressible as

\[
f_0^{nn} = 2h_F^{(n)}(C_0^n + C_1^n) + 2(C_1^n + C_0^n) + \frac{d C_0^n}{d n_b} 4 n_b + 4 \frac{d C_0^n}{d n_b} (n_n - n_p) + \frac{d^2 C_0^n}{d n_b^2} n_n^2 + \frac{d^2 C_0^n}{d n_b^2} (n_n - n_p)^2
\]

\[
f_0^{np} = f_0^{pn} = (h_F^{(n)} + h_F^{(p)})(C_0^n - C_1^n) + 2(C_0^n - C_1^n) + \frac{d C_0^n}{d n_b} 4 n_b + \frac{d^2 C_0^n}{d n_b^2} n_n^2 + \frac{d^2 C_0^n}{d n_b^2} (n_n - n_p)^2
\]

\[
f_1^{nn} = 2(C_0^n + C_1^n) h_F^{(n)}
\]

\[
f_1^{pp} = 2(C_0^n + C_1^n) h_F^{(p)}
\]

\[
f_1^{np} = f_1^{pn} = 2(C_0^n - C_1^n) h_F^{(n)} h_F^{(p)}
\]

These formulae agree with those of Bender et al. for the limiting case of symmetric nuclear matter (using standard notations \( f_1^{nn} = f_1^{pp} = f_t + f_0 \) and \( f_1^{np} = f_t - f_0 \)) and generalize the results of Blaizot \& Haensel \[14\] for asymmetric nuclear matter to any energy density functional of the form \[18\].

It should be remarked in particular that for any such functional \[19\], the parameters \( f_1^{nn} \) and \( f_1^{pp} \) are related by

\[
\frac{f_1^{nn}}{f_1^{pp}} = \left( \frac{n_n}{n_p} \right)^{2/3}
\]

The corresponding dimensionless \( \ell = 1 \) Landau parameters can be expressed in compact form as

\[
\mathcal{F}_1^{qq'} = 3 \bar{\alpha}_{qq'} \sqrt{n_q m_q^{(q')} n_{q'} m_{q'}}
\]

in which the coefficients \( \bar{\alpha}_{qq'} \) are defined by

\[
\bar{\alpha}_{nn} = \bar{\alpha}_{pp} = \frac{2}{h^2} (C_0^n + C_1^n)
\]

\[
\bar{\alpha}_{np} = \bar{\alpha}_{pn} = \frac{2}{h^2} (C_0^n - C_1^n)
\]

\[
f_0^{pp} = 2h_F^{(p)}(C_0^p + C_1^p) + 2(C_1^p + C_0^p) + \frac{d C_0^p}{d n_b} 4 n_b + 4 \frac{d C_0^p}{d n_b} (n_n - n_p) + \frac{d^2 C_0^p}{d n_b^2} n_n^2 + \frac{d^2 C_0^p}{d n_b^2} (n_n - n_p)^2
\]

We conclude this section by remarking that in a general case of asymmetric nuclear matter (i.e., with \( n_n \neq n_p \)) the \( \ell = 1 \) Landau parameters can be uniquely determined in terms solely of the effective masses as

\[
\mathcal{F}_1^{nn} = 3 \frac{n_n m_{n1} m_{n2}}{n_p^2 - n_n^2 m} \left[ n_p \left( 1 - \frac{m_n}{m_{n1}} \right) - n_n \left( 1 - \frac{m_n}{m_{n2}} \right) \right]
\]

\[
\mathcal{F}_1^{np} = 3 \frac{n_n m_{n1} m_{n2}}{n_p^2 - n_n^2 m} \left[ n_p \left( 1 - \frac{m_n}{m_{n1}} \right) - n_n \left( 1 - \frac{m_n}{m_{n2}} \right) \right]
\]

\[
\mathcal{F}_1^{pp} = 3 \frac{n_n m_{n1} m_{n2}}{n_p^2 - n_n^2 m} \left[ n_p \left( 1 - \frac{m_n}{m_{n1}} \right) - n_n \left( 1 - \frac{m_n}{m_{n2}} \right) \right]
\]

