Nonextensive statistical features of the Polish stock market fluctuations

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Abstract

The statistics of return distributions on various time scales constitutes one of the most informative characteristics of the financial dynamics. Here we present a systematic study of such characteristics for the Polish stock market index WIG20 over the period 04.01.1999–31.10.2005 for the time lags ranging from one minute up to one hour. This market is commonly classified as emerging. Still on the shortest time scales studied we find that the tails of the return distributions are consistent with the inverse cubic power-law, as identified previously for majority of the mature markets. Within the time scales studied a quick and considerable departure from this law towards a Gaussian can however be traced. Interestingly, all the forms of the distributions observed can be comprised by the single $q$-Gaussians which provide a satisfactory and at the same time compact representation of the distribution of return fluctuations over all magnitudes of their variation. The corresponding nonextensivity parameter $q$ is found to systematically decrease when increasing the time scales. The temporal correlations quantified here in terms of multifractality provide further arguments in favour of nonextensivity.

Key words: Financial markets, $q$-Gaussian distributions, Tsallis statistics
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1 Introduction

Making the quantification of financial fluctuations is a real interdisciplinary challenge. The related well identified stylised fact is the so-called inverse cubic power-law [1] which applies to developed stock markets [2,3,4,5], to the commodity market [6], as well as to the most traded currency exchange rates [7]. The emerging stock markets are commonly considered to be governed by a somewhat different dynamics which often [8,9] results in exponential tails of...
the return distributions. Of course, both the above types of distributions are Lévy unstable and thus for the sufficiently long time lags they may converge towards a Gaussian. The distribution with an exponential tail might correspond to an intermediate stage between a distribution with the power-law asymptotics and a very large time lag limit - a Gaussian [10]. Such a scenario corresponds for instance to the Heston model [11].

In order to elaborate more on this sort of issues we systematically study the character of fluctuations of the Polish stock market as represented by the WIG20 index. This equity market started trading on April 16, 1991 and the presently most often quoted corresponding index is WIG20 (Warszawski Index Giełdowy - Warsaw Stock Market Index), introduced in 1994, comprising capitalization weighted prices of the 20 largest companies. The high quality electronic processing and recording of all the transactions started in the beginning of 1999. The analysis presented here thus covers the time period since January 4, 1999 until October 31, 2005.

The daily trading closing hour during this period was 4:00pm and since 17.11.2000 4:10pm. The opening hours have been changed two times. On 4.1.1999 until 30.7.1999 (Period1) it was 1pm, then until 16.11.2000 (Period2) it was 12am, and then 10am (Period3) with the closing at 4:10pm.

As it can be seen from Fig. 1 in the whole time period inspected here, even though representing an emerging market, the WIG20 has been following the overall world trend - more in terms of the phase than in the amplitude how-
ever. During the first two years of the period considered its behaviour closely resembles the Nasdaq. Starting in 2003 it however by far overperforms the two world major indices: the Nasdaq and the S&P500. As natural, the original WIG20 is quoted in the Polish Zloty (PLN). Converting systematically the PLN into the US$ - to make this comparison even more informative - results in an even larger gain as can be easily seen from Fig. 1. This is due to a parallel sizable PLN appreciation in the period considered.

2 Conventional log-log analysis

For the time series $W(t)$ representing the index value at time $t$ we use the commonly accepted definition of returns as

$$ R \equiv R(t, \Delta t) = \ln W(t + \Delta t) - \ln W(t). $$

(1)

As another standard procedure, we calculate the normalized returns $r \equiv r(t, \Delta t)$ defined as

$$ r = \frac{R - \langle R \rangle_T}{v}, $$

(2)

where $v \equiv v(\Delta t)$ is the standard deviation of returns over the period $T$

$$ v^2 = \langle R^2 \rangle_T - \langle R \rangle_T^2 $$

(3)

and $\langle \ldots \rangle_T$ denotes a time average.

