SELFGRAVITATING GAS SPHERES IN A BOX AND RELATIVISTIC CLUSTERS: RELATION BETWEEN DYNAMICAL AND THERMODYNAMICAL STABILITY

Gennady S. BISNOVATYI-KOGAN\textsuperscript{1,2} and Marco MERAFINA\textsuperscript{3}

ABSTRACT

We derive a variational principle for the dynamical stability of a cluster as a gas sphere in a box. Newtonian clusters are always dynamically stable and, for relativistic clusters, the relation between dynamical and thermodynamical instabilities is analyzed. The boundaries between dynamically and thermodynamically stable and unstable models are found numerically for relativistic stellar systems with different cut off parameters. A criterion based on binding energy curve is used for determination of the boundary of dynamical stability.

Subject headings: dense matter — galaxies: star clusters — hydrodynamics — instabilities — relativity

1. Introduction

It is well known that Newtonian stellar clusters with effective adiabatic power $\gamma = 5/3$ are always dynamically stable (see Zel’dovich & Novikov 1971). An isolated cluster, in which temperature tends to a constant value all over the radius, suffers gravothermal catastrophe (see Antonov 1962; Lynden-Bell & Wood 1968) at which any finite object evolves into a

\textsuperscript{1}Space Research Institute (IKI)
Profsoyuznaya 84/32, Moscow 117997, Russia
E-mail: gkogan@iki.rssi.ru

\textsuperscript{2}Joint Institute of Nuclear Researches, Dubna, Russia

\textsuperscript{3}Department of Physics, University of Rome “La Sapienza”
Piazzale Aldo Moro 2, I-00185 Rome, Italy
E-mail: marco.merafina@roma1.infn.it
model with highly concentrated core and very extended envelope, where radius and central density tend to infinity. If we remove a demand of constant temperature and consider a rapid dynamical (adiabatic) perturbation, then the cluster does not react drastically and returns always to its mechanical equilibrium.

An interesting analysis of dynamical and thermodynamical instability of stellar clusters made by Chavanis (2002a, 2002b) contains some sections that show this problem is still not quite clear, instead of several publications devoted to this topic (see Merafina 1999; Lightman & Shapiro 1978).

In section 2 we derive in Newtonian gravity the variational principle for investigation of dynamical stability of a stellar cluster or a gas in a spherical box and discuss properties of trial functions which may be used for a stability analysis. For Newtonian gravity, we find conditions $\gamma > 4/3$ for stability of an extended cluster with $\rho_e/\rho_0 \ll 1$ and the condition $\gamma > 0$ for stability of a very hot body in a box at $P/\rho \gg GM/R$ with almost constant pressure $P$ and density $\rho$. The pressure along the adiabate is supposed to follow the relation $P = k\rho^\gamma$.

Relations between dynamical and thermodynamical stability of relativistic clusters are analyzed in sections 3 and 4. Numerical results about stability analysis of relativistic stellar clusters with different cut off parameters are represented in section 5.

The oscillatory behavior of the mass $M$ of the cluster as a function of a central density $\rho_0$, at fixed temperature $T$, indicates increasing number of thermodynamically unstable modes. Similar oscillatory dependence of $M(\rho_0)$, at fixed parameter $W_0$, shows (approximately) increasing number of dynamically unstable modes. Dynamically stable models with arbitrarily large central redshifts $z_c$ exist only at temperature $T \lesssim 0.06$. All such models are thermodynamically unstable.

## 2. Newtonian clusters and gas spheres

Let us consider development of dynamical perturbations in the cluster, where characteristic time is usually much shorter than any other time, including the time of energy exchange with an outer thermostat. In dynamical (rapid) perturbations, where local entropy is conserved and there is no time to smooth the temperature, the relations between $\delta T$, $\delta \rho$ and $\delta P$ follow adiabatic relations

\begin{align}
\frac{\delta P}{P} = & \frac{5}{3} \frac{\delta \rho}{\rho} = \frac{5}{2} \frac{\delta T}{T},
\end{align}

(1)
so the pressure perturbation, where in mechanical equilibrium we have

\[
\frac{dP}{dr} = \frac{kT \, d\rho}{m_s \, dr},
\]

should be taken as

\[
\delta P = \gamma \frac{P}{\rho} \delta \rho \quad \text{(where } \gamma = 5/3) .
\]

The dynamical stability analysis (Chandrasekhar 1964; Bisnovatyi-Kogan 2001) is usually performed for a star with zero density and pressure on the boundary. For a cluster in a box with nonzero \( \rho \) and \( P \) at the edge, the boundary conditions are different, but the expression for frequency, following from the variational principle, remains the same. It is easy to show, using equations in Lagrangian coordinates in which dynamical equations at spherical symmetry are written as

\[
\frac{\partial v}{\partial t} + \frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{Gm}{r^2} = 0
\]

\[
\frac{\partial r}{\partial t} = v
\]

\[
\frac{\partial r}{\partial m} = \frac{1}{4\pi \rho r^2},
\]

that, for linear perturbations of the static model with \( \delta r \sim e^{i\sigma t} \), we obtain, using Eq. (3),

\[
-\sigma^2 \delta r + \delta \left[ \frac{1}{\rho} \frac{dP}{dr} \right] - \frac{2Gm}{r^3} \delta r = 0
\]

\[
\frac{d\delta r}{dm} = -\frac{1}{2\pi \rho r^2} \frac{\delta r}{r} - \frac{1}{4\pi \rho r^2} \frac{\delta \rho}{\rho}, \quad \frac{\delta \rho}{\rho} = -2 \frac{\delta r}{r} - \frac{d\delta r}{dr}
\]