IV. ENTRAINMENT MATRIX AND EFFECTIVE MASSES

Substituting the expressions \[37\] of the Landau parameters obtained in the previous section, the entrainment matrix elements \[8\] can be seen to be expressible as

\[
\rho_{qq'} = \rho_q \frac{m}{m_q} \delta_{qq'} + \bar{\alpha}_{qq'} \rho_q \rho_{q'}
\]

or more explicitly in terms of the mass densities \( \rho_n \) and \( \rho_p \)

\[
\rho_{nn} = \rho_n (1 - \bar{\alpha}_{np} \rho_p)
\]

\[
\rho_{pp} = \rho_p (1 - \bar{\alpha}_{np} \rho_n)
\]
\[
\rho_{np} = \bar{\alpha}_{np} \rho_n \rho_p. \tag{46}
\]

It is readily seen that the formulae (44) imply basic property of the entrainment matrix, namely \( \rho_{nn} + \rho_{np} = \rho_n \) and \( \rho_{pp} + \rho_{np} = \rho_p \), which guarantees the Galilean invariance of the two fluid model.

Two other kinds of effective masses, different from the Landau effective masses \( m_q^\rho \), have been introduced in the literature. Effective nucleon masses can be defined from the mass density matrix elements by setting

\[
\frac{\rho_{pq}}{\rho_q} = \frac{m}{m_q^\rho}, \tag{47}
\]

in such a way that in the proton momentum rest frame (\( V_p = 0 \)) we have \( \pi_n = m_q^\rho v_n \) and similarly in the neutron momentum rest frame, \( \pi_p = m_q^\rho v_p \).

These effective masses have a very simple density dependence as shown on the formulae

\[
\frac{m_q^\rho}{m} = \frac{1}{1 - \bar{\alpha}_{np} \rho_p}, \tag{48}
\]

\[
\frac{m_q^n}{m} = \frac{1}{1 - \bar{\alpha}_{np} \rho_n} \tag{49}
\]

The \( \pi \)-effective masses differ from the Landau quasiparticle effective masses, and are related to the latter ones by

\[
\frac{m}{m_q^\pi} = \frac{m}{m_q^n} + \bar{\alpha}_{qq} \rho_q, \tag{50}
\]

due to the non-vanishing quasiparticle interactions. The extra term on the right hand side can be interpreted as resulting from the backflow induced by the motion of the quasiparticles.

Alternatively one can introduce effective masses \( m_q^\pi \) such that in the proton rest frame (meaning \( v_p = 0 \)) we have \( \pi_n = m_q^\pi v_n \) and similarly in the neutron rest frame \( \pi_p = m_q^\pi v_p \). These effective masses are given by

\[
\frac{m_q^n}{m} = \frac{1}{1 - \bar{\alpha}_{np} \rho_n}, \tag{51}
\]

\[
\frac{m_q^p}{m} = \frac{1}{1 - \bar{\alpha}_{np} \rho_p}. \tag{52}
\]

where \( \rho_b = \rho_n + \rho_p \). The effective masses of the different kinds are related by

\[
m_q^n - m = (m_q^n - m) \left[ 1 + \frac{n_n}{n_p} \left( \frac{m_q^n}{m} - 1 \right) \right]^{-1}, \tag{53}
\]

\[
m_q^p - m = (m_q^p - m) \left[ 1 + \frac{n_p}{n_n} \left( \frac{m_q^p}{m} - 1 \right) \right]^{-1}. \tag{54}
\]

These formulae show that, as pointed out by Prix et al. [15], in the limit of very small proton fraction \( n_p/n_b \ll 1 \), as relevant in the liquid core of neutron stars, we shall have \( m_q^n \sim m_q^n \sim m \) and \( m_q^p \sim m_q^p \). We shall compute more accurate values of the effective masses in section VI.

In studies of neutron star cores, the non diagonal entrainment matrix elements \( \rho_{np} \) have often been parametrised as

\[
\rho_{np} = -\epsilon \rho_n, \tag{55}
\]

in which the dimensionless parameter \( \epsilon \) was taken as a constant [16, 17, 18]. Other authors [15, 19, 20] have suggested instead to set the dimensionless parameters defined by

\[
\epsilon_q = 1 - \frac{m_q^q}{m}, \tag{56}
\]

\[
\epsilon_n = \frac{\bar{\alpha}_{np} \rho_p}{1 - \bar{\alpha}_{np} \rho_b}, \tag{57}
\]

\[
\epsilon_p = \frac{\bar{\alpha}_{np} \rho_n}{1 - \bar{\alpha}_{np} \rho_b}. \tag{59}
\]

The effective masses and entrainment parameters seem to diverge at some points of the \( \rho_n - \rho_p \) plane. However as will be shown in the next section, once stability constraints are imposed, these apparent singularities disappear.