The cumulative distribution functional (cdf) of $\Delta t = 1\text{min}$ moduli of WIG20 returns collected from the whole period 1999 - 2005 specified above is shown in panel (a) of Fig. 2. Interestingly, this distribution displays very similar behaviour as for many mature markets analysed before. Even the tails of this distribution reveal scaling

$$ P(r > x) \sim x^{-\alpha}, $$

(4)

consistent in addition with the inverse cubic power law ($\alpha = 3$). The remaining (b), (c) and (d) panels show cdf’s separately for the Period1, Period2 and Period3. As one can see, even though these periods correspond to different phases - from less to more advanced - of the Polish Stock Market development, the return distribution characteristics remain essentially invariant. This indicates that the fluctuation characteristics of an emerging market do not
Fig. 2. Cumulative distributions of moduli of the WIG20 normalised returns ($\Delta t = 1$ min, (○)) for four periods: a) the whole period 4 Jan 1999 – 31 Oct 2005; b) Period1 from 4 Jan 1999 to 30 Jul 1999; c) Period2 from 2 Aug 1999 to 16 Nov 2000; d) Period3 from 17 Nov 2000 to 31 Oct 2005. Dashed line corresponds to a Gaussian distribution.

have to differ from those of a mature one. A distribution whose tails follow the inverse cubic power law has a finite second moment and is thus Lévy unstable. In the present context this means that the return distributions for the sufficiently large time lags $\Delta t$ are expected to eventually start converging towards a Gaussian due to effects in the spirit of the Central Limit Theorem (CLT). Of course, fluctuations typically carry some higher order time correlations - quantifiable for instance in terms of multifractals [12,13] - therefore this convergence may be much slower than for uncorrelated stochastic processes to which the conventional CLT refers. For the stock market fluctuations such effects are however identified in the literature [3,4] for the time scales of the order of a few days or even a few hours for the more recent data [5]. How the corresponding situation develops for our WIG20 data when the time lag $\Delta t$ increases is shown in Fig. 3.

Departure from the inverse cubic scaling can already be seen for time lags
Fig. 3. Cumulative distributions of moduli of the WIG20 normalized returns (symbols) from 4 Jan 1999 to 31 Oct 2005 for several time scales: 1 min (△), 2 min (○), 4 min (▽), 8 min (□), 16 min (●), 32 min (□), and 60 min (■). Dashed line corresponds to a Gaussian distribution.

larger than 10 minutes and for Δt = 60 min the scaling regime is hardly visible with the tail of the distribution being closer to a Gaussian.

Whether this much faster convergence with Δt towards a normal distribution is characteristic to WIG20 fluctuations or it just reflects the fact that one deals here with even more recent data than in ref. [5] remains an open question. Based on empirical arguments collected from the world leading stock markets the hypothesis put forward in that reference says that when going from past to present the same Δt measured by a conventional clock time effectively corresponds to increasing time lags of an internal market time and this originates from an increasing speed of the information processing.

3 Nonextensive statistical approach

The fat tails in the financial return distributions and the complex character of the underlying temporal correlations [12,13,14,15] indicate that the con-
ventional concept of ergodicity may break down in the financial dynamics. Under such circumstances the generalised formalism of nonextensive statistical mechanics may offer an appropriate framework to quantify the corresponding statistics. At present the most consistent seems the one based on the generalised entropy which for a set of $N$ events $\{x_i\}$ characterised by the probabilities $\{p_i\}$ reads

$$S_q = - \sum_{i=1}^{N} p_i^q \ln_q p_i,$$

as postulated by Tsallis [16]. Here $\ln_q x$ denotes the $q$-logarithm function

$$\ln_q x = (x^{1-q} - 1)/(1 - q).$$

The parameter $q$ in Eq. 5 is the so called nonextensivity parameter. For $q = 1$ this equation expresses the standard Boltzmann-Gibbs entropy.

