Fedosov Observables on Constant Curvature Manifolds and the Klein-Gordon Equation

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Abstract

In this paper we construct the set of quantum mechanical observables in the Fedosov ∗-formalism (a coordinate invariant way to do quantum mechanics on any manifold \(\mathcal{M}\)) of a single free particle that lives on a constant curvature manifold with metric signature \((p,q)\). This was done for most but not all constant curvature manifolds. We show that the algebra of all observables in \(n = p + q\) dimensions is \(SO(p+1,q+1)\) in a nonperturbative calculation. A subgroup of this group is identified as the analogue of the Poincaré group in Minkowski space i.e. it is the space of symmetries on the manifolds considered. We then write down a Klein-Gordon (KG) equation given by the equation \(\hat{p}_\mu \hat{p}^\mu |\phi\rangle = m^2 |\phi\rangle\) for the set of allowed physical states. This result is consistent with previous results on AdS. Furthermore we lay out the standard scheme for the free KG field from the single particle theory. Furthermore we argue that this scheme will work on a general space-time.

1 Introduction

The Fedosov ∗-formalism yields a quantization procedure that is invariant under all smooth transformations and reproduces ordinary quantum mechanics in the case ordinary flat space (either \(\mathbb{E}^3\) or Minkowski space). In this paper we construct the set of observables in the Fedosov ∗-formalism (a generalization of the Moyal ∗ that is coordinate invariant) for a large class of constant curvature manifolds equipped with a metric \(g\) we denote \((\mathcal{M}_C,g)\). We define \((\mathcal{M}_C,g)\) to be a constant curvature manifold imbeddable in \(n + 1\) dimensional pseudo-euclidean space \((\mathbb{R}^{n+1},\eta)\) by hyperboloids where \(n = \dim \mathcal{M}_C\) and \(\eta\) is the flat metric on \(\mathbb{R}^{n+1}\). This encompasses a huge class of constant curvature manifolds including the familiar de Sitter (dS) and Anti-de Sitter (AdS) space-times in General Relativity (GR) in 1 + 3 dimensions. It also includes the case of the sphere which was done
previously by the authors in [1]. This, in other words, is a straightforward generalization of this
previous paper.

The main results of this paper are that we show explicitly that the algebra of observables for
\((\mathcal{M}_C, g)\) is the none other than the Lie algebra \(\text{SO}(p + 1, q + 1)\) where the signature of \(g\) is \((p, q)\). The invariant \(p_{\mu}p^{\mu} = m^2\) is promoted to a constraint on the set of allowable states \(\hat{p}_{\mu} \hat{p}^{\mu} |\phi\rangle = m^2 |\phi\rangle\). The condition is called the Klein-Gordon (KG) equation and the operator \(\hat{p}_{\mu} \hat{p}^{\mu}\) is calculated to be a difference of a Casimir invariant of the Lie algebra of the symmetry group of the hyperboloid \(\text{SO}(p + 1, q)\) or \(\text{SO}(p, q + 1)\) as well as one from the full group of observables \(\text{SO}(p + 1, q + 1)\). These subgroups take the same role as the Poincaré group in Minkowski space in their respective cases (either \(C > 0\) or \(C < 0\)). This is manifestly consistent with previous analyses of free fields on dS/AdS in [3] [4] [5] [6] where explicit representations are given for spin 0 and spin 1/2 for the AdS group \(\text{SO}(2, 3)\).

Furthermore, we argue that for a general manifold \(\mathcal{M}\) the equation \(\hat{p}_{\mu} \hat{p}^{\mu} |\phi\rangle = m^2 |\phi\rangle\) will, in a completely analogous way, play the role of the Klein-Gordon equation in flat space on Lorentzian manifolds. Once the single particle theory is hammered out one only needs to define the Hilbert space of the free field as the Fok space of the complete infinite tensor product space of single particle states. Of course after this one needs to deal whether it is a fermionic or bosonic particle to know the proper commutation relations between their creation and annihilation operators.

1.1 Outline

In section 2 we will go over the basic formulas, notation and general picture of the geometry of \(\mathcal{M}_C\). It is a straightforward generalization of dS/AdS. For sections 3 through 7 we will essentially follow the same steps as in [1] which is the implementation of the Fedosov algorithm. \((\mathcal{M}, g)\) will represent a general manifold with metric (it may or may not satisfy the Einstein equation) while \(\mathcal{M}_C\) will represent the constant curvature manifolds we are considering. In section 8 we will promote the invariant \(p^{\mu}p_{\mu} = m^2\) on \((\mathcal{M}, g)\) to define the Klein-Gordon (KG).

2 The Geometry of dS/AdS

We employ the abstract index notation for this paper. Greek letters are numerical indices while latin letters are abstract ones. Furthermore we use the convention that the lower-case are the indices of \(\mathcal{M}_C\) (these run from 1, \ldots, \(n\)) and capital-ones are the indices of the phase-space \(T^*\mathcal{M}_C\) (these run from 1, \ldots, \(2n\)). The abstract indices that are not written will be form indices so that multiplication of them implies a wedging \(\wedge\) of the forms.

*We will make some exceptions to our index convention as needed, however it will always be explicitly stated exactly when and where these exceptions are used.

We start with the phase space of a single classical particle confined to a constant curvature manifold
with metric \((\mathcal{M}_C, g)\) that is imbedded in \((\mathbb{R}^{n+1}, \eta)\) where \(\dim \mathcal{M}_C = n\) and \(\eta\) is a pseudoeuclidean metric. The imbedding specifically is the hyperboloid:

\[
x^\mu x_\mu = \eta_{\mu\nu} x^\mu x^\nu = \frac{1}{C}
\]

\(\eta\) induces a metric on \(\mathcal{M}_C\) called \(g\) and explicitly \(g_{\mu\nu} = \eta_{\mu\nu} - C x_\mu x_\nu\) which is easily obtained by the constraint above. Also we will always raise and lower the lower-case indices or \(\mathcal{M}_C\) indices (greek or latin) by the metric of the imbedding space \(\mathbb{R}^{n+1} \eta\).

We make the convention that the positive signature directions are the "time" directions while the negative ones are the "space" directions. If the signature of \(g\) denoted by \(\text{sign}(g)\) is \((p, q)\) then for \(C > 0\), \(\eta\) is a pseudoeuclidean metric of signature \((p, q+1)\) or explicitly:

\[
\eta = \text{diag}(1, \ldots, 1, -1, \ldots, -1)_{p+1, q}
\]

If, however, \(C < 0\), \(\eta\) is a pseudo-euclidean metric of signature \((p, q+1)\). This is because for \(C > 0\) the hyperboloid is "time"-like i.e. it has normal vectors pointing in a combination of the \(p+1\) positive signature directions thus the induced metric has a signature of one less "time" dimensions from the imbedding. For the case of \(C < 0\) the hyperboloid is space-like and thus the induced metric has a signature of one less "space" dimensions i.e. it has normal vectors pointing in a combination of the \(q+1\) negative signature directions.

A good way to visualize these spaces is to look at the \(1 + 3\) dimensions which gives us the familiar de Sitter (dS) and Anti-de Sitter (AdS) space-times for \(C < 0\) and \(C > 0\) respectively. The picture, of course, generalizes very naturally. The embeddings in these cases are:

\[
(x^0)^2 - (x^4)^2 - \mathbf{z} \cdot \mathbf{z} = 1/C, \quad C < 0 \quad \text{(dS)}
\]

\[
(x^0)^2 + (x^4)^2 - \mathbf{z} \cdot \mathbf{z} = 1/C, \quad C > 0 \quad \text{(AdS)}
\]

where:

\[
\mathbf{z} = (x^1, x^2, x^3)
\]

We notice that in the case of dS the definition of time must be \(x^0\) and in AdS it must be the 0-4 angle \(\theta\). We immediately notice a problem in this embedding of AdS: If we follow a world line starting at \(\theta = 0\) and ending at \(\theta = 2\pi\) we arrive back at our starting point. We reason that we cannot reach the past by going far into the future. This is to avoid serious paradoxes of what must be a pathological space-time.