\[
\delta \left[ \frac{1}{\rho} \frac{dP}{dr} \right] = \delta \left[ 4\pi r^2 \frac{dP}{dm} \right] = \frac{2dP}{\rho \, dr} \frac{\delta r}{r} + \frac{\gamma}{\rho \, dr} \left( P \frac{\delta \rho}{\rho} \right).
\]
The equation for a perturbation $\delta r(m)$, taking into account Eqs. (7), (8) and the equilibrium equations, is written as

$$-\sigma^2 \delta r + \frac{4}{\rho} \frac{dP}{dr} \frac{\delta r}{r} - \frac{\gamma}{\rho} \frac{d}{dr} \left[ P \left( \frac{2}{r} \frac{\delta r}{r} + \frac{d\delta r}{dr} \right) \right] = 0 .$$

(10)

The Eq. (10) for a cluster in a box should be solved at boundary conditions

$$\delta r(0) = \delta r(M) = 0 , \quad r(M) = R ,$$

(11)

with a variable outer density and pressure $\delta \rho(M) \neq 0$, $\delta P(M) \neq 0$. This is different from the boundary conditions of stellar oscillations which, in adiabatic approximation, are

$$\left\{ \begin{array}{l}
\delta r(0) = 0 \\
\delta \rho(M) = -\rho \left( \frac{2}{r} \frac{\delta r}{r} + \frac{d\delta r}{dr} \right) \bigg|_{m=M} = 0 , \\
\end{array} \right.$$  

(12)

with a variable outer boundary radius $\delta r(M) \neq 0$.

Instead of solving Eq. (10), let us derive a variational principle, which gives a possibility of a simple stability analysis. Integrating Eq. (10) over the mass of the cluster, after multiplying by $\delta r$, and accepting the normalization of the linear perturbation function in the form

$$\int_0^M \delta r^2 dm = A ,$$

(13)

we obtain from Eq. (10), after partial integration with boundary conditions (11), the following expression for the squared frequency

$$\sigma^2 = \frac{1}{A} \int_0^M \frac{P}{\rho} \left[ \gamma \left( \frac{2}{r} \frac{\delta r}{r} + \frac{d\delta r}{dr} \right)^2 - 4 \left( \frac{\delta r}{r} \right)^2 - 8 \frac{\delta r}{r} \frac{d\delta r}{dr} \right] dm .$$

(14)

This is exactly the same expression which takes place for a star with another boundary conditions (12).
Variational principle over-estimates the values of $\sigma^2$ for different trial functions $\delta r(m)$, giving the minimal exact value for the eigenfunction of the oscillations. So, variational principle may prove the existence of instability, but only approximately permits to make a judgement about the stability of the system. It is important to use only those trial functions which satisfy the boundary conditions of the eigefunction.

It is known for stars with boundary conditions (12), that the linear trial function gives almost an exact result for stability boundary. We have from Eq. (14) for $\delta r = \alpha r$

$$\sigma^2 = \frac{9\alpha^2}{A} \int_0^M \frac{P}{\rho} (\gamma - 4/3) \, dm$$

that corresponds to stability boundary at $\gamma = 4/3$. This boundary is the exact value for stars with constant $\gamma$ because at $\gamma = 4/3$ the trial function $\delta r = \alpha r$ is also an exact eigenfunction (Zel’dovich & Novikov 1971).

For a cluster in the box the linear eigenfunction is not valid, because it does not satisfy outer boundary conditions. Nevertheless, for clusters with low ratio $\rho_e/\rho_0$ (where $\rho_e$ and $\rho_0$ are the external and central density, respectively) the exact fulfilment of the outer boundary condition is unimportant and, at $\rho_e/\rho_0 \to 0$, the condition (15) is approximately valid also as a criterion for dynamical stability of a cluster in a box. On the other hand, the dynamic response of a stellar cluster with quasi-maxwellian distribution function to global radial perturbations is similar to the response for adiabatic perturbations in a star with the same adiabatic index, which is $\gamma = 5/3$ in the nonrelativistic cluster. In fact, for non-relativistic spherical stellar clusters, we obtain dynamical stability because also barotropic stars with the same density distribution function are dynamically stable (Antonov 1960). This result leads in many cases to choose a parallel treatment in stability analysis for these physically different systems (Binney & Tremaine 1987). Similar correspondence exists even in the investigation of thermodynamical stability (Bettwieser & Sugimoto 1985). Therefore, nonrelativistic clusters with $\gamma = 5/3$ in the process of gravothermal catastrophe are, always, dynamically stable. Chavanis (2002a) considered perturbations at constant temperature and obtained dynamical instability in this case. We should stress that perturbations developing in the dynamical time do not preserve constant temperature, which could be reached only by contact with an external thermostat. Therefore the increment of “dynamic” instability at constant temperature is determined by the time of heat exchange between the cluster and thermostat, but not by the time of dynamical processes inside the clusters.

For clusters in a smaller box the outer boundary condition plays more important role and, for small boxes, where the gravity is less important (in the limit $P = \text{constant}$), using
the trial function $\delta r = \alpha r$ gives an highly inadequate result. Using a trial function with a correct boundary condition is necessary here, what, as evidently expected, reduce the demand on $\gamma$ for stability of the object in the box. Therefore, only a relativistic gas in the box may become dynamically unstable, because the Newtonian clusters have always $\gamma = 5/3$.