The optimization of this generalized entropic form under appropriate constraints [16,17], in the continuous form, yields the following $q$-Gaussian form for the distribution of probabilities

$$p(x) = N_q e^{-B_q (x - \bar{\mu}_q)^2},$$

where

$$N_q = \begin{cases} \frac{\Gamma\left(\frac{3-q}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \sqrt{\frac{1-q}{\pi}} B_q & \text{for } q < 1 \\ \frac{\Gamma\left(\frac{1}{3-q}\right)}{\Gamma\left(\frac{1}{2}\right)} \sqrt{\frac{x}{(q-1)B_q}} & \text{for } 1 < q < 3 \end{cases},$$

$$\bar{\mu}_q = \int x \frac{[p(x)]^q}{\int [p(x)]^q dx} dx \equiv \langle x \rangle_q,$$

$$B_q = \left((3 - 1) \sigma_q^2\right)^{-1},$$

and $e^x_q$ denotes the $q$-exponential function defined as

$$e^x_q = [1 + (1 - q) x]^{\frac{1}{1-q}}.$$

Another argument which makes this distribution attractive from the present perspective is that for $q > 1$ it asymptotically ($x >> 1$) develops a power law form $p(x) \sim x^{-\frac{1}{1-q}}$. In particular, for $q = 3/2$, on the level of the cumulative distribution, it recovers the inverse cubic power law. This is an especially nice aspect of the functional form expressed by the Eq. 6 because it at the same time provides a compact form for the probability distribution for any value of $x$. Indeed, the first attempts [18] of the applicability of this form
to describe the probability distributions of the financial fluctuations turn out quite promising. For all these reasons in the following we explore a possibility to describe the empirical WIG20 return distributions presented in Fig. 3 by a family of the $q$-Gaussians. In order however to attain a better stability of this analysis, instead of directly using the Eq. 6 we convert it to the cumulative form by defining

$$P_{\pm}(x) = \mp \int_{x_{\pm\infty}}^{x} p(x') dx'$$

(8)

where the + and - signs correspond to the right and left wings of the distribution correspondingly. By using here the Eq. 6 one obtains

$$P_{\pm}(x) = N_q \left( \frac{\sqrt{\pi} \Gamma \left( \frac{1}{2}(3-q) \beta \right)}{2 \Gamma(\beta) \sqrt{\beta}} \pm (x - \bar{\mu}_q)_{2F1}(\alpha, \beta; \gamma; \delta) \right),$$

(9)

where, 
$$\alpha = \frac{1}{2}, \beta = \frac{1}{q-1}, \gamma = \frac{3}{2}, \delta = -B_q(q-1) (\bar{\mu}_q - x)^2$$ and $2F1(\alpha, \beta; \gamma; \delta)$ is the Gauss hypergeometric function defined by the following power series expansion:

$$2F1(\alpha, \beta; \gamma; \delta) = 1 + \frac{\alpha \beta}{1! \gamma} \delta + \frac{\alpha(\alpha+1) \beta(\beta+1)}{2! \gamma(\gamma+1)} \delta^2 + \ldots = \sum_{k=0}^{\infty} \frac{\delta^k}{k! (\gamma)_k} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k}$$

Fig. 4 shows the cumulative variant of the WIG20 data points for the same sequence of the time lags as in Fig. 3 and the corresponding best theoretical fits in terms of Eq. 9. The result appears very encouraging. For a given $\Delta t$ one obtains a good theoretical representation for the empirical probability distribution over the whole interval of changes of the returns. The only inaccuracy is at small $\Delta t$ for a few positive and even more negative extreme events whose probability is somewhat lower than what the overall global fit provides. Nevertheless, the obtained $q$-values for the smallest $\Delta t$ are close to $3/2$, as consistent with the inverse cubic power law. With increasing $\Delta t$ the best fit $q$-values systematically decrease and the corresponding $q$-Gaussians provide amazingly reasonable representation for the empirical data on all the time scales considered.
We start with calculation of the singularity spectra for the data. The latter of these stylized facts was already discussed in preceding sections. For the Polish stock market data, these higher-order correlations can lead to some well-known stylized facts such as nonlinear correlations observed in empirical signals. In financial markets, both space and time correlations, are the most characteristic features of the complex and nonextensive systems. Such interactions can manifest themselves as nonlinear correlations observed in empirical signals. In financial data, these higher-order correlations can lead to some well-known stylized facts like, e.g., the multifractality, the persistent memory in volatility, the leverage effect and the fat tails of the distributions of returns. For the Polish stock market data, the latter of these stylized facts was already discussed in preceding sections and here we present results for the former ones.