V. STABILITY OF THE STATIC GROUND STATE AND CONSTRAINTS ON THE ENTRAINMENT PARAMETERS

Since the momentum of each nucleon is a linear combination of both the neutron and proton currents, this means that the corresponding dynamical contribution to the Lagrangian density of the system \( \Lambda_{\text{dyn}} = E_{\text{dyn}} \) is a bilinear symmetric form of the currents. It is therefore readily seen that this dynamical contribution is expressible in terms of the Andreev-Bashkin entrainment matrix elements as

\[
E_{\text{dyn}} = \frac{1}{2} (\rho_{nn} V_n^2 + 2 \rho_{np} V_n \cdot V_p + \rho_{pp} V_p^2). \tag{60}
\]

As a result the total energy density \( E \) of the fluid mixture can be written as the sum of the dynamical contribution,
\[\mathcal{E}_{\text{dyn}}, \text{ and an internal static contribution } \mathcal{E}_{\text{ins}} \text{ which only}
\] depends on the densities: \( \mathcal{E} = \mathcal{E}_{\text{dyn}} + \mathcal{E}_{\text{ins}} \).

The static ground state of the system is stable if the term \( \mathcal{E}_{\text{dyn}} \) is strictly positive. This means that the
entrainment matrix must be positive definite (the minimum
energy state thus being obtained by the vanishing of the
superfluid velocities or equivalently of the currents, i.e.
\( \mathcal{E}_{\text{dyn}} = 0 \)), which means that its eigenvalues must be
strictly positive. This condition entails that the matrix elements
\( \rho_{nn} + \rho_{pp} > 0 \),
\[\rho_{np} < \rho_{pn} \rho_{pp} . \tag{62}\]

These conditions lead to constraints on the \( \ell = 1 \)
Landau parameters (using the other constraint that the Landau
effective masses \( m_n^{\oplus} \) and \( m_p^{\oplus} \) have to be positive)
\[\mathcal{F}_1^{nn} > -3, \quad \mathcal{F}_1^{pp} > -3 \tag{63}\]

\[\left(1 + \frac{1}{3} \mathcal{F}_1^{nn}\right) \left(1 + \frac{1}{3} \mathcal{F}_1^{pp}\right) > \left(\frac{\mathcal{F}_1^{np}}{3}\right)^2 . \tag{64}\]

The stability conditions can also be expressed in terms of
effective masses \( \varepsilon_q \),
\[\frac{m_q^m}{m} > \frac{n_q}{n_b} , \tag{65}\]
or equivalently
\[\frac{m_q^m}{m} < \frac{n_b}{n_q} . \tag{66}\]

In terms of the dimensionless entrainment parameters \( \varepsilon_q \),
these conditions can be expressed as
\[\varepsilon_q < 1 - \frac{n_q}{n_b} . \tag{67}\]

It should be remarked that the previous inequalities are
very general and have to be satisfied in any two fluid
models. In the present case, these conditions also impose
a constraint on the energy functional \( \mathcal{E}(\rho_{nn}, \rho_{pp}) \)
from which the entrainment matrix is derived. Since the conditions
\( \rho_{nn} + \rho_{pp} > 0 \) and \( \rho_{np} < \rho_{pn} \rho_{pp} \) must be satisfied for any neutron and proton
densities, this leads to the following requirement
\[C_1^q \leq C_1^q . \tag{68}\]

In the particular case of Skyrme functionals, this last
condition reads
\[t_1(2 + x_1) + t_2(2 + x_2) \geq 0 . \tag{69}\]

Whenever this condition is fulfilled, it can be seen from
equations \( 48, 49, 51, 52 \) that for any neutron and proton densi-
ties, the effective masses \( m_n^q \) and \( m_p^q \) are therefore positive
and smaller than the bare nucleon mass
\[0 < m_n^q, m_p^q \leq m . \tag{70}\]

Combining the latter inequality with \( 55 \) shows in particular that
\[n_q/n_b < m_n^q/m \leq 1 . \tag{71}\]

Besides, since the proton fraction is very small inside
neutron stars, it can be seen from \( 68 \) and the definitions
\( 15, 16, 17, 18 \) that in this case the neutron
effective masses are always larger than the proton ones
\[m_n^q > m_p^p , \quad m_p^n > m_p^p . \tag{72}\]

Likewise, it can be shown that \( \varepsilon_q \geq 0 \) which, in associa-
tion with \( 67 \), yields
\[0 \leq \varepsilon_q < 1 - \frac{n_q}{n_b} . \tag{73}\]

It is thus found that the entrainment parameters are
well behaving functions of nucleon densities.