Using a trial function for a box with radius $R$ and constant pressure $P_0$ as

$$\delta r = \alpha r(1 - r/R)$$ \hspace{1cm} (16)

we get from Eq. (14)

$$\sigma^2 = \frac{4\pi\alpha^2}{A} P_0 \int_0^R \left[ \gamma (3 - 4r/R)^2 - 4(1 - r/R)^2 - 8(1 - r/R)(1 - 2r/R) \right] r^2 dr =$$

$$= \frac{4\pi\alpha^2}{A} P_0 R^3 \left[ \gamma \left( \frac{9}{3} - \frac{24}{4} + \frac{16}{5} \right) - \frac{12}{3} + \frac{32}{4} - \frac{20}{5} \right] = \frac{4\pi\alpha^2}{5A} \gamma P_0 R^3$$ \hspace{1cm} (17)

where $\gamma$ is constant all over the configuration.

The stability criterion $\gamma > 0$ corresponds to the dynamical stability of a weakly gravitating gas in the box under the condition of a high pressure $P_0/\rho \gg GM/R$. The criterion $\gamma > 0$ remains valid for a more general trial function $\delta r = \alpha r^n(1 - r/R)$, for any value of $n > 0$. We see that all instabilities of the Newtonian isothermal clusters with different boundary conditions considered by Antonov (1962), Lynden-Bell & Wood (1968) and Chavanis (2002a) are related only to thermodynamical instabilities, being the clusters always dynamically stable.

3. Relativistic models of selfgravitating systems

Gaseous systems in a box

The relation between dynamical and thermodynamical stability is much tenser in the relativistic case, when increased gravitational force may lead to dynamical instability for any equation of state.

The analysis made by Chavanis (2002b) demonstrates the development of dynamical instability for stars with ultrarelativistic equation of state in the box at increasing box size. We should clearly distinguish between two limits of the ultrarelativistic equation of state.
considered by Chavanis.

The first one corresponds to the deep interior of the neutron star where, in conditions of ultrarelativistic degeneracy and strong nuclear forces, we have Eq. (18) with $1/13 < q < 1$ (see Ambartsumyan & Saakyan 1961; Zel’dovich 1962a). The parameter characterizing the dynamical stability is the adiabatic index (see Harrison et al. 1965)

$$\gamma = \frac{P + \rho c^2}{Pc^2} \left( \frac{\partial P}{\partial \rho} \right)_S = q + 1,$$

(19)

where, for dynamically stable configurations, we have

$$\gamma > \gamma_{cr}.$$  

(20)

In Newtonian limit we have always $\gamma_{cr} = 4/3$ but, when the effects of general relativity make the gravity stronger, $\gamma_{cr}$ becomes larger than $4/3$ (see Kaplan 1949; Chandrasekhar 1964; Merafina & Ruffini 1989). Then, isolated ultrarelativistic stars for which $P \sim \rho c^2$ are always dynamically unstable, while stable existence of such configurations is possible, in principle, only inside a box with fixed radius or fixed external pressure. Clearly, in conditions of zero temperature there is no sense to discuss the thermodynamical stability of the system. Therefore, the analysis made in Section 3 and 4 by Chavanis (2002b) relates only to the dynamical stability of such star (see also Yabushita 1974).

The loss of stability, dynamical as well as thermodynamical, may be found from the linear series of equilibrium models, where the extremum of mass in the appropriate curve $M(\rho_0)$ determines the appearance or disappearance of unstable mode. For dynamical stellar stability this “static” criterion was formulated by Zel’dovich (1963) and, for thermodynamical instability, was used by Lynden-Bell & Wood (1968). Analysis of Chavanis (2002b) has shown that dynamical instability of relativistic sphere in a box with fixed radius happens exactly in the first maximum of the function $M(\rho_0)$. Contrary to that, Yabushita (1974), who investigated dynamical stability of gas spheres in a box with fixed external pressure in general relativity (GR), had found the loss of stability in a point well before the maximum of the curve $M(\rho_0)$. It is important to note that not every kind of linear series of models may be used for investigation of dynamical stability. In the case of an isolated star with a zero boundary pressure it is necessary to have a fixed distribution of specific entropy.
$S(m/M)$ along the series of models $M(\rho_0)$, where the first maximum of $M(\rho_0)$ denotes the loss of stability relative to the global mode without nodes (see Zel’dovich 1963). Analysis of dynamical stability of stars in a box with fixed radius and fixed external pressure had shown that only the first maximum of the curve $M(\rho_0)|_{r_e}$ corresponds to the loss of dynamical stability, but maximum on the curve $M(\rho_0)|_{P_b}$ is situated after the point of the loss of dynamical stability. The spheres in a box with constant external pressure considered by Yabushita (1974) are not isolated from the surrounding medium, which produces a work when the box is changing its volume. So the comparison of two models with equal masses and different central densities has no sense because these models have different energies due to different volumes of the box. Therefore, the first maximum on the sequence with constant $P_b$, considered by Yabushita does not correspond to the onset of instability in spite of constant specific entropy of the matter considered along the series. The coincidence of the maximum of the curve $M(\rho_0)|_{P_b}$ with the point of the loss of dynamical stability happens only at $q = 0$, that formally corresponds to zero external pressure and no external work.