We start with calculation of the singularity spectra $f(\alpha)$ that allows us to
Fig. 5. Singularity spectra $f(\alpha)$ for original (solid line) and randomized (dashed line) time series of 1 minute WIG20 returns. The spectrum for randomized data was averaged over 100 independent generations.

quantify the fractal properties of the data. We use the well-known method of multifractal detrended fluctuation analysis (MFDFA) which, according to our experience, gives the most reliable outcomes [22]. Technically, MF-DFA can be briefly sketched as follows [23]. For the time series $G$ of the returns $g(i), i = 1, \ldots, N$ one calculates the signal profile

$$Y(i) = \sum_{k=1}^{i} (g(k) - \langle g \rangle), \ i = 1, \ldots, N$$ (10)

where $\langle \ldots \rangle$ denotes the time-average of $G$. Now $Y(i)$ is divided into $M$ disjoint segments of length $n$ starting from the beginning of $G$ and $M$ equivalent segments starting from the end of $G$. For each segment $\nu, \nu = 1, \ldots, 2M$, the local trend is to be calculated by least-squares fitting the polynomial $P_{\nu}(l)$ of order $l$ to the data, and then the variance

$$F^2(\nu, n) = \frac{1}{n} \sum_{j=1}^{n} \{Y[(\nu - 1)n + j] - P_{\nu}(l)(j)\}^2.$$ (11)

For the financial data the polynomial order as low as $l = 2$ can be used. The variances (11) have to be averaged over all the segments $\nu$ and finally one gets
the $q$th order fluctuation function

$$F_q(n) = \left\{ \frac{1}{2M_s} \sum_{\nu=1}^{2M_s} [F^2(\nu, n)]^{q/2} \right\}^{1/q}, \quad q \in \mathbb{R}. \quad (12)$$

The function $F_q(n)$ must be calculated for many different segments of lengths $n$. If the signal is fractal, the fluctuation function reveals power-law scaling

$$F_q(n) \sim n^{h(q)} \quad (13)$$

for large $n$. The family of the scaling exponents $h(q)$ (the generalized Hurst exponents) can be then obtained by observing the slope of log-log plots of $F_q$ vs. $n$. If $h(q) = \text{const}$ then the signal under study is monofractal; it is multifractal otherwise. From the spectrum of the generalized Hurst exponents, one can calculate the singularity strength $\alpha$ and the singularity spectrum $f(\alpha)$ using the following relations (e.g. [23]):

$$\alpha = h(q) + q h'(q) \quad \text{and} \quad f(\alpha) = q[\alpha - h(q)] + 1, \quad (14)$$

where $h'(q)$ denotes the derivative of $h(q)$ with respect to $q$.

Fig. 5 shows the singularity spectrum for the high-frequency WIG20 returns with $\Delta t = 1 \text{ min}$ (solid line). On this time scale the number of data points is sufficiently large to allow a reliable study of this kind [22]. From the shape of $f(\alpha)$ curve it is evident that the signal under study has multifractal properties. This statement can receive an additional support from the shape of the spectrum for randomized data. The average spectrum for 100 independent realizations of the randomized data is also displayed in Fig. 5 (dashed line). This spectrum differs substantially from the original one and is much closer to monofractal than the multifractal spectrum of the original series. As expected, the randomized data reveal no linear correlations and the spectrum is located at 0.5 which is in contrast with the spectrum for the original data showing a visible trace of such correlations. This example shows again [13] that these are the time correlations that constitute the main source of multifractality in the stock market dynamics.

The $f(\alpha)$ spectrum for our data indicates the existence of the correlations of different types; two basic ones are shown in Fig. 6. The volatility autocorrelation (Fig.6(a)) looks typically with its long temporal decay resembling its counterparts for the developed markets like, e.g., the American and the German ones [24,5]. Also the negative crosscorrelation between the returns and the volatility (the leverage effect, Fig. 6(b)) looks similar to other data [26,25].