VI. APPLICATION TO NEUTRON STAR MATTER IN BETA EQUILIBRIUM

In the previous section we have obtained general for-
mlaue for the entrainment parameters and the associated
effective masses for nuclear matter with arbitrary asym-
metry. In the present section we shall apply these for-
mlaue to construct a model of neutron star core. We
shall use the SLy4 Skyrme force which has been specifi-
cally devised for astrophysical purposes \( 22, 23, 24, 25 \).

In the framework of a compressible liquid drop model
based on the SLy4 Skyrme energy functional, Douchin
& Haensel \( 21 \) found that the bottom edge of the crust
corresponds to the baryon density \( n_{\text{edge}} \simeq 0.076 \text{ fm}^{-3} \).

In the following we shall consider the density domain
\( n_{\text{edge}} < n_b < 3n_s \), where \( n_s = 0.16 \text{ fm}^{-3} \) is the nuclear
saturation density.

We assume that the liquid core is composed of an ho-
mogeneous plasma of neutrons, protons and electrons
and muons for baryon densities beyond some critical
threshold in beta equilibrium
\[n \leftrightarrow p^+ + e^- + \bar{\nu}_e, \quad \mu^- \leftrightarrow e^- + \nu_\mu + \bar{\nu}_e . \tag{74}\]

This means that the chemical potentials of the various
species are related by (assuming that neutrinos escaped
from the star)
\[\mu_n = \mu_p + \mu_e, \quad \mu_e = \mu_\mu . \tag{75}\]
In the Hartree-Fock approximation, chemical potentials of nucleons are equal to the corresponding Fermi energies \((q = n, p)\) including rest mass energy

\[
\mu_q = m_qc^2 + \frac{\hbar^2 k_q^2}{2m_q} + U_q .
\]  

(76)

Considering the leptons as ideal relativistic Fermi gases, the lepton chemical potentials are given by \((l = e, \mu)\)

\[
\mu_l = \sqrt{m_l c^2 + \frac{\hbar^2 k_l^2}{32\pi^2 n_l}} .
\]  

(77)

Charge neutrality requires that

\[
n_p = n_e + n_\mu .
\]  

(78)

\[
\mathcal{E}_0\{n_n, n_p\} = n_0 m c^2 + \left(\frac{\hbar^2}{2m} + C_0 n_b\right) \frac{3}{5} (3\pi^2)^{2/3}(n_n^{5/3} + n_p^{5/3}) + C_0 n_b^2 + C_1 (n_n - n_p)^2 + C_2 (n_n - n_p) \frac{3}{5} (3\pi^2)^{2/3}(n_n^{5/3} - n_p^{5/3}) ,
\]  

(80)

which appears in the perturbed hydrodynamical equations and which is therefore important for the study of oscillation modes. In the present case, this quantity is simply given by

\[
\mathcal{E}_\text{ent} = \frac{1}{2} \rho_n \delta \pi_n (\delta v)^2 .
\]  

(81)

where \(\delta v\) is the velocity difference between neutrons and protons. The entrainment term is negligibly small compared to the static term \(\mathcal{E}_\text{ent} \ll \mathcal{E}_0\) even for the fastest pulsars and can therefore be neglected.

The muons are present in matter when the electron chemical potential \(\mu_e\) exceeds the muon mass \(m_\mu c^2 \approx 105\) MeV. This occurs at a baryon density \(n_b \approx 0.12\) fm\(^{-3}\). In equilibrium, the composition of the liquid core is therefore completely determined by the baryon density \(n_b\) and is shown on figure 11.