In order to have a possibility to judge about the onset of dynamical instability from the linear series of models in presence of constant external pressure $P_b$ at the stellar boundary with a nondimensional radius $\xi_b$ (usual Emden coordinates), we must take into account the work of the external pressure and use the function

$$E(\rho_0) = M + \frac{P_b V}{c^2}, \quad (21)$$

where $V$ is the total volume of the model, instead of $M(\rho_0)$. It is easy to show, using Yabushita results and notations, that for these models the extrema of the functions $E(\rho_0)|_{P_b}$ or $P_b(\xi_b)|_E$ coincide exactly with the point of the loss of stability for different modes, with increasing $\rho_0$ or $\xi_b$ (see also Chavanis 2003).

**Star clusters with cutoff**

The second limit to which relativistic analysis of Chavanis (2002b) is applied concerns “isothermal” relativistic star clusters, with a local constant temperature (Bisnovatyi-Kogan & Zel’dovich 1969; Bisnovatyi-Kogan & Thorne 1970). Oppenheimer-Volkoff equations describing the equilibrium of a relativistic cluster are given by

$$\begin{align*}
\frac{dP}{dr} &= -\frac{G (P + \rho c^2)(M_r c^2 + 4\pi Pr^3)}{c^2 r (r c^2 - 2GM_r)} \\
\frac{dM_r}{dr} &= 4\pi r^2 \rho,
\end{align*} \quad (22)$$
with \( P(0) = P_0 \) and \( M_r(0) = 0 \). Pressure \( P \) and total energy density \( \rho c^2 \) are expressed as integrals in momentum space with the distribution function (29), \( M_r \) is the mass inside the lagrangian radius \( r \) (Bisnovatyi-Kogan et al. 1993, 1998). The Schwarzschild-type metric was chosen

\[
ds^2 = e^\nu c^2 dt^2 - e^\lambda dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),
\]

(23)

where the metric coefficients are defined by the expressions

\[
\begin{align*}
\nu &= \text{exp} \left( 2 \int_r^{\infty} \frac{dP}{dr} \frac{dr}{P + \rho c^2} \right), \\
\lambda &= \left( 1 - \frac{2GM_r}{rc^2} \right)^{-1}.
\end{align*}
\]

(24)

In our calculations we have used the variable \( W \) instead of \( P \), which leads to equilibrium equation (Merafina & Ruffini 1989)

\[
\frac{dW}{dr} = -\frac{G}{c^2} \left( 1 - \frac{\beta W}{\beta} \right) \frac{M_r c^2 + 4\pi P r^3}{r(rc^2 - 2GM_r)},
\]

(25)

with the condition for integration \( W(0) = W_0 \). Here \( \beta = T_R/mc^2 \) and \( W \) is defined by (27). Therefore, each model is uniquely determined by choosing the parameters \( W_0 \) and \( T \) (or \( \beta \)).

These clusters are not in exact thermodynamical equilibrium, so the investigation of the behavior of the linear series of models of such clusters in a box does not give, rigorously speaking, correct results about neither thermodynamical nor dynamical stability. Nevertheless at \( T \lesssim mc^2 \), when the cluster is almost nonrelativistic and gravitational potential \( \varphi \ll c^2 \), the local temperature is almost constant all over the cluster and therefore the analysis of Chavanis (2002b) can give a valid presentation about thermodynamical stability of such system. In order to investigate dynamical instability of such clusters, the equations for small perturbations should be solved or variational principle may be used: this method, which is more complicated in the relativistic case than in the Newtonian one, was derived, for relativistic stars, by Chandrasekhar (1964) and Harrison et al. (1965); for relativistic clusters, the variational principle derived by Ipser & Thorne (1968) and completed by Fackerell (1970) was used by Bisnovatyi-Kogan & Thorne (1970) for the investigation of dynamical stability of the “isothermal” clusters.
As it was noted by Lynden-Bell & Wood (1968 and references therein), the behavior of a star cluster in a box is related, in some sense, to the isolated cluster with cutoff in the energy distribution. Relativistic Maxwellian clusters with cutoff have been first introduced by Zel’dovich & Podurets (1965) and their dynamical stability was studied by Bisnovatyi-Kogan et al. (1993, 1998). Due to relativistic gravity, the loss of dynamical stability happens at $\langle \gamma \rangle < \gamma_{cr}$ and the structure of the marginally stable model strongly depends on the cutoff parameter. In the relativistic cluster, the lower adiabatic index $\gamma < 5/3$ is connected with relativistic motion of an ideal nondegenerate gas of stars: it is decreasing with the increase of the temperature. Contrarily to that, a fully degenerate highly non ideal nuclear matter is present in the interior of a neutron star which may have even larger adiabatic index $\gamma > 5/3$.

An analogy exists between the cutoff parameter $W_0$ used by Merafina & Ruffini (1989) and the quantity $v_1 = m(\varphi_0 - \varphi_e)/T$ used by Lynden-Bell & Wood (1968), where $\varphi_0$ and $\varphi_e$ are the Newtonian gravitational potential in the center and at the edge of the cluster respectively. In a full equilibrium cluster in the box with the density distribution $\rho = \rho_0 \exp [m\varphi_0 - m\varphi(r)]/T$, the quantity $-v_1$ represents the logarithm of the ratio of densities in the center and at the edge of the cluster

$$-v_1 = \frac{m(\varphi_e - \varphi_0)}{T} = \ln \frac{\rho_0}{\rho_e}.$$  \hspace{1cm} \text{(26)}

Note that density is a finite value at the outer boundary. The ratio $\rho_0/\rho_e$ is also called “density contrast”. The value $W_0$ is defined as

$$W_0 = \left( \frac{\epsilon_{cut}}{T_r} \right)_{r=0},$$  \hspace{1cm} \text{(27)}

where $T_r = T e^{-\nu(r)/2}$ is the local thermodynamical temperature and the constant $T$ is the temperature for an infinitely-remote observer. The metric coefficient $\nu(r)$ has an usual meaning as in the Schwarzschild metric (Landau & Lifshitz 1962). It is easy to show (see Bisnovatyi-Kogan et al. 1993) that, in Newtonian limit, $W_0$ formally reduces to $-v_1$ in the first relation to the right side of Eq. (26) but, because of deviation from thermodynamical equilibrium implied by the cutoff, the outer edge of the cluster corresponds to zero density.