On the other hand, standard deviation of the returns (i.e., the time-averaged...
Fig. 6. (a) Volatility autocorrelation calculated for 1 min WIG20 returns after removing the daily pattern and (b) crosscorrelation between 1 minute returns and volatility (b). Zero level in (b) is denoted by dashed line.

volatility) presented in Fig. 7 as a function of $\Delta t$ shows a somehow distinct behaviour than that observed for other markets [3,5]. Actually, for the first half of the considered time interval we observe three different scaling regions (see Fig. 7(a)): subdiffusive (scaling exponent $\delta = 0.45$) for the smallest time scales ($\Delta t \leq 3$ min), superdiffusive ($\delta = 0.60$) for medium time scales ($4 \leq \Delta t \leq 56$ min), and the region of approximately normal diffusion ($\delta = 0.51$) for the longest time scales ($\Delta t \geq 64$ min). For the second half of the considered time interval, corresponding to newer data, this scheme noticeably changes (Fig. 7(b)), mainly due to the broadening of the normal diffusion region which now covers $\Delta t > 10$ min, and the shortening of the now hardly identifiable weak superdiffusive region ($3 < \Delta t < 10$ min) with $\delta = 0.52$. This effect resembles the one observed for the American market [3,5] where the crossover
point between the superdiffusion regime and the normal diffusion regime is
shifted towards shorter time scales when going from past to present. It is
noteworthy that this result goes in parallel with a faster convergence to the
normal distribution of the return fluctuations observed in the previous section.

An interesting feature of the Polish stock market is the existence of the subd-
iffusive region for short time scales. This effect can be related to the antiper-
sistence property of WIG20 fluctuations on minute time scales. However, it
remains unclear whether this antipersistence is unique to WIG20 or there is
similar behaviour of other stock indices like, e.g., DJIA or DAX30, likely on
time scales even shorter than 1 minute due to more sizeable volume traded
in those latter cases. In parallel with the superdiffusion region, this subdiffu-
sive regime in Fig. 7(b) weakens its character towards normal diffusion with
$\delta = 0.48$. Only the crossover point between the subdiffusive and the superdif-
fusive region remains unchanged.

Since the scaling and multifractal properties of a system are associated with
the long-range correlations, our results presented in this Section can be con-
sidered as yet another argument supporting the description of financial data
in terms of the nonextensive statistical mechanics.

5 Summary

The main conclusion to be drawn from the analysis presented above is that the
formalism of nonextensive statistical mechanics based on the generalised Tsal-
lis entropy seems to offer the best theoretical framework to compactly quantify
the probability distributions of the stock market fluctuations on various time
scales. This observation may appear helpful in formulating a generalised CLT
when the summed stochastic variables - like the financial once - are not com-
pletely uncorrelated. It also gives further support to the option pricing [19,20]
models based on the $q$-Gaussians [21]. This analysis has been performed for
the Warsaw Stock Market WIG20 high frequency recordings. Even though
commonly considered as emerging this market displays fluctuation and corre-
lation characteristics typical for the mature markets. The returns distribution
for the small time lags conforms well to the inverse cubic power law. However,
the departure from this law when increasing the time lags is seen to be faster
than for any market presented so far in the literature. None of those previous
studies for other markets refers however to such a recent time period and it
is likely that going from past to present effectively contracts the time scales
characteristic to the internal market dynamics and a similar effect may be
seen for the other markets as well. In fact a trace of such an effect has already
been noticed in ref. [5]. This is an interesting issue for further study. We show
that data from the Polish stock market exhibits long-range nonlinear correla-
Fig. 7. Standard deviation of WIG20 fluctuations for different time scales ranging from $\Delta t = 1$ to $\Delta t = 256$ minutes. Time series of returns were divided into two parts of approximately equal length covering older (a) and newer (b) time interval. Dashed lines represent fits in terms of $\sigma(\Delta t) \sim \Delta t^{\delta}$. Vertical lines indicate crossover points ($\times$) between different diffusion regimes.

These effects which can be seen in terms of the multifractality and the higher-order correlations in returns. Existence of such effects provides further arguments for considering the nonextensive statistical mechanics an inspiring framework for quantifying the stock market dynamics.

References