The resolution to this dilemma is to go to the covering space of the hyperboloid by "unidentifying" (or not identifying them in the first place) the values \(0, \pm 2\pi, \pm 4\pi, \ldots\). This is done by breaking the hyperboloid into leaves (labelled by \(n\)) and so if we follow a world-line starting at \(\theta = 0\) when we get to \(2\pi\) we will be in a different leaf of the covering space and thus not at our original point. The picture is described by first imagining that we have infinitely many hyperboloids. We then cut them length-wise, flatten them out and put each successive one above the other. Thus the topology of time is \(\mathbb{R}\) not an \(S^1\).

\[\text{[1]}\]

\(\text{[1]}\) We have figures but cannot, for our lives, figure out how to upload them.
3 The Phase-Space Connection for $T^* (dS/AdS)$

We now introduce a Levi-Civita connection $\nabla$ and subsequent curvature given the metric $g$ on a general manifold $\mathcal{M}$:

$$\nabla_\sigma f(x) = \frac{\partial f}{\partial x^\sigma}$$

$$\nabla_\sigma (dx^\mu) = \Gamma^\mu_{\nu\sigma} dx^\nu$$

$$\nabla_\sigma \left( \frac{\partial}{\partial x^\mu} \right) = -\Gamma^\nu_{\mu\sigma} \frac{\partial}{\partial x^\nu}$$

$$\nabla_\mu \nabla_\nu (dx^\rho) = R^\rho_{\nu\sigma\mu} dx^\sigma$$

where $R^\rho_{\nu\sigma\mu}$ is the Riemann tensor. We employ the notation that $\nabla = dx^\rho \nabla_\rho$ and define a basis of covectors or forms $\Theta^\Sigma \in T^* T^* \mathcal{M}$:

$$\Theta^\Sigma = (\theta^\sigma, \alpha_\sigma)$$

where the $\theta$’s are the first $n+1$ $\Theta$’s, the $\alpha$’s are the last $n+1$ $\Theta$’s and they are defined to be:

$$\theta^\sigma = dx^\sigma$$

$$\alpha_\sigma = dp_\sigma - \Gamma^\nu_{\sigma\rho} \theta^\rho p_\nu$$

The reason $\alpha_\sigma$ is defined in such a way is because $p_\sigma$ is a covector in the cotangent space to any point on the manifold $\mathcal{M}$ and thus $\alpha_\sigma(= \nabla p_\sigma)$ is a two-index tensor on it.

Following the example in our previous paper,[1] the objects needed are the phase space connection $D$ and the symplectic form $\omega$ on $T^* \mathcal{M}$. A phase-space connection’s action on all functions $f(x, p) \in T^* \mathcal{M}$ and on the $\Theta$’s are:

$$Df = df = \frac{\partial f}{\partial x^\mu} dx^\mu + \frac{\partial f}{\partial p_\mu} dp_\mu$$

$$D \otimes \Theta^\Delta = \Gamma^\Delta_{\Sigma\Psi} \otimes \Theta^\Sigma = \Gamma^\Delta_{\Sigma\Psi} \Theta^\Psi \otimes \Theta^\Sigma$$

in such a way as to preserve the symplectic form $\omega = dp_\mu \wedge dx^\mu$ on $T^* \mathcal{M}$ $(D \otimes \omega = 0)$ where $D = \Theta^\Psi D_\Psi, D_\Psi \Theta^\Delta = \Gamma^\Delta_{\Sigma\Psi} \Theta^\Sigma$ and $\Gamma^\Delta_{\Sigma\Psi}$ is the Christoffel symbol in this basis.

Additionally we impose that $D$ be torsion-free ($D^2 f = 0$) and that it corresponds to the Levi-Civita connection on $\mathcal{M}$ when it acts on functions of $x$ and $dx$. Of course we extend to vectors and higher tensors by the Leibnitz rule.

In the specific case of $\mathcal{M}_C (T^* \mathcal{M}_C)$ we employ the convention that the lower-case (upper-case) indices be of the embedding space $\mathbb{R}^{n+1} (T^* \mathbb{R}^{n+1})$ running from $1, \ldots, n+1 (1, \ldots, 2(n+1))$ instead of $1, \ldots, n (1, \ldots, 2n)$. We note before continuing that the calculation of the Fedosov observables is inherently $n$ space-time dimensional. The $(n+1)^{th}$ coordinate is merely for convenience. We see this fact manifest itself by the two conditions (e.g. $x^\mu x_\mu = 1/C$ and $x^\mu p_\mu = 0$) on the $2(n+1)$ coordinates of the phase-space every step of the way.

We specifically for the case of $T^* \mathcal{M}_C$ imbend $\mathcal{M}_C \subset (\mathbb{R}^{n+1}, \eta)$ hence the natural objects and quantities on it to be:
The induced $\mathcal{M}_C$ metric $g$ by the $(\mathbb{R}^{n+1}, \eta)$ embedding metric $\eta$.

The induced $T^*\mathcal{M}_C$ symplectic form $\omega$ by the $T^*\mathbb{R}^{n+1}$ embedding symplectic form.

Also the equations defining $T^*\mathcal{M}_C$ inside of $T^*\mathbb{R}^{n+1}$, $x^\mu x_\mu = \eta_{\mu\nu} x^\mu x^\nu = 1/C$ and $x^\mu p_\mu = 0$.

A torsion-free connection $D = \Theta^\Sigma D_\Sigma$ on $T^*\mathbb{R}^{n+1}$ that preserves all of the above conditions along with the symplectic form $\omega$ and there subsequent derivatives. In other words,

$$D^l \otimes g = D^l \otimes \omega = D^l (x^\mu x_\mu) = D^l (x^\mu p_\mu) = 0$$

for all positive integers $l$ where $g = g_{\mu\nu} dx^\mu \vee dx^\nu$, $\omega = \omega_{\Delta\Sigma} \Theta^\Delta \wedge \Theta^\Sigma$ and $\vee, \wedge$ are the symmetric, antisymmetric tensor products respectively that we will omit because it will be clear when we mean the one or the other.

The configuration space ($\mathcal{M}_C$) metric, Christoffel symbol and Riemann tensor are for our specific case are:

$$g_{\mu\nu} = \eta_{\mu\nu} - C x^\mu x_\nu$$  \hspace{1cm} (g)

$$\Gamma_{\mu\nu\sigma} = -C x^\mu g_{\nu\sigma}$$  \hspace{1cm} (G)

$$R^{\mu}_{\nu\rho} = -C \delta^\mu_{[\nu} g_{\rho]}$$  \hspace{1cm} (R)

The phase-space connection we use for $T^*\mathcal{M}_C$ is:

$$D x^\mu := \theta^\mu$$  \hspace{1cm} (D)

$$D p_\mu := dp_\mu = \alpha_\mu + \Gamma^\nu_{\mu\rho} \theta^\rho p_\nu$$

$$D \otimes \theta^\mu = \Theta^\Sigma \otimes D_\Sigma (\theta^\mu) := \theta^\sigma \otimes \Gamma^\mu_{\nu\sigma} \theta^{\nu'}$$

$$D \otimes \alpha_\mu = \Theta^\Sigma \otimes D_\Sigma \alpha_\mu := -\frac{4}{3} R^\psi_{\langle \mu\sigma \rangle\beta} \theta^\sigma \otimes \theta^3 p_\psi - \Gamma^\nu_{\mu\beta} \theta^\sigma \otimes \alpha_\nu$$

And its corresponding curvature:

$$D^2 x^\mu = 0$$  \hspace{1cm} (D^2)

$$D^2 p_\mu = 0$$

$$D^2 \otimes \theta^\mu = \Theta^\Delta \Theta^\Sigma \otimes D_{\langle \Delta D_\Sigma \rangle} \theta^\mu = \theta^\psi \theta^\sigma \otimes R^\mu_{\nu\psi\sigma} \theta^{\nu'}$$

$$D^2 \otimes \alpha_\mu = \frac{4}{3} \theta^\sigma \left( C^\psi_{\mu\beta\sigma} p_\psi \theta^{\nu'} + R^\nu_{\langle \mu\beta \rangle\sigma} \alpha_\nu \right) \otimes \theta^3 - R^\nu_{\mu\sigma\beta} \theta^\sigma \theta^3 \otimes \alpha_\nu$$

where $C^\psi_{\mu\alpha\beta} := \nabla_s R^\psi_{\langle \mu\alpha\beta \rangle s}$.

We also have the conditions:

$$\eta_{\mu\nu} x^\mu \theta^{\nu'} = 0 \quad x^\mu \alpha_\mu = 0$$

and the definitions:

$$\theta^\mu := dx^\mu$$

$$\alpha_\mu := dp_\mu - \Gamma^\nu_{\mu\rho} \theta^\rho p_\nu$$
4 Introducing the $\hat{y}$'s

Following Fedosov, we are going to introduce some machinery namely the operators $\hat{y}$'s to calculate the observables on general manifold $\mathcal{M}$. However, unlike Fedosov who defines these $\hat{y}$'s as covectors equipped with a Moyal-like product between them we choose a different starting point. We define the $\hat{y}$'s at fixed point to be a Heisenberg algebra $[\hat{y}^\Delta, \hat{y}^\Sigma] = i\hbar \omega^\Delta \omega^{\Delta \Sigma}$ where $\omega^\Delta \omega^{\Delta \Sigma}$ is the inverse of $\omega_{\Delta \Sigma}$ with $\omega_{\Delta \Sigma} \omega_{\Sigma \Psi} = \delta^\Delta_\Psi$. This can be seen more clearly by the employment of Darboux coordinates. Darboux's theorem says that in the neighborhood of each point of $q \in T^* \mathcal{M}$ there exist $2n$ local coordinates $(\tilde{s}^1, \ldots, \tilde{s}^n, \tilde{k}_1, \ldots, \tilde{k}_n)$, called canonical or Darboux coordinates, such that the symplectic form $\omega$ may be written by means of these coordinates as $\omega = \sum_{i,j} d\tilde{s}^i d\tilde{k}^j$. Thus in this coordinate system at $q$ the $\hat{y}$'s are expressed as $2n$ operators $(\tilde{s}_1, \ldots, \tilde{s}_n, \tilde{k}_1, \ldots, \tilde{k}_n)$ which have the commutators $[\tilde{s}^i, \tilde{s}^j] = [\tilde{k}^i, \tilde{k}^j] = 0$, $[\tilde{s}^i, \tilde{k}^j] = i\hbar \delta^i_j$ where $i$ and $j$ run from 1 through $2n$. And so at each point the $\hat{y}$'s establish a Heisenberg algebra which acts on a Hilbert space. More explicitly $\hat{y}$'s are huge (infinite dimensional) matrices:

$$
\hat{y}^\Delta = \begin{pmatrix}
\hat{y}^\Delta_1 (x, p) & \hat{y}^\Delta_2 (x, p) & \cdots \\
\hat{y}^\Delta_1 (x, p) & \hat{y}^\Delta_2 (x, p) & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
$$

where we remind the reader that $\Delta$ runs from 1 through $2n + 2$.