Thermodynamical stability of the isothermal cluster in Newtonian gravity with a truncated distribution function may be characterized by the curve $E_b(W_0)$ at constant $T$, where its maximum denotes the loss of thermodynamical stability, $E_b = -Em/MT$ is a nondimensional specific binding energy of the cluster, $E = T + U$ is the total energy of the cluster (kinetic and gravitational energy). The value $E_{re}/GM^2$, plotted in Lynden-Bell & Wood.
paper (1968) for an isolated equilibrium cluster in a box as a functions of $-v_1$ at constant $r_e$, has the same meaning and characterizes the thermodynamical stability of such cluster. The plot $GM/r_eT$ as a functions of $-v_1$, also at fixed $M$ and $r_e$, going along the sequence of models with varying $T$ determines, indeed, the thermodynamical stability of the cluster in a thermal bath. The loss of stability in such a cluster (at $-v_1 = 3.47$, with density contrast $\rho_0/\rho_e = 32.125$) happens before that of the isolated one (at $-v_1 = 6.55$, with density contrast $\rho_0/\rho_e = 708.61$). In the isolated cluster perturbations are developed at constant energy but, for the perturbed cluster in the bath, the temperature is preserved, so in the last case the cluster occurs to be less stable.

The definition of the thermodynamical stability or instability of an open cluster with a cutoff and zero density at the edge is, generically speaking, senseless, because all such clusters are “thermodynamically unstable”. The relaxation in these systems leads to approaching a local Maxwellian distribution function without cutoff, equivalent to the formation of a thermodynamically unstable isothermal gas sphere. Existence of maxima on the curve $E_{b,T}(W_0)$ or, equivalently, on the curve $E_{b,T}(\rho_0)$, where henceforth $E_b$ will simply indicate the specific binding energy $E_b/N$, may signify, instead, the appearance of an additional “thermodynamically” unstable mode which is developed without an increase of cutoff parameter and only due to a global fluctuation of the parameter $T$. In reality, thermodynamical instability of both types is governed by the so-called “two-body relaxation time” (Ambartsumyan 1938; Spitzer 1940)

$$\tau_b = 8.8 \cdot 10^5 \sqrt{\frac{N R^3}{m/M_\odot}} \ln N - 0.45 \text{ yr},$$

where $m$ is the star mass, $R$ is the radius of the cluster expressed in parsec and $N$ is the total number of stars in the cluster, and therefore, in this sense, all dynamically stable stellar clusters in space are thermodynamically unstable. On the other hand, the local value of $\tau_b$ is proportional to $1/n$, where $n$ is the number density of stars, and the influence of cutoff is strong near the outer boundary, where local $\tau_b$ has a maximal value. Moreover, after the point of the loss of thermodynamical stability, the instability begins to develop also in the central regions, where local $\tau_b$ has a minimal value, much lower than at the edge of the cluster. Therefore, the loss of thermodynamical stability determined by the curve $E_{b,T}(W_0)$ or $E_{b,T}(\rho_0)$ is important for the cluster with a cutoff almost as well as for the cluster in the box.

The analogy between our truncated clusters and isothermal cluster in the box appears if we surround the truncated cluster by the box and let them relax to thermal equilibrium. In this sense we exclude the global thermodynamical instability of the open cluster and
the curve $E_{b,T}(\rho_0)$ or $E_{b,T}(W_0)$ gives informations about thermodynamical stability of the cluster in a box with a radius equal to the radius of the open truncated cluster. Comparison of Newtonian curve of specific binding energy $E_b/N$ (see Fig. 1) for open clusters considered in the paper of Bisnovatyi-Kogan et al. (1998) with the corresponding one for clusters in a box, indicated in Fig. 2 of the paper of Lynden-Bell & Wood (1968), shows a good correspondence between first extrema of these curves which lay at $-v_1 = 6.55$, for clusters in a box, and at $W_0 = 6.42$, for open clusters with truncated Maxwellian distribution function. This similarity may be seen also from the comparison of these curves as a whole, where approximate coincidence of two subsequent extrema is visible.

4. Relativistic and nonrelativistic oscillations in the curve $E_b(\rho_0)$

Stability criteria for relativistic models

The curve $E_b(\rho_0)$ characterizing the stability of the star shows a strikingly similar oscillatory behavior for dynamical and thermodynamical types of instability. This oscillatory behavior was first analyzed by Dmitriyev & Kholin (1963) in the curve $M(\rho_0)$ for cold neutron stars. They have shown that the star loses its dynamical stability relative to the global contraction in correspondence to the maximum of the curve $M(\rho_0)$ and each new maximum and minimum leads to appearance of a new unstable mode (see also Misner and Zapolsky 1964). The detailed analysis of this curve was done by Harrison et al. (1965), where the dependence $M(R)$ was also used: this choice is even more useful for analysis of instability, because the behavior of the spiral curve $M(R)$ permits to distinguish unambiguously between the appearance of a new mode of instability or the removal of the unstable mode, both of which may happen in the extremum of the curve $M(\rho_0)$. The analysis of Chavanis (2002b), made for a simple relativistic equation of state $P = q\epsilon$ in a box, is very similar to the considerations of Dmitriyev & Kholin (1963) and Harrison et al. (1965) made for neutron stars.