The dimensionless entrainment parameters as defined by (40) and (45), which have been widely used in neutron star simulations, are represented on figures 4 and 6 respectively. The \(*\)- and \(\pi\)-effective masses are shown on figures 4 and 6 respectively. Due to the increase of the proton fraction with the baryon density (see figure 11), the differences between the two kinds of effective masses \(m_*^\alpha\) and \(m_\pi^\alpha\), which are negligible at the crust-core boundary, become significant in deeper layers. We have also plotted on figure 6 the Landau effective masses for comparison. As can be seen on those figures, the various definitions of “effective mass” do not coincide. This concept should therefore be carefully employed and the definition that has been adopted should always be clearly specified.

We have finally shown on figure 4 the dimensionless determinant of the entrainment matrix

\[
\Upsilon = \frac{\rho_{nn} \rho_{pp} - \rho_{np}^2}{\rho_{n} \rho_{p}} ,
\]  

(82)

In equations (76) and (77), we have neglected the deviations in the chemical potentials due to the existence of non vanishing currents since the relative velocities are typically very small compared to the velocities of the various constituents. For completeness, let us mention that the internal energy density of the nucleons can be decomposed in the form

\[
\mathcal{E}_\text{int} = \mathcal{E}_0 + \mathcal{E}_\text{ent}
\]  

(79)

in which \(\mathcal{E}_0\) is the functional (19) evaluated in the static ground state with the distribution function (26) including the rest mass energies.
In the present paper we considered only spin-unpolarized nuclear matter. We therefore did not discuss spin and spin-isospin instabilities that plague many Skyrme forces at supranuclear densities \([30, 31]\). However, let us notice that for the SLy4 used in our calculations, the ferromagnetic instability appears above the baryon density \(0.5 \text{ fm}^{-3}\), which is beyond the upper limit in our figures.

VII. CONCLUSION

Analytical expressions for the entrainment matrix and related effective masses of a neutron-proton superfluid mixture at zero temperature have been obtained within the non relativistic energy density functional theory. In contrast to recent investigations within relativistic mean field models \([4]\), the entrainment parameters have been found to be expressible by very simple formulae which could be easily implemented in dynamical simulations of neutron star cores. We have also clarified the link between various definitions of effective masses that have been introduced in the literature.

We have applied these formulae for Skyrme forces in
In order to evaluate the entrainment matrix in the standard model of the liquid core of neutron stars, composed of a mixture of neutrons, protons, electrons and possibly muons in beta equilibrium. Comparing the results with different Skyrme forces, we have found that the entrainment parameters are quite sensitive to the adopted parametrisation. The observations of the entrainment effects in neutron stars could therefore provide new constraints on the construction of phenomenological nucleon-nucleon interactions and shed light on the properties of strongly asymmetric nuclear matter.

**APPENDIX A: SKYRME ENERGY DENSITY FUNCTIONAL COEFFICIENTS**

The energy functional deduced from the Skyrme effective interaction in the Hartree-Fock approximation has a similar form as equation (19). The coefficients in the energy functional [10] can thus be expressed in terms of the parameters of the Skyrme interaction as follows. As a result of local phase invariance of the Skyrme forces [32], the coefficients $C^T_T$ and $C^T_T$ are related by

$$C^T_T = -C^T_T. \quad (A1)$$

In terms of the parameters of the Skyrme interaction the coefficients of the energy functional are given by [10]

$$C^0_0\{n_b\} = \frac{3}{8}t_0 + \frac{3}{48}t_3n_0^\gamma \quad (A2)$$

$$C^1_0\{n_b\} = -\frac{1}{4}t_0\left(\frac{1}{2} + x_0\right) - \frac{1}{24}t_3\left(\frac{1}{2} + x_3\right)n_0^\gamma \quad (A3)$$

$$C^0_T = \frac{3}{16}t_1 + \frac{1}{4}t_2\left(\frac{5}{4} + x_2\right) \quad (A4)$$

$$C^1_T = -\frac{1}{8}t_1\left(\frac{1}{2} + x_1\right) + \frac{1}{8}t_2\left(\frac{1}{2} + x_2\right). \quad (A5)$$

**ACKNOWLEDGMENTS**

Nicolas Chamel acknowledges financial support from the Lavoisier program of the French Ministry of Foreign Affairs. This research has been partially supported by the Polish MNiI Grant No. 1 P03D-008-27 and the PAN/CNRS LEA Astro-PF.