**Defining Properties of $\hat{y}$:**

$$
[y^\Delta, \hat{y}^\Sigma] = i\hbar \omega^\Delta \omega^{\Delta \Sigma}
$$

$$
D \hat{y}^\Delta = \Gamma^\Delta_\Sigma \hat{y}^\Sigma = \Gamma^\Delta_\Sigma \Theta^\Psi \hat{y}^\Sigma, \quad \Theta^\Sigma = (\theta^\sigma, \alpha_\sigma)
$$

The $\hat{y}$'s commute with the set of quantities $\{x, p, dx, dp, g, \omega, \hbar, i\}$ where $i$ is the complex unit.

*Note that the action of the phase-space connection on $\hat{y}$ is the same as the one on $\Theta$ ($D \otimes \Theta^\Delta = \Gamma^\Delta_\Sigma \Theta^\Psi \otimes \Theta^\Sigma$) and so we regard it as a basis of operator or matrix-valued covectors. This tells us how to parallel transport the Heisenberg algebra (the $\hat{y}$'s) at one point to the Heisenberg algebra of every other point in a consistent way.

**Introducing terminology:**

In this paper when we say $f$ is a function/form we define it to be a complex Taylor series in its variables. Explicitly:

$$
f (u, \ldots, v) = \sum_{l,j,s} f_{j_1 \ldots j_l} u^{j_1} \cdots v^{j_l} \quad (j$'$s$ are powers not indices)$$

where $u$ and $v$ are arbitrary.

---

2Note that these $2n$ coordinates and are different from the $2n + 2$ embedding coordinates $(x^\mu, p_\mu)$.

3The set of all of these type of functions is sometimes called the enveloping algebra of its arguments.
So if \( f \) is a function/form of some subset or all of the quantities \( x, p, dx, dp, \omega, \hbar \) and \( i \) it then commutes with the \( \hat{y} \)'s and will be called a complex-valued function/form. On the contrary an matrix-valued function/form is a complex Taylor series in \( \hat{y} \) and possibly some subset or all of the quantities \( x, p, dx, dp, \omega, \hbar \) and \( i \).

So if \( f(x, p, dx, dp, \omega, \hbar, i) \) is a complex-valued function/form it then commutes with the \( \hat{y} \)'s. More explicitly with the matrix indices written (which are exceptions to our index conventions):

\[
\left( \hat{y}^{\Delta} \hat{y}^{\Sigma} \right)_{jk} = \sum_{l} \hat{y}^{\Delta}_{jl} \hat{y}^{\Sigma}_{lk}
\]

\[
\left( [\hat{y}^{\Delta}, f] \right)_{jk} := \hat{y}^{\Delta}_{jk} f - f \hat{y}^{\Delta}_{jk} = 0
\]

On the contrary a matrix-valued function/form does not. From now on we will not write the matrix indices explicitly.

**End Goal:**

The idea for Fedosov’s introduction of the \( \hat{y} \)'s is to associate to each \( f(x, p) \in C^\infty (T^*M) \) a unique observable \( \hat{f}(x, p, \hat{y}) \):

\[
\hat{f}(x, p, \hat{y}) = \sum_{i} f_{\Delta_1 \ldots \Delta_i} \hat{y}^{\Delta_1} \ldots \hat{y}^{\Delta_i} (\hat{f})
\]

**Important Note:** Most of the rest of the sections will be dedicated to finding an \( \hat{f} \) (i.e. the coefficients \( f_{\Delta_1 \ldots \Delta_i} \)) for each \( f(x, p) \in C^\infty (T^*M) \) up to some "reasonable" ambiguity as is discussed in [1].

Specifically for \( T^*MC \) we have the induced symplectic form \( \omega \) of \( T^*\mathbb{R}^{n+1} \) onto \( T^*MC \) being:

\[
\omega = \alpha_\mu \theta^\mu = (\delta^\nu_\mu - Cx_\mu x^\nu) \alpha_\mu \theta^\nu
\]

We make the convention\(^4\):

\[
\hat{y}^{\Sigma} = (s^\sigma, k_\sigma)
\]

where the \( s \)'s are the first \( n + 1 \) \( \hat{y} \)'s and the \( k \)'s are the last \( n + 1 \) \( \hat{y} \)'s.

From the definition of \( \hat{y} \) the commutation relations:

\[
[s^\mu, s^\nu] = 0 = [k_\mu, k_\nu], [s^\mu, k_\nu] = i\hbar(\delta^\mu_\nu - Cx^\mu x_\nu)
\]

along with the conditions:

\[
\eta_{\mu\nu} x^\mu s^\nu = x^\mu k_\mu = 0
\]

We may assume w.l.o.g. that \( \eta_{\mu\nu} x^\mu s^\nu = x^\mu k_\mu = 0 \) because we observe that the only part of \( s \) and \( k \) that affect the commutators are the parts that are perpendicular to \( x \). The irrelevance of the part of \( s \) and \( k \) parallel to \( x \) stems from the above relations because \( [x_\mu s^\nu, k_\mu] = 0 \) and \( [s^\mu, k_\nu x^\nu] = 0 \)

\(^4\)Note that the indices go from 1 to \( 2n + 2 \) and are different from the \( 2n \) operators defined above by \( (\hat{s}_1, \ldots, \hat{s}_n, \hat{k}_1, \ldots, \hat{k}_n) \). The difference between them is the same as the difference between the embedding co-ordinates \( (x^1, \ldots, x^{n+1}, p_1, \ldots, p_{n+1}) \) and \( (\hat{x}^1, \ldots, \hat{x}^n, \hat{p}_1, \ldots, \hat{p}_n) \).
and so we could always subtract off the part of $s$ and $k$ parallel to $x$ and get the same commutators. Since $\eta_{\mu\nu}x^\mu s^\nu = x^\mu k_\mu = 0$ we have $n$ independent operators which is required since (one for each direction on $T^*\mathcal{M}_C$).

The action of the connection and curvature acting on $s^\mu & k_\mu$ on $T^*\mathcal{M}$ (not just $T^*\mathcal{M}_C$) are written down directly from the phase-space connection and curvature $(D)$ and $(D^2)$:

$$Ds^\mu = \Gamma^\mu_{\nu\sigma} \theta^\sigma s^\nu$$

$$D^2 s^\mu = R^\mu_{\nu\psi\theta} \theta^\psi \theta^\theta s^\nu$$

$$Dk_\mu = -\frac{4}{3} R^\psi_{(\mu\sigma)\theta} \theta^\sigma s^\nu p_\psi - \Gamma^\nu_{\mu\sigma} \theta^\sigma k_\nu$$

$$D^2 k_\mu = \frac{4}{3} \theta^\sigma \left( C^\omega_{\nu\beta\sigma} p_\nu \theta^\nu + R^\rho_{(\mu\beta)\sigma} \alpha_\sigma \right) s^\beta - R^\rho_{\mu\sigma\beta} \theta^\sigma \theta^\beta k_\nu$$

where we reiterate that $C^\omega_{ab} := \nabla_s R^\omega_{(ab)e}$.

5 Constructing the global derivation $\hat{D}$

Following Fedosov, we now introduce a global derivation as a matrix commutator $\hat{D} = [\hat{Q}, \cdot]$ which is central to constructing the coefficients $f_{\Delta_1 \cdots \Delta_l}$ in equation (5) for each $f(x, p) \in C^\infty(T^*\mathcal{M})$. One possible physical motivation for $\hat{D}$ is that in the next section we will require that all observables $\hat{f}$ must satisfy the equation $(D - \hat{D})\hat{f}(x, p, \hat{y}) = 0$ (cond $\hat{f}$). We see that on $\hat{f} \hat{D}$ is an infinitesimal translation matrix operator equivalent to $D$. We then reason that matrix operators corresponding to infinitesimal translations on the cotangent bundle should exist i.e. $\hat{D}$. The reason that we require that they must exist is because we are constructing the set of all physical matrix operators on states and certainly infinitesimal translations are in this set. If this reasoning is correct then the equation (cond $\hat{f}$) must be satisfied for all observables $\hat{f}$. Also the case of $T^*\mathbb{R}^n$ may provide some insight since it is the overlap of this formalism and quantum mechanics using the Moyal * (see in [1] for the example of $T^*\mathbb{R}^n$).