The dependence of the total mass $M$ of the star on its total barion number $N_b$, which has always a positive derivative and shows an angle on the curve $M(N_b)$ corresponding to the extremum of the curve $M(\rho_0)$ obtained by Chavanis (2002b) for a star in the box, was also demonstrated by Zel'dovich (1962b) for a cold neutron star.

Similar oscillations for the thermodynamical instability have been obtained by Lynden-Bell & Wood for an isothermal cluster in the box. In the paper of Bisnovatyi-Kogan et al. (1998) the oscillatory behavior was found in the curve $M_T(\rho_0)$ with a related spiral behavior.
of the curve $M_T(\alpha)$, where the parameter $\alpha$ determines the cutoff of the distribution function\(^1\)

\[ E \leq mc^2 - \alpha T/2, \quad f \sim e^{-E/T}. \tag{29} \]

This oscillatory (spiral) behavior, indicated by Bisnovatyi-Kogan et al. (1998) for different fixed values of temperature $T$ (Newtonian case corresponds to $T \to 0$), shows appearance of modes of “thermodynamical” instability of clusters with a cutoff, relative to perturbations with heat redistribution inside the cluster and change of the cutoff parameter $\alpha$. More detailed curves $M_T(\rho_0)$ and $M_T(\alpha)$ are shown in Figs. 2 and 3, where results are represented in nondimensional coordinates from Bisnovatyi-Kogan et al. (1998) with using the same calculation scheme. Note that criterion for thermodynamical stability based on the curve $M_T(\rho_0)$ works sufficiently well only for large $T \gtrsim 0.1$. For smaller temperatures the criterion based on the curve $E_{b,T}(\rho_0)$ should be used like in Newtonian clusters (see next section).

Several approximate criteria had been suggested by Bisnovatyi-Kogan et al. (1993) for investigation of dynamical stability of relativistic clusters. All these criteria work almost equally well for moderate values of $\alpha \lesssim 1.5$. At larger $\alpha$, the criterion based on the evaluation of extremum of the curve $M_\alpha(\rho_0)$ is not appropriate because of appearance of increasing number of loops (see Fig. 2) which are connected with multiple intersections of the vertical line $\alpha = \text{constant}$ with the spiral curves of Fig. 3, corresponding to very low (zero) temperature. The first loop appears at $\alpha \simeq 1.5$, at $1.9 \lesssim \alpha \lesssim 1.5$ there are two loops and, in general, there is an even number of loops, except boundaries at $\alpha \simeq 1.5, 1.9, 2.0, \ldots, 2.02$ at which there is an odd number of loops $N_l = 3, 5, \ldots, \infty$. At $\alpha < \alpha_\infty \simeq 2.02$ there is a curve $M_\alpha(\rho_0)$ going to infinity and at $\alpha > \alpha_\infty$ all curves $M_\alpha(\rho_0)$ are represented only by loops at finite densities. Another important value is $\alpha_{lim} \simeq 2.87$ so that there are no solutions for clusters with a cutoff parameter $\alpha > \alpha_{lim}$ (see Bisnovatyi-Kogan et al. 1998). The particular values of $\alpha$ given above characterize the Newtonian curve $M_T(\alpha)$, at $T \to 0$. In fact, the values $\alpha_\infty(T)$ and $\alpha_{lim}(T)$, changing for different values of temperature $T$ (see center of spirals and maximum values of $\alpha$ for each curve in Fig. 3), in the Newtonian limit of $T \to 0$ reach the particular values $\alpha_\infty = 2.02$ and $\alpha_{lim} = 2.87$ (see Bisnovatyi-Kogan et al. 1998).

The criteria based on evaluation of entropy and adiabatic invariants are still valid, but their application is too complicated. Analysis made by Bisnovatyi-Kogan et al. (1998) had shown that the most convenient criterion of the dynamical stability of relativistic clusters should be based on the investigation of the curves of dependence of the specific binding energy

\(^1\)Newtonian clusters with another type of cutoff had been studied by Katz (1980).
of the cluster $E_b$ on the central density $\rho_0$, at constant value of $W_0$. Note that binding energy of the relativistic cluster is equal to the total energy of the cluster in Newtonian case, where rest mass energy does not appear in the definition of $E$.

5. Numerical results

*Turning point analysis for relativistic star clusters*

To analyze the stability of relativistic clusters we need to calculate the specific binding energy of the equilibrium models. We have calculated two families of curves, which characterize dynamical and thermodynamical stability of relativistic clusters with different cutoff parameters. The curves $E_b(\rho_0)$ of specific binding energy at constant temperature $T$ (Fig. 4) characterize the thermodynamical stability, while the curves $E_b(\rho_0)$ at constant $W_0$ (Fig. 5) give information about dynamical stability of the cluster. Relativistic expression of specific binding energy is $E_b = (Nm - M)/Nm$, where $N$ is the total number of stars in the cluster given by

$$N = 4\pi \int_0^R \frac{nr^2 dr}{\sqrt{1 - \frac{2GM_r}{rc^2}}}$$  \hspace{1cm} (30)

and $m$ is the mass of a single star (all stars have the same mass). The number density $n$ is expressed as integral in momentum space with similar calculation procedure used for obtaining the pressure $P$ and total energy density $\rho c^2$ (Bisnovatyi-Kogan et al. 1993, 1998).