Define the derivation $\hat{D}$ by the graded commutator:

$$\hat{D} = [\hat{Q}, \cdot] = [\hat{Q}_\Delta \Theta^\Delta, \cdot] \quad (\hat{D})$$

$$\hat{Q}_\Delta = \sum \hat{Q}_{\Delta_1 \cdots \Delta_l} \hat{y}^{\Delta_1} \cdots \hat{y}^{\Delta_l}$$

5Graded commutators have the property that $[\hat{Q}_A \Theta^A, w] = [\hat{Q}_A, w] \Theta^A = (\hat{Q}_A w - w \hat{Q}_A) \Theta^A$ where $w$ is an $l$-form with coefficients $w_{A_1 \cdots A_l}$ which are complex-valued functions of the variables $x, p$ and $\hat{y}$. 8
where $\Theta^\Sigma = (\theta^\sigma, \alpha_\sigma)$ and $Q_{\Delta_1 \cdots \Delta_l}$ are complex-valued functions of $x$ and $p$ that need to be determined. We reiterate that complex-valued functions are not matrices hence they commute with the $\hat{y}$’s.

Again following Fedosov, we can partially determine the functions $Q_{\Delta_1 \cdots \Delta_l}$ by the mysterious equation:\footnote{Fedosov adds an additional condition that makes his $\hat{D}$ unique from a fixed $D$ being $\hat{d}^{-1} r_0 = 0$ where $\hat{d}^{-1}$ is what he calls $\delta^{-1}$ (an operator used in a de Rham decomposition) and $r_0$ is the first term in the recursive solution. We regard this choice as being artificial and thus omit it from the paper.}

\[
\left( D - \hat{D} \right)^2 \hat{y}^\Delta = 0 \quad \text{(cond $\hat{D}$)}
\]

The physical motivation for this equation is still unclear and may lurk in the work of Fedosov. One reason for the above requirement is that in the next section we want to solve the equation \footnote{This is the same as the condition of Fedosov $\Omega - Dr + \hat{r}^2 = 0$. See [1] and [2].} for $\hat{f}$ and the above is an integrability condition for the solvability of this equation.

We rewrite the condition $\left( D - \hat{D} \right)^2 \hat{y}^\Delta = 0$ as:

\[
\left( D - \hat{D} \right)^2 \hat{y}^\Delta = \left[ \Omega - D\hat{Q} + \hat{Q}^2, \hat{y}^\Delta \right] = 0
\]

where $\Omega$ is the phase-space curvature as a commutator.\footnote{On a technical note: we ran the Fedosov algorithm a few times to help us see what form the ansatz should take. Also remember that when we require $\Omega - Dr + \hat{r}^2 = 0$ modulo terms that commute with the $\hat{y}$’s.}

From now on we let:

\[
\Omega - D\hat{Q} + \hat{Q}^2 = 0 \quad \text{(}$\hat{Q}$\text{)}
\]

and keep it in the back of our minds that we could add something that commutes with all $\hat{y}$’s to $\Omega - D\hat{Q} + \hat{Q}^2$.

To emphasize the importance of this equation the reader should note that the whole Fedosov $\ast$ hinges on this $\hat{Q}$ existing. We know a solution exists perturbatively in general (Fedosov has the recursive solution for it in [2] on p.144), however convergence issues still remain. We have found that solving for $\hat{Q}$ to be the hardest point of the computation of the Fedosov $\ast$ because of the need for the right ansatz and the nonlinear equation \footnote{This is the same as the condition of Fedosov $\Omega - Dr + \hat{r}^2 = 0$. See [1] and [2].} above that it must solve.

Fedosov at this point would implement an algorithm to construct $\hat{Q}$ perturbatively, however rather than do this we will make an ansatz for $\hat{Q}$ by using the work done in [1] and some ingenuity. This will give us an exact solution for $\hat{Q}$.\footnote{On a technical note: we ran the Fedosov algorithm a few times to help us see what form the ansatz should take. Also remember that when we require $\Omega - Dr + \hat{r}^2 = 0$ modulo terms that commute with the $\hat{y}$’s.}

Our ansatz is:

\[
\hat{Q} = (s^\mu \alpha_\mu - k_\mu \theta^\mu) + j^\mu (x, s) \alpha_\mu + k_\nu f^\nu_\mu (x, s) \theta^\mu + p_\nu \left( \left( D + f^\sigma_\rho \theta^\rho \hat{\partial}_\sigma \right) j^\nu - \frac{2}{3} s^\rho s^\sigma R^\nu_{(\rho \sigma)} \theta^\mu \right) + h_\mu (x, s) \theta^\mu \quad \text{(}$\hat{Q}$\text{ ansatz)}
\]
along with condition:

\[ k_\nu \left( R^\nu_\rho s^\rho + \left( D + f^\sigma \theta^\rho \partial_\rho \right) f^\nu_\mu \theta^\mu \right) = 0 \]  

(\text{cond } \hat{Q})

These solve equation \([\hat{Q}]\) where \((\text{cond } \hat{Q})\) is integrable for \( f^\nu_\mu \). This ansatz has not used anything about our specific case of \( T^*M_C \) and would be valid for any cotangent bundle \( T^*M \).

Specifically for the case of \( T^*M_C \) the solution for \( \Omega \) is:

\[
\Omega = -R^\nu_\mu \theta^\beta k_\nu s^\mu + \frac{4}{3} \theta^\sigma \left( C^\psi_\mu \beta \rho \sigma p_\psi \theta^\rho + R^\nu_\mu (\beta \sigma) \rho \alpha_\nu \right) s^\beta s^\mu
\]

we verify that it gives the curvature as commutators:

\[
\frac{1}{i\hbar} [\Omega, s^\mu] = D^2 s^\mu = R^\mu_\nu \psi^\rho \theta^\sigma s^\nu
\]

\[
\frac{1}{i\hbar} [\Omega, k_\mu] = D^2 k_\mu = \frac{4}{3} \theta^\sigma \left( C^\psi_\mu \beta \rho \sigma p_\psi \theta^\rho + R^\nu_\mu (\beta \sigma) \rho \alpha_\nu \right) s^\beta - R^\nu_\mu \beta \theta^\sigma \theta^3 k_\nu
\]

The solution that was found for our example of \( T^*M_C \) using the above ansatz \([\hat{Q} \text{ ansatz}]\) and condition \((\text{cond } \hat{Q})\):

\[
\hat{Q} = s^\mu \alpha_\mu + z_\mu f^\mu + \frac{C}{3} (p_\nu s^\rho s_\rho \theta^\nu - p_\rho \theta^\rho u)
\]

\[
f^\mu = -\theta^\mu - C s^\mu (s_\rho \theta^\rho)
\]

where \( z_\mu := k_\mu + p_\mu \), \( u = \eta_\mu s^\mu s^\nu \) and \( p_\mu x^\mu = \eta_\mu s^\mu x^\nu = k_\mu x^\mu = \alpha_\mu x^\mu = \eta_\mu \theta^\mu x^\nu = 0 \).

### 6 Finding \( \hat{x} \) and \( \hat{p} \) in terms of \( x, p, \) and \( \hat{y} \)

At this point in Fedosov’s algorithm we have all the tools in place to associate an observable \( \hat{f} \) to every \( f \in C^\infty (T^*M) \). Following Fedosov we require that every observable \( \hat{f} (x, p, \hat{y}) \) must satisfy the equation:

\[
\left( D - \hat{D} \right) \hat{f} (x, p, \hat{y}) = 0
\]

where \( f_{\Delta_1 \cdots \Delta_i} \) are some unknown functions of \( x \) and \( p \) such that:

\[
\ell_0 \left( \hat{f} (x, p, \hat{y}) \right) = f (x, p)
\]

\( \ell_0 \) (short for leading order in \( \hat{y} \) and \( \hbar \)) picks out the term which has no \( \hat{y} \)'s and no \( \hbar \)'s in it. Explicitly:

\[
\hat{f} (x, p, \hat{y}) = f (x, p) + \mathcal{O} (\hat{y}, \hbar)
\]
An additional condition we impose is that there exists an ordering $\sigma$ of the $\hat{y}$'s in equation (1) so the coefficients $f_{A_1\cdots A_l} \rightarrow \tilde{f}_{A_1\cdots A_l}$:

$$\hat{f}(x, p, \hat{y}) = \sum_{l} \tilde{f}_{A_1\cdots A_l} \sigma (\hat{y}^{A_1}, \cdots, \hat{y}^{A_l})$$

such that $\tilde{f}_{A_1\cdots A_l}$ have no $\hbar$'s in them.\(^9\)

And so the condition to solve (up to some "reasonable" ambiguity) for an observable $\hat{f}$ for every $f \in C^\infty(T^*M)$ is:

$$\left(D - \hat{D}\right) \hat{f}(x, p, \hat{y}) = 0 \quad \ell \alpha \left(\hat{f}(x, p, \hat{y})\right) = f(x, p) \quad \text{(cond $\hat{f}$)}$$

If we have determined our $D$ and $\hat{D}$ we can find solutions (see [1]) for the operators $\hat{x}^\mu$ and $\hat{p}_\mu$ (i.e. their coefficients $b_{\Delta_1\cdots\Delta_l}^\mu$ and $c_{\Delta_1\cdots\Delta_l}^\mu$):

$$\hat{x}^\mu = \sum_{l} b_{\Delta_1\cdots\Delta_l}^\mu \hat{y}^{\Delta_1} \cdots \hat{y}^{\Delta_l} \quad (\hat{x})$$

$$\hat{p}_\mu = \sum_{l} c_{\Delta_1\cdots\Delta_l}^\mu \hat{y}^{\Delta_1} \cdots \hat{y}^{\Delta_l} \quad (\hat{p})$$

where $b_{\Delta_1\cdots\Delta_l}^\mu$ and $c_{\Delta_1\cdots\Delta_l}^\mu$ are complex-valued functions of $x$ and $p$ (which are the coefficients $f_{\Delta_1\cdots\Delta_l}$ in equation (1) where $f = x$ or $f = p$ respectively) and will be determined by the equations:

$$\left(D - \hat{D}\right) \hat{x}^\mu = 0 \quad \ell \alpha (\hat{x}^\mu) = x^\mu \quad \text{(cond $\hat{x}$)}$$

$$\left(D - \hat{D}\right) \hat{p}_\mu = 0 \quad \ell \alpha (\hat{p}_\mu) = p_\mu \quad \text{(cond $\hat{p}$)}$$

If we invert the equations [4] and [5] once we have solved for the coefficients $b_{\Delta_1\cdots\Delta_l}^\mu$ and $c_{\Delta_1\cdots\Delta_l}^\mu$ to get $\hat{y}$ as matrix-valued function of $x, p, \hat{x}$ and $\hat{p}$ (i.e. $\hat{y}^\Delta = \hat{y}^\Delta (x, p, \hat{x}, \hat{p})$) and then substitute it into the equation for an arbitrary observable (1) and get:

$$\hat{f}(\hat{x}, \hat{p}) = \sum_{l m} f_{\mu_1\cdots\mu_m}^{\nu_1\cdots\nu_m} \hat{x}^{\mu_1} \cdots \hat{x}^{\mu_l} \hat{p}_{\nu_1} \cdots \hat{p}_{\nu_m} \quad \text{(soln $\hat{f}$)}$$

where $f_{\mu_1\cdots\mu_l}$ are constant coefficients (just $\left(D - \hat{D}\right)$ on the equation).

However, once have our $\hat{x}$ and $\hat{p}$ there is the ambiguity of how to order each variable when you map a function $f(x, p)$ to $\hat{f}(\hat{x}, \hat{p})$. For example does the function $f(x, p) = x^1 p_1$ go to $\hat{x}^1 \hat{p}_1$, $\hat{p}_1 \hat{x}^1$ or some linear combination of the two? We should expect this in any well defined quantization

\(^9\)This last condition is not in Fedosov and is additional. It is so the correspondance of the Fedosov $*$ and Fedosov observables is 1-1 because in the projection to the $*$ everything that has an $\hbar$ is dropped. Keeping terms like this would add an artificial arbitrariness to $\hat{x}$ and $\hat{p}$. See [1].
procedure because such ordering ambiguities arise in quantum mechanics. We will, for now, regard the ordering of each \( f \) to be undetermined.\(^{10}\)

Fedosov at this point would implement an algorithm to construct \( \hat{x} \) and \( \hat{p} \) perturbatively\(^{2}\) [p.146]. We instead try to find exact solutions to them.\(^{11}\) Specifically for the case of \( T^*M_\mathcal{C} \) we make the ansatz for both \( \hat{x} \) and \( \hat{p} \):

\[
\hat{x}^\mu = f(u) x^\mu + h(u) s^\mu
\]

\[
\hat{p}_\mu = z_\nu s^\nu x_\mu g(u) + z_\mu j(u)
\]

where \( u := \eta_{\mu\nu} s^\mu s^\nu \) and \( z_\mu := k_\mu + p_\mu \).

We require that both \( \hat{x} \) and \( \hat{p} \) satisfy:

\[
\left( D - \hat{D} \right) \hat{x}^\mu = D\hat{x}^\mu - \left[ \hat{Q}, \hat{x}^\mu \right] = 0
\]

\[
\left( D - \hat{D} \right) \hat{p}_\mu = D\hat{p}_\mu - \left[ \hat{Q}, \hat{p}_\mu \right] = 0
\]

By solving the subsequent differential equations we obtain the solutions (along with adding an arbitrary term proportional to \( \hat{x} \) to \( \hat{p} \)):

\[
\hat{x}^\mu = \left( x^\mu + s^\mu \right) \frac{1}{\sqrt{Cu + 1}} \quad (\hat{x} \text{ soln})
\]

\[
\hat{p}_\mu = (-Cz_\nu s^\nu x_\mu + z_\mu) \sqrt{Cu + 1} + C (A + nih) \hat{x}_\mu \quad (\hat{p} \text{ soln})
\]

where \( u = s_\mu s^\mu \), \( z_\mu := k_\mu + p_\mu \), \( A \) is an arbitrary constant and with the computed conditions:

\[
\ell_0 (\hat{x}^\mu) = x^\mu, \quad \ell_0 (\hat{p}_\mu) = p_\mu
\]

\[
\hat{x} \cdot \hat{x} = \frac{1}{C}, \quad \hat{x} \cdot \hat{p} = \hat{p} \cdot \hat{x} - nih = A \quad (\hat{x}\hat{p} \text{ conds})
\]

Note: The reason for the additional term \( C (A + nih) \hat{x}_\mu \) in the definition of \( \hat{p} \) will be clear in the next section because the commutators will not depend on the arbitrary constant \( A \). Moreover the operator \( \hat{x} \cdot \hat{p} \) is a Casimir invariant of the full group of observables and it is clearly illustrated in this freedom to add the additional term to \( \hat{p} \).

7 The Commutators \([\hat{x}_\mu, \hat{x}_\nu], [\hat{p}_\mu, \hat{p}_\nu], [\hat{x}_\mu, \hat{p}_\nu]\)

Once we have \( \hat{x}^\mu \) and \( \hat{p}_\mu \) i.e. the coefficients \( b^\mu_{\Delta_1 \cdots \Delta_l} \) and \( c_{\mu\Delta_1 \cdots \Delta_l} \) we work out the commutation relations \([\hat{x}^\mu, \hat{x}_\nu], [\hat{x}^\mu, \hat{p}_\nu] \) and \([\hat{p}_\mu, \hat{p}_\nu] \) using the formulas (5) and (6) in the previous section. In general we will get:

\[
[\hat{x}^\mu, \hat{x}_\nu] = i\hbar f^{\mu\nu}_{\Lambda} (\hat{x}, \hat{p})
\]

\(^{10}\) Fedosov chooses Weyl ordering.

\(^{11}\) We, again, ran the Fedosov algorithm a few times to help us see what for the ansatz should take.
\[
\begin{align*}
[\hat{x}^\mu, \hat{p}_\nu] &= i\hbar \delta^\mu_\nu (\hat{x}, \hat{p}) \\
[\hat{p}_\mu, \hat{p}_\nu] &= i\hbar \delta_{\mu\nu} (\hat{x}, \hat{p})
\end{align*}
\]
where:
\[
\begin{align*}
\hat{f}^{\mu\nu}(\hat{x}, \hat{p}) &= \sum_{l_{m}} f^{\mu\nu_{1}\ldots\nu_{m}}_{\mu_{1}\ldots\mu_{l}} \hat{x}^{\mu_{1}} \ldots \hat{x}^{\mu_{l}} \hat{p}_{\nu_{1}} \ldots \hat{p}_{\nu_{m}} = f^{\mu\nu}(x, p) + O(\hat{y}, \hbar) \\
\hat{w}^{\mu}_{\nu}(\hat{x}, \hat{p}) &= \sum_{l_{m}} w^{\mu\nu_{1}\ldots\nu_{m}}_{\mu_{1}\ldots\mu_{l}} \hat{x}^{\mu_{1}} \ldots \hat{x}^{\mu_{l}} \hat{p}_{\nu_{1}} \ldots \hat{p}_{\nu_{m}} = w^{\mu\nu}(x, p) + O(\hat{y}, \hbar) \\
\hat{h}_{\mu\nu}(\hat{x}, \hat{p}) &= \sum_{l_{m}} h_{\mu\nu_{1}\ldots\nu_{m}}^{\mu_{1}\ldots\mu_{l}} \hat{x}^{\mu_{1}} \ldots \hat{x}^{\mu_{l}} \hat{p}_{\nu_{1}} \ldots \hat{p}_{\nu_{m}} = h_{\mu\nu}(x, p) + O(\hat{y}, \hbar)
\end{align*}
\]
The coefficients \( f^{\mu\nu_{1}\ldots\nu_{m}}_{\mu_{1}\ldots\mu_{l}} \), \( w^{\mu\nu_{1}\ldots\nu_{m}}_{\mu_{1}\ldots\mu_{l}} \) and \( h_{\mu\nu_{1}\ldots\nu_{m}}^{\mu_{1}\ldots\mu_{l}} \) are constants because the left-hand-side of the commutator equations is killed by \( D - \bar{D} \) and thus the algebra is closed. For quantization on cotangent bundle and knowing that the phase-space connection has no \( p \)'s or \( dp \)'s when acting on configuration space forms we know that \( \hat{x} = \hat{x}(x, s) \) so that \( [\hat{x}^\mu, \hat{x}^\nu] = 0 \).