The temperature is increasing along each curve in Fig. 5, tending to a finite constant value for large values of central density $\rho_0$. The loss of stability, characterized by the first maximum, takes place only for $W_0 \leq 15.5$. In correspondence of this critical value $W_0 \approx 15.5$, the temperature, at large $\rho_0$, reaches a limiting value $T_a = 0.0635$. This means that no dynamic instabilities are present for $T \lesssim 0.06$. At $W_0 = 16$, for example, the limiting temperature is equal to $T_a = 0.597$ and specific binding energy $E_b(\rho_0)$ increases monotonously until the asymptotic value $E_{b,a} = 0.0312$.

At large $\rho_0$, for models with very large central redshift $z_c$, there is an asymptotic value $E_{b,a}$ of specific binding energy for each value of $W_0$. Plotting the function $E_{b,a}(W_0)$ from Fig. 5 we obtain a more precise boundary of the dynamical stability $W_{0,a} = 15.8$. Due to monotonic dependence of asymptotic (at large $\rho_0$) values of limiting temperature $T_a$ on the parameter $W_0$, similar curve $E_{b,a}(T_a)$ from Fig. 4 shows the appearance of dynamically unstable clusters at $T_a \gtrsim 0.06$. The limiting curves of specific binding energy $E_{b,a}(W_0)$ and $E_{b,a}(T_a)$ are represented in Figs. 6a and 6b respectively. In the equilibrium configurations
with very large central redshift, the temperature is decreasing monotonously with the increase of $W_0$, as may be shown in Fig. 7. Note that the curve $T_a(W_0)$ is approximated with a good precision by the power-law relation

$$T_a = \frac{0.937}{(W_0)^{0.989}}. \quad (31)$$

Combining results of numerical investigation of dynamical and thermodynamical stability are represented in Figs. 8, 9 and 10. In Figs. 8a and 8b dynamically stable and unstable regions are represented in the planes $(T, \rho_0)$ and $(T, z_c)$, respectively. The results plotted in the plane $(T, z_c)$ are analogous to ones of the work of Merafina (1999). In Figs. 9a and 9b thermodynamically stable and unstable regions are represented in the planes $(T, \rho_0)$ and $(T, z_c)$, respectively. Also in this case the results plotted in the plane $(T, z_c)$ are analogous to ones of Merafina (1999). Summary of numerical results on dynamical and thermodynamical stability analysis is given in Figs. 10a and 10b, where different regions are represented in the planes $(T, \rho_0)$ and $(T, z_c)$, respectively. It is important to stress the coincidence of boundaries between dynamically and thermodynamically stable and unstable configurations at large temperatures. Using approximate criteria of dynamical stability we cannot definitely judge if these boundaries coincide exactly or there is a small difference between them, however the behavior of specific binding energy $E_b/N$ let us enough confidence in this result.

Analysis of models with $\alpha$ approaching $\alpha_\infty$, when maximal densities are expected, has shown that they have a very extended halo so, even at very high densities, the cluster could remain to be dynamically stable with local Newtonian properties. Similar situation should appear when taking into account of degeneracy (Merafina & Ruffini 1990). In the Newtonian limit, only nonrelativistic degeneracy is expected, so the situation is not changed qualitatively. The logarithmic curve $M(\rho_0)$ for a fully degenerate nonrelativistic gas is a monotonic line $M \sim \rho_0^{1/2}$, like in a polytropic star with index $n = 3/2$ and $\gamma = 5/3$, which does not show neither thermodynamical nor dynamical instabilities. The thermodynamical instabilities on the curve $M_T(\rho_0)$ appear only at finite temperatures.

Consideration of fully degenerate ultrarelativistic particles with equation of state $P = q\epsilon$, made by Chavanis (2002b), has sense only for a limited box because these configurations are dynamically unstable with an open outer boundary. The curve $E_b(\rho_0)$ for a given size of the box gives results about the onset of dynamical instability in such configurations and has an analogy with the corresponding curve at constant $W_0$.

In the problem of stellar stability the curve $M_S(\rho_0)$, where $S$ is the specific entropy of an isentropic star, is used in the static criterion and its maximum determines the boundary of dynamical stability of a star (Zel’dovich 1963). The curve $M(\rho_0)$ at constant “entropy” was
also used for an approximate estimation of the boundary of dynamical stability of truncated stellar cluster (Bisnovatyi-Kogan et al. 1993), but such approach is very cumbersome. Using the same curve at constant $\alpha$ or $W_0$ is much easier and give very close results. It happens however, that at large values of $W_0 \gtrsim 11$ or at large enough $\alpha \gtrsim 2$, the curve $M(\rho_0)$ is not valid anymore for the estimation of the dynamical stability. The corresponding curve at constant $\alpha$ becomes discontinuous and irrelevant. Using the curve $E_b(\rho_0)$ instead of $M(\rho_0)$ permits to make such estimation in all range of values of the parameter $W_0$. We have obtained numerically that at $W_0 \lesssim 11$ maxima of the curves $E_b(\rho_0)$ and $M(\rho_0)$ coincide exactly but, at larger $W_0$, maximum of the curve $M(\rho_0)$ disappears while the maximum of the curve $E_b(\rho_0)$ is remaining and shifting to infinite central density at $W_0 = 15.8$. So that all models at larger $W_0$ are dynamically stable. Similar relation between $E_b(\rho_0)$ and $M(\rho_0)$ takes place for the curves at constant $T$. At large values of $T \gtrsim 1$ maxima of these curves coincide and the curves themselves tend to an asymptotic behavior (see Figs. 2 and 4). At smaller temperatures the maximum of $E_b(\rho_0)$ is shifting to larger densities (see Fig. 4) and maximum of $M(\rho_0)$ moves in the opposite side of smaller densities. Following Lynden-Bell & Wood (1968) we accept that loss of thermodynamical stability is connected with the maximum of the curve $E_{b,T}(\rho_0)$.