In our case of \( T^*\mathcal{M}_C \) we find (putting the \( i\hbar \) back in):
\[
[\hat{x}^\mu, \hat{x}^\nu] = 0 \\
[\hat{x}^\mu, \hat{p}_\nu] = i\hbar (\delta^\mu_\nu - C\hat{x}^\mu \hat{x}^\nu) \\
[\hat{p}_\mu, \hat{p}_\nu] = 2i\hbar C \hat{x}^\mu \hat{p}_\nu
\]
along with the conditions:
\[
\hat{x}^\mu \hat{x}_\mu = \frac{1}{C}, \quad \hat{p}_\mu \hat{x}^\mu + n\hbar = \hat{x}^\mu \hat{p}_\mu = A
\]
We now define:
\[
\hat{M}_{\mu\nu} = \hat{x}_{[\mu} \hat{p}_{\nu]} = \hat{p}_{[\nu} \hat{x}_{\mu]} = \left( -C z_{[\rho} s^\rho x_{[\nu} + z_{[\nu} \right) (x_{[\mu]} + s_{\mu]})
\]
because we argue below that it is a more "natural":
\[
2C \hat{x}^\mu \hat{M}_{\mu\nu} = \hat{p}_\nu - \hat{x}^\mu \left( C \hat{p}^\mu \hat{p}_\mu \right)
\]
and this is the part of the momentum that is perpendicular to \( \hat{x}^\mu \) i.e. we easily compute the conditions:
\[
\hat{x}^\nu \left( \hat{x}^\mu \hat{M}_{\mu\nu} \right) = 0, \quad \left( \hat{x}^\mu \hat{M}_{\mu\nu} \right) \hat{x}^\nu = 0
\]
The leading order term is found to be:
\[
\ell_0 \left( \hat{M}_{\mu\nu} \right) := x_{[\mu} p_{\nu]} = x_{[\mu} p_{\nu]} = M_{\mu\nu}
\]
We recognize that \( \hat{M} \) and \( \hat{x} \) are the more "natural" variables than \( \hat{x} \) and \( \hat{p} \) because \( \hat{p}_\mu \hat{x}^\mu = A - n\hbar \) and \( \hat{x}^\mu \hat{p}_\mu = A \). These are very "unnatural" since there is no reason why it shouldn’t be \( \hat{p}_\mu \hat{x}^\mu = A \) and \( \hat{x}^\mu \hat{p}_\mu = A + n\hbar \). \( \hat{M} \) projects out the part of the momentum \( \hat{p} \) that is parallel to \( \hat{x} \). Therefore the part of \( \hat{p} \) parallel to \( \hat{x} \) is irrelevant and we have the definitions:
\[
\hat{x}^\mu = (x^\mu + s^\mu) \frac{1}{\sqrt{Cu + 1}} \quad (\hat{x} \hat{M} \text{ solns})
\]
\[ \hat{M}_{\mu \nu} = \hat{x}_{[\mu} \hat{p}_{\nu]} = \hat{p}_{[\mu} \hat{x}_{\nu]} = -C z_{\rho s} x_{[\mu s_{\rho]} + z_{\nu x_{\mu]} + z_{\rho s_{\nu}]} \]

We compute the commutation relations:

\[ [\hat{x}^\mu, \hat{x}^\nu] = 0 \quad (\hat{x} \hat{M}) \]

\[ [\hat{x}_\mu, \hat{M}_{\nu \rho}] = i h [x_{[\rho \eta]} x_{\mu]} \]

\[ [\hat{M}_{\mu \nu}, \hat{M}_{\rho \sigma}] = i h \left( \hat{M}_{[\mu \eta \nu]} \right) \]

subject to the conditions:

\[ \hat{x}^\mu \hat{x}_\mu = \frac{1}{C}, \quad \hat{x}^\mu \hat{M}_{\mu \nu} = \hat{M}_{\mu \nu} \hat{x}^\mu = 0 \quad (\hat{x} \hat{M} \text{ cons}) \]

We then see that the \( M \)'s generate \( SO(p + 1, q) \) in the case of \( C > 0 \) because \( \text{sign}(\eta) = (p + 1, q) \). Similarly the \( M \)'s generate \( SO(p, q + 1) \) in the case of \( C < 0 \) because \( \text{sign}(\eta) = (p, q + 1) \). We expected to see these groups in the group of observables because they are the symmetry groups for hyperboloids defined by \( x^\mu x_\mu = 1/C \).

The enveloping algebra of these operators gives the algebra of observables on \( T^* \mathcal{M}_C \): a general element being:\(^{12}\)

\[ \hat{f} \left( \hat{x}, \hat{M} \right) = \sum_{l m} f_{\mu_1 \ldots \mu_{2m}}^{\nu_1 \ldots \nu_{2m}} \hat{x}^\mu_1 \ldots \hat{x}^\mu_l \hat{M}^{\nu_1 \nu_{2m}} \ldots \hat{M}_{(2m - 1) \nu_{2m}} \]

where the coefficients \( f_{\mu_1 \ldots \mu_{2m}}^{\nu_1 \ldots \nu_{2m}} \) are constants.

It turns out that commutation relations \((\hat{x} \hat{M})\) are equivalent to:

\[ [\hat{M}_{\mu' \nu'}, \hat{M}_{\rho' \sigma'}] = i h \left( \hat{M}_{\rho'[\mu' \eta_{\nu']} \sigma'] - \hat{M}_{\sigma'[\mu' \eta_{\nu']} \rho'} \right) \quad (\hat{M}') \]

where we use the notation that the primed indices run from \( 1, \ldots, n + 2 \). Thus the \( \hat{M}' \)'s (i.e. the \( \hat{M}_{\mu' \nu' \prime} \)'s) form the Lie Algebra of \( SO(p + 1, q + 1) \), so \( (p + 1, q + 1) \) for both \( C > 0 \) and \( C < 0 \)! (See Appendix B for proof)

Of course we still have the conditions:

\[ \hat{x}^\mu \hat{x}_\mu = \frac{1}{C}, \quad \hat{x}^\mu \hat{M}_{\mu \nu} = \hat{M}_{\mu \nu} \hat{x}^\mu = 0 \]

The extra \( n + 1 \) generators of \( \hat{M} \) being:

\[ \hat{M}^{(n+2)\mu'} = -\hat{M}_{(n+2)\mu'} = \frac{1}{2 \sqrt{|C|}} \hat{p}_{\mu'} = \frac{1}{2 \sqrt{|C|}} \left( C \hat{x}^{\nu} \hat{M}_{\nu \mu'} - C \frac{i}{h} \hat{x}^{\nu} x_{\mu'} \right) \quad \text{for} \quad \mu' = 0, \ldots, n + 1 \]

\[ \hat{M}^{(n+2)(n+2)} = 0 \]

\(^{12}\)Of course there is still an ordering ambiguity (i.e. is it \( \hat{x}^1 \hat{M}_{12} \) or \( \hat{M}_{12} \hat{x}^1 \)) which we do not address here. We think it is ok because the same problem exists in ordinary quantum mechanics and should be present (in some form or another) in any generalization of it.
along with the extra components of $\eta$ being:

$$\eta_{(n+2)(n+2)} = -\frac{C}{|C|}$$

$$\eta_{(n+2)\mu'} = 0 \quad \text{for} \quad \mu' \neq n + 2$$

It is a straightforward computation to verify that the commutation relation $[\hat{M}_{\mu'\nu'}, \hat{M}_{\rho'\sigma'}]$ is the above.

The Summary of the Results:

We now have the following scheme worked out exactly:

For $\text{sign} (g) = (p, q)$ and $C > 0$:

$$\implies \text{sign} (\eta) = (p + 1, q) \quad , \quad \hat{M} \in \mathfrak{so} \ (p + 1, q)$$

$$\implies \text{sign} (\eta') = (p + 1, q + 1) \quad , \quad \hat{M}' = (\hat{M}, \hat{x}) \in \mathfrak{so} \ (p + 1, q + 1)$$

For $\text{sign} (g) = (p, q)$ and $C < 0$:

$$\implies \text{sign} (\eta) = (p, q + 1) \quad , \quad \hat{M} \in \mathfrak{so} \ (p, q + 1)$$

$$\implies \text{sign} (\eta') = (p + 1, q + 1) \quad , \quad \hat{M}' = (\hat{M}, \hat{x}) \in \mathfrak{so} \ (p + 1, q + 1)$$

Since $\hat{M}'$’s generate a well known group $SO \ (p + 1, q + 1)$ (with some well known representations [4] [5]) and they yield a consistent definition of the original $\hat{y}$’s. Inverting the formulas (8) and (9) to get $\hat{y}$:

$$s^\mu = \frac{\hat{x}^\nu x^\nu - x^\mu}{C (\hat{x}^\nu x^\nu)} \quad \text{and} \quad k_\mu = C (\hat{x}^\nu x^\nu) (\hat{p}_\mu - C (\hat{p}_\rho x^\rho) x_\mu) - p_\mu$$

where $\hat{x}$ and $\hat{p}$ are Hilbert space operators. Thus we have the consistent interpretation of $\hat{y} = (s, k)$ as an element of some Hilbert space with coefficients which are functions of $(x, p)$.

8 The Klein-Gordon (KG) Equation on dS/AdS

We observe that in Minkowski space the quantization of a single particle starts with the invariant:

$$p_\mu p^\mu - m^2 = 0$$

and promotes it to a constraint on the set of physically allowed states where $m$ is the rest mass of the particle.

The idea is to restrict the full state space to the physically relevant ones. So in Minkowski space we have the requirement that a quantum particle of definite mass must satisfy the eigenvalue equation:

$$(\hat{p}_\mu \hat{p}^\mu - m^2) |\phi\rangle = 0$$
where $[\hat{x}^{\mu}, \hat{x}^{\nu}] = 0$, $[\hat{x}^{\mu}, \hat{p}_{\nu}] = i\hbar \delta^{\mu}_{\nu}$, $[\hat{p}_{\mu}, \hat{p}_{\nu}] = 0$.

Consider first the general manifold with metric $(\mathcal{M}, g)$ that is a solution to the Einstein equation. Once the Fesosov observables have been constructed the relevant equation for us to consider is the invariant:

$$p_{\mu}p^{\mu} - m^2 = 0$$

and promote it to the constraint to be our Klein-Gordon (KG) equation for a general manifold $\mathcal{M}$:

$$(\hat{p}_{\mu}\hat{p}^{\mu} - m^2) |\phi\rangle = 0$$  \hspace{1cm} \text{(KG)}$$

once we have constructed $\hat{x}$ and $\hat{p}$.

In the rest of the section we should think of $\mathcal{M}_C$ as being dS or AdS, because these are space-times in General Relativity (GR) in $1 + 3$ dimensions however we want to construct KG on a general $\mathcal{M}_C$ because it is a straightforward generalization.

Having the KG equation we may rewrite $\hat{p}_{\mu}\hat{p}^{\mu}$ by expressing it in terms of the Casimir invariants of the group of observables generated by the $\hat{M}$’s either $SO(p, q + 1)$ or $SO(p + 1, q)$ and of the full group of $SO(p + 1, q + 1)$ (the group generated by the $x$’s and the $\hat{M}$’s). The Casimir invariants of the subgroup are:

\[ \hat{M}^2 = \hat{M}_{\mu\nu} \hat{M}^{\mu\nu} \]
\[ \hat{M}^4 = \hat{M}_{\mu_1\mu_2} \hat{M}^{\mu_2\mu_3} \hat{M}_{\mu_3\mu_4} \hat{M}^{\mu_4\mu_1} \]
\[ \vdots \]
\[ \hat{M}^N = \hat{M}_{\mu_1\mu_2} \hat{M}^{\mu_2\mu_3} \cdots \hat{M}_{\mu_{N-1}\mu_N} \hat{M}^{\mu_N\mu_1} \]

where $N$ is the integer part of $\frac{p+q+1}{2}$ i.e. the rank of the group $SO(p + 1, q)$ or $SO(p, q + 1)$.

The Casimir invariants of the full group $SO(p + 1, q + 1)$ are

\[ \hat{M}^2 = \hat{M}_{\mu'\nu'} \hat{M}^{\mu'\nu'} \]
\[ \hat{M}^4 = \hat{M}_{\mu_1'\mu_2'} \hat{M}^{\mu_2'\mu_3'} \hat{M}_{\mu_3'\mu_4'} \hat{M}^{\mu_4'\mu_1'} \]
\[ \vdots \]
\[ \hat{M}^{N'} = \hat{M}_{\mu_1'\mu_2'} \hat{M}^{\mu_2'\mu_3'} \cdots \hat{M}_{\mu_{N'-1}'\mu_{N'}} \hat{M}^{\mu_{N'}\mu_1'} \]

where $N'$ is the integer part of $\frac{p+q+2}{2}$ i.e. the rank of the group $SO(p + 1, q + 1)$.

Using the equation $\hat{M}_{\mu\nu} = \hat{x}_{[\mu} \hat{p}_{\nu]}$ we compute directly:

$$\hat{M}^2 = -\frac{1}{2} (\hat{x} \cdot \hat{p} - i\hbar n) (\hat{x} \cdot \hat{p})$$

$$\hat{M}^2 = \frac{1}{2C} \hat{p}_{\mu} \hat{p}^{\mu} + \hat{M}^2$$

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where $[\hat{x}^\mu, \hat{p}_\mu] = i\hbar (\delta^\mu_\mu - 1) = n = p + q$ is the dimension of $\mathcal{M}$.

KG equation takes the form of:

$$
\left(2C \left( \hat{M}^2 - \hat{M}'^2 \right) - m^2 \right) |\phi\rangle = 0 \quad (\mathcal{M}_C \text{ KG})
$$

$$
\hat{M}'^2 = -\frac{1}{2} (\hat{x} \cdot \hat{p} - i\hbar n) (\hat{x} \cdot \hat{p})
$$

$$
\hat{x}_\mu \hat{x}^\mu = \frac{1}{C}, \quad \hat{x} \cdot \hat{p} = \text{Casimir}
$$

The case of $\text{sign}(g) = (1, 3)$ when $C > 0$ and $C < 0$ $\mathcal{M}_C$ are the well known de Sitter (dS) and Anti de Sitter (AdS) space-times in General Relativity respectively.

In the case of spin 0 particles the operator $\hat{M}^2$ becomes the Laplace-Beltrami operator $\Box = (-g)^{1/2} \partial_\mu g^{\mu\nu} (-g)^{-1/2} \partial_\nu$ and $\hat{x}^\mu \hat{p}_\mu \rightarrow -i\hbar x^\mu \partial_\mu$ so let $\phi(x) := \langle x|\phi \rangle$ then:

$$
(2C\Box - C (k - n) k - m^2) \phi(x) = 0
$$

where $x^\mu \partial_\mu \phi = k\phi$. This equation is the free wave equation on AdS that is studied in [4] and therefore the results given here are consistent with what has been done previously.

### 9 A Basic Scheme For a Free Quantum KG Field From the Single Particle Theory

For this section we assume that the representation of the group of all observables is unitary. Also we need our Latin letters for indices that do not represent space-time or phase-space indices so we eliminate our index conventions. In other words all indices here represent numerical values which are neither space-time or phase-space ones.

For a single particle state $|q\rangle \in \mathcal{H}$ that satisfies the equation (KG) we have the following notation:

$$
|q\rangle = |q_1, \ldots, q_N\rangle
$$

In other words $\{q_1, \ldots, q_N\}$ is the set of $N$ (unrelated to the old $N$ in the previous section) state labels of a single particle i.e. quantum numbers. For the case of $\mathcal{M}_C = \text{AdS}$ the $\hat{M}'$s generate the group $SO(2, 3)$ ($\hat{M}$'s generate $SO(2, 4)$) and states can be labelled by the eigenvalues of operators $M_{04}, \hat{M}_{12}, J^2 = \left(\hat{M}_{23}\right)^2 + \left(\hat{M}_{13}\right)^2 + \left(\hat{M}_{12}\right)^2$ which all mutually commute. The labels $\{q_1, \ldots, q_N\}$ represent real physical quantities such as momentum, spin, species, etc. For the example of AdS $M_{04} = E$ is the energy, $\hat{M}_{12} = J_z$ is the z-component of angular momentum and $J^2$ is the angular momentum vector squared $J_x^2 + J_y^2 + J_z^2$. Spin 0 and spin 1/2 representations have been studied in [3] [6].

We also require that they form an orthonormal basis for the Hilbert space:

$$
\langle q|q'\rangle = \delta_{qq'} := \delta_{q_1 q'_1} \cdots \delta_{q_N q'_N}
$$
For multiparticle states we define the (either fermionic or bosonic) creation/annihilation operators $a_q^\dagger / a_q$ of the state $|q\rangle$. We let the Hilbert space of multiparticle states $\mathcal{H}$ be the complete infinite tensor product space of single particle states as one does:

$$\mathcal{H} = \prod_i \otimes \mathcal{H}_i$$

Of course we know that this space is not separable therefore we use the Fok space resolution by always working in the subspace $\mathcal{H}_0$ of $\mathcal{H}$ of only a finite number of particles.\[9]\]

10 Conclusions

We have thus explicitly constructed an exact solution to the commutators of the Fedosov observables on $T^*\mathcal{M}_C$ and showed that it was the Lie algebra of $SO(p+1, q+1)$. In other words we took the phase space of a single classical particle confined to $\mathcal{M}_C$, quantized it and got the set of all observables to be $SO(p+1, q+1)$. $SO(p+1, q)$ for $C > 0$ and $SO(p, q+1)$ for $C < 0$ is completely analogous the Poincaré group in Minkowski space in the sense that particles are vectors in the representation spaces upon which this group acts. We also derived an ansatz which maybe the form of $\hat{D}$ for a general manifold $\mathcal{M}$.