6. Conclusions

The visible similarity between the oscillations of the curve $E_b(\rho_0)$ for relativistic clusters and the Newtonian isothermal stellar cluster in the box may have a different physical nature: in the first case there are two types of oscillations, reflecting the loss of dynamical and thermodynamical stability, while the cluster in the box is dynamically stable and oscillations are connected only with the onset of thermodynamical instability leading to gravothermal catastrophe.

The analogy between the open cluster with a cutoff in the Maxwellian energy distribution function and an isothermal cluster in a box is not complete because the first one is thermodynamically unstable everywhere while the second one is only after the Antonov’s point, at density contrast $\rho_0/\rho_e \gtrsim 709$. Nevertheless, both kinds of oscillations are present in the structure of relativistic clusters with a cutoff when we consider clusters with different cutoff parameter $\alpha$ and a variable value $W_0$, which plays the role of the ratio $\rho_0/\rho_e$ (density contrast) or $-v_1$ for the cluster in a box. The Figs. 4 and 5 illustrate the behavior of both oscillations for the function $E_b(\rho_0)$, where oscillations of the curves at constant $T$ add thermodynamically unstable modes and oscillations of the curves at constant $W_0$ give a picture of the dynamical stability of the cluster. Calculations show that Newtonian truncated clusters
lose their thermodynamical stability at $W_0 = 6.42$, very close to the value $-v_1 = 6.55$ for clusters in a box.

We have obtained that only the curve $E_b(\rho_0)$ at constant $T$ is valid for the estimation of thermodynamical stability, while the corresponding curve $M(\rho_0)$ may be misleading. In the case of dynamical stability both curves give identical results at lower values of $W_0 \lesssim 11$ while, at larger ones, only the curve $E_b(\rho_0)$ may be used.
REFERENCES


Antonov, V.A. 1960, AZh, 37, 918


Dmitriyev, N., Kholin, S. 1963, Voprosy Kosmogonii, 9, 254


Lightman, A.P., Shapiro, S.L. 1978, Rev. Mod. Phys., 50, 437
Merafina, M. 1999, Odessa Astron. Publ., 12, 210
Spitzer, L. 1940, MNRAS, 100, 396
Zel’dovich, Ya.B. 1962a, Soviet Phys. JETP, 14, 1143
Zel’dovich, Ya.B. 1962b, Soviet Phys. JETP, 15, 446
Zel’dovich, Ya.B. 1963, Voprosy Kosmogonii, 9, 157
Zel’dovich, Ya.B., Podurets, M.A. 1965, AZh, 42, 963

This preprint was prepared with the AAS LaTeX macros v5.2.
Fig. 1.— Specific binding energy $E_b/N$ as a function of $W_0$, representing the points of loss of thermodynamical stability of different modes for Newtonian clusters with truncated Maxwellian distribution.
Fig. 2.— Mass $M$ of equilibrium configurations in clusters with a cutoff as a function of central density $\rho_0$ for different values of temperature $T$ (dotted lines) and cutoff parameter $\alpha$ (continuous line). The curves representing dependence $M(\rho_0)$ at constant $\alpha$ have several branches at $\alpha \gtrsim 1.5$; three branches for $\alpha = 2.0$ are represented.
Fig. 3.— Mass $M$ of equilibrium configurations in clusters with a cutoff as a function of parameter $\alpha$ for different values of temperature $T$. 
Fig. 4.— Specific binding energy $E_b/N$ of equilibrium configurations in clusters with a cutoff as a function of central density $\rho_0$ for different values of temperature $T$. Each extremum corresponds to appearance of new thermodynamically unstable modes.
Fig. 5.— Specific binding energy $E_b/N$ of equilibrium configurations in clusters with a cutoff as a function of central density $\rho_0$ for different values of parameter $W_0$. First maxima, corresponding to loss of dynamical stability, are present only on curves with $W_0 \leq 15.5$. Each extremum corresponds to appearance of new dynamically unstable modes.
Fig. 6.— Specific binding energy $E_b/N$ of equilibrium configurations in clusters with a cutoff for very large central densities $\rho_0$ and central redshifts $z_c$ as a function of $W_0$ (Fig. 6a, left side) and $T_a$ (Fig. 6b, right side). The maximum, indicating the loss of dynamical stability, corresponds to $W_0 = 15.8$ and $T \simeq 0.06$, respectively. The limiting value of binding energy is $E_{b,a} = 0.0312$. 
Fig. 7.— Asymptotic values of temperature $T$ as a function of $W_0$ in the limiting clusters with very high central redshifts.
Fig. 8.— Regions of dynamical stability and instability in the plane \((T, \rho_0)\) (Fig. 8a, left side) and in the plane \((T, z_c)\) (Fig. 8b, right side).
Fig. 9.— Regions of thermodynamical stability and instability in the plane \((T, \rho_0) \) (Fig. 9a, left side) and in the plane \((T, z_c) \) (Fig. 9b, right side).
Fig. 10.— Regions of dynamical and thermodynamical stability and instability in the plane \((T, \rho_0)\) (Fig. 10a, left side) and in the plane \((T, z_c)\) analogous to results of Merafina in 1999 (Fig. 10b, right side).