By promoting the classical invariant $p_\mu p^\mu - m^2$ of the classical particle to an operator constraint we obtained a generalization of the Klein-Gordon equation (KG) as a condition on the set of physically allowed states given arbitrary unitary representation of the group observables on $\mathcal{M}$. The subsequent equation in the case of $\mathcal{M}_C$ turned out to involve the Casimir invariants of the relevant subgroups corresponding to $SO(p+1, q)$ for $C > 0$ and $SO(p, q+1)$ for $C < 0$ as well as the full group $SO(p+1, q+1)$. Hence by simple tensoring (i.e. the simple pasting) of single particle states we obtained, in a straightforward way, multiparticle states. To get the seperable state space we use the usual Fok resolution. These then are the essential building blocks of a free KG quantum field theory.

11 Acknowledgements

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\[13\] This is done by requiring that the particles are indistinguishable. See \[12\].
Our starting ansatz for $\hat{Q}$ is:

$$\hat{Q} = (k_\nu f_\mu (x, s) + p_\nu g_\mu (x, s) + h_\mu (x, s)) \theta^\mu + j^\mu (x, s) \alpha_\mu$$

where the tensors $f, g, h$ and $j$ are some functions of $x$ and $s$ that need to be determined. We believe this would be the right ansatz, in general, for a cotangent bundle. The reason is that the phase-space connection and curvature are only linear in $p$ and $\alpha$ so it seems reasonable to have an ansatz linear in $k$ and $p$ for $\hat{D}$.

A useful identity in our calculation is:

$$Df (x, s) = (\partial_\mu f) dx^\mu + (\hat{\partial}_\mu f) s^\nu \Gamma^\mu_\rho^\nu \theta^\rho$$

where $\partial_\mu := \frac{\partial}{\partial x^\mu}$, $\hat{\partial}_\mu := \frac{\partial}{\partial s^\mu}$ and $f (x, s)$ is an arbitrary Taylor series in $x$'s and $s$'s.

Equation (Q ansatz1) is satisfied if the following two equations hold:

$$g_\nu^\mu (x, s) \theta^\mu = \left( (D + f^\sigma \theta^\sigma \hat{\partial}_\sigma) j^\nu - \frac{2}{3} s^\sigma s^\rho R^\nu_{\rho\sigma\mu} \theta^\rho \right)$$

and

$$k_\nu \left( R^\nu_\rho s^\rho + \left( D + f^\sigma \theta^\sigma \hat{\partial}_\sigma \right) f^\nu_\mu \theta^\mu \right) = 0$$

$$R^\mu_\nu := R^\mu_{\nu\sigma\rho} \theta^\sigma \theta^\rho$$

where $h_\mu$ and $j_\mu$ are arbitrary so far. It seems that the Fedosov prescription requires that we now impose the linear terms in $\hat{y}$ in the Taylor expansions of $h_\mu$ and $j_\mu$ to be $-k_\mu$ and $s^\mu$ respectively. The reason is that first we notice that:

$$\omega_{\Delta x} \hat{y}^\Delta \Theta^\Sigma = s^\mu \alpha_\mu - k_\mu \theta^\mu$$

and second we want $\hat{Q}$ to be a global object and not tied to any coordinate patch we must make it out of tensors like $\omega$. In general the only other term we have that could give linear terms in $k$ besides the symplectic form are other invariant tensor contractions thus we are very limited to possible substitutes (e.g. terms like $k^\mu \alpha_\mu$ and $s_\mu \theta^\mu$). Another argument in favor of this is that this is precisely the flat space solution and we are expanding about a known solution. However we think that other linear terms would lead to either gauge equivalent solutions or to trivial solutions to the * product if they lead to one at all. An argument in favor of this is Fedosov’s iterative construction of $\hat{Q}$. Fedosov’s algorithm shows us that terms of a certain order or powers in $\hat{y}$ to certain degree depend on the previous orders or powers in $\hat{y}$. But this argument is, of course, moot and we choose to leave this for further discussion.
We thus now send \( j, h \rightarrow j^\mu + s^\mu, h - k^\mu \) and obtain a more refined ansatz:

\[
\hat{Q} = (s^\mu \alpha_\mu - k^\mu \theta^\mu) + j^\mu (x, s) \alpha_\mu + \left( k^\mu f^\nu_{\mu} (x, s) + p^\nu \left( \left( D + f^\rho_{\nu} \theta^\rho \hat{\partial}_\sigma \right) j^\nu - \frac{2}{3} s^\rho s^\sigma R^\nu_{(\rho \sigma)\mu} \theta^\mu \right) + h^\mu (x, s) \right) \theta^\mu
\]

along with condition:

\[
k^\nu \left( R^\rho_{\nu} s^\rho + \left( D + f^\rho_{\nu} \theta^\rho \hat{\partial}_\sigma \right) f^\nu_{\mu} \theta^\mu \right) = 0 \quad (\text{cond } \hat{Q})
\]

### 13 Appendix B: \( SO(p + 1, q + 1) \)

By a little reorganization we will see that \( x \)'s and the \( p \)'s generate the group \( SO(p + 1, q + 1) \).

Proof:

Of course we have the subgroup of \( SO(p + 1, q) \) generated by the \( M \)'s which gives us the bulk of \( SO(p + 1, q + 1) \).

So we need an extra \( n \) generators.

We notice that:

\[
2 \hat{x}^\nu \hat{M}_{\nu \mu} = 2 \hat{x}^\nu \hat{x}_{[\nu} \hat{p}_{\mu]} = 2 \left( \frac{1}{C} \hat{p}_\mu - (\hat{x} \cdot \hat{p}) \hat{x}_\mu \right)
\]

\[
\implies \hat{p}_\mu = C \hat{x}^\nu \hat{M}_{\nu \mu} - C (\hat{x} \cdot \hat{p}) \hat{x}_\mu
\]

We think of \( p_\mu \) as a function of \( x \) and \( M \) and the Casimir invariant \( \hat{x} \cdot \hat{p} \).

Let's make the convention that the primed indices run from 0, ..., \( n + 2 \) so we define the extra \( n \) generators for \( SO(p + 1, q + 1) \) by:

\[
\hat{M}_{(n+2)\mu'} = -\hat{M}_{\mu'(n+2)} = \frac{1}{2 \sqrt{|C|}} \hat{p}_\mu = \frac{1}{2 \sqrt{|C|}} \left( C \hat{x}^\nu \hat{M}_{\nu \mu'} - C \frac{1}{i \hbar} \frac{\hat{x} \cdot \hat{p}}{\hat{x}_\mu} \right) \quad \text{for } \mu' = 0, \ldots, n + 1
\]

\[
\hat{M}_{(n+2)(n+2)} = 0
\]

and the extra components of \( \eta \):

\[
\eta_{(n+2)(n+2)} = -C/|C|
\]

\[
\eta_{(n+2)\mu'} = 0 \quad \text{for } \mu' \neq n + 2
\]

We then compute the commutation relation:

\[
\left[ \hat{M}_{\mu' \rho' \sigma'}, \hat{M}_{\rho' \sigma'} \right]_\ast = i \hbar \left( \hat{M}_{\rho' \sigma'} [\mu', \eta_{\nu' \sigma'}] - \hat{M}_{\rho' \sigma'} [\mu', \eta_{\nu' \rho'}] \right)
\]

Which is equivalent to the commutation relations:

\[
[\hat{x}^\mu, \hat{x}^\nu] = 0
\]
\[
\left[ \hat{x}_\mu, \hat{M}_{\nu} \right] = i \hbar \hat{x}_{[\nu} \hat{g}_{\rho]} \mu
\]
\[
\left[ \hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma} \right] = i \hbar \left( \hat{M}_{\rho[\mu} \eta_{\nu]\sigma} - \hat{M}_{\sigma[\nu} \eta_{\rho]\mu} \right)
\]
since we may invert the relations by the formula for \( \hat{x} \):
\[
\hat{x}_\mu = \left( 2 \hat{M}_{\nu\rho} \hat{p}^\rho + ((\hat{x} \cdot \hat{p}) + i \hbar) \hat{p}_\mu \right) \frac{1}{(\hat{p}^2 + i \hbar C (\hat{x} \cdot \hat{p}))}
\]
\[
\hat{M}_{(n+2)\mu'} = -\hat{M}_{\mu'(n+2)} = \frac{1}{2 \sqrt{|C|}} \hat{p}_{\mu'} = \frac{1}{2 \sqrt{|C|}} \left( C \hat{x}^\mu \hat{M}_{\mu'} - \frac{C}{i \hbar} \hat{x} \cdot \hat{p} \right) \text{ for } \mu' = 0, \ldots, n + 1
\]

14 References


