General Relativistic Effects of Gravity in Quantum Mechanics

--- a Case of Ultra-Relativistic, Spin 1/2 Particles ---

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We present a general relativistic framework for studying gravitational effects in quantum mechanical phenomena. We concentrate our attention on the case of ultra-relativistic, spin-1/2 particles propagating in Kerr spacetime. The two-component Weyl equation with the general relativistic corrections is obtained in the case of slowly rotating, weak gravitational field. Our approach is also applied to neutrino oscillations in the presence of the gravitational field. The relative phase of two different mass eigenstates is calculated in radial propagation, and the result is compared with the previous works.

§1. Introduction

It had been thought that physical phenomena in which gravitational effects and quantum effects appear simultaneously were far beyond our reach, before Colella, Overhauser and Werner\textsuperscript{1) made an elegant experiment using a neutron interferometer (hence this kind of experiment is called the COW experiment). The COW experiment was the first experiment that measures the Newtonian gravitational effect on a wave function. This effect and the detectability were first suggested by Overhauser and Colella\textsuperscript{2) and the next year the effect was verified by Colella et al.\textsuperscript{1) Although their analysis, which was based on inserting the Newtonian gravitational potential into the Schrödinger equation, was so simple, this experiment is conceptually very important in the history of quantum theory.

Recently, gravitational effects on another physical phenomenon, neutrino oscillations, have been much discussed\textsuperscript{3), 4), 5), 6), 7), 8) The COW experiment and this have common ground that gravitational effects appear in the quantum interference. However, there are some differences between the two. In the former case, the spatial spread of the wave function plays a significant role, whereas in the latter case, the existence of different mass eigenstates and the linear superposition are important. Moreover, it is another important difference whether the related particle is non-relativistic or ultra-relativistic.

It seems that a controversy about the gravitationally induced neutrino oscillation phases arises. Ahluwalia and Burgard\textsuperscript{3} state that the phases amount to roughly 20% of the kinematic counterparts in the vicinity of a neutron star. Nevertheless, the definition of neutrino energy and the derivation of the phases were not clear in the original paper\textsuperscript{3}. On the other hand, the other groups\textsuperscript{11, 12, 13} have obtained similar results for radially propagating neutrinos; the results seem to be different from that

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in Ref. 3. However, the authors of Ref. 4 assume that different mass eigenstates are produced at different times. This assumption seems to be questionable because the relative phase between the two different mass eigenstates initially becomes arbitrary. These papers except Ref. 7 are based on previous work, in which the classical action is taken as a quantum phase. Therefore, effects arising from the spin of the particle are not considered in these papers. On the other hand, the authors of Ref. 7 use the covariant Dirac equation, but they also calculate the classical action along the particle trajectory in the end.

In this situation, we shall provide another framework different from the previous works for studying general relativistic gravitational effects on spin-1/2 particles with non-vanishing mass such as massive neutrinos (the experimental confirmation which shows that neutrinos have nonzero mass is not yet obtained. However, recent experimental reports seem to suggest neutrinos to be massive). We do not merely calculate the classical action along the particle trajectory, but start from the covariant Dirac equation. Our approach will allow us to discuss the effects of the coupling between the spin and the gravitational field. In particular, we consider the propagation of the particle in the Kerr geometry, by which the external field of a rotating star can be described. We shall perform our calculations in the slowly rotating, weak gravitational field, and derive the two-component Weyl equations with the corrections arising from the non-vanishing mass and the gravitational field. Furthermore, we shall discuss neutrino oscillations in the presence of the gravitational field.

The organization of this paper is as follows. In §2, we shall assume that the external field of a rotating object is described by the Kerr metric, and discuss the covariant Dirac equation in this field. In §3, we shall derive the Weyl equations with general relativistic corrections for a ultra-relativistic particle. The application to neutrino oscillations in the presence of the gravitational field will be discussed in §4. Finally, we shall give a summary and conclusion in §5.

§2. Covariant Dirac equation in Kerr geometry

In this section, we consider the covariant Dirac equation in the presence of the gravitational field arising from a rotating object. We shall derive an equation for the time evolution of spinors, which describe particles with spin-1/2, in the last part of this section.

2.1. Covariant Dirac equation

To begin with, we briefly review the covariant Dirac equation. The natural generalization of the Dirac equation into curved space-time gives

\[ i\hbar \gamma^\mu \left( \frac{\partial}{\partial x^\mu} - \Gamma^\mu_\mu \right) - mc = 0, \]  

(2.1)

where \( \gamma^\mu \) are the covariant Dirac matrices connected with space-time through the relations

\[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \]  

(2.2)
and $\Gamma_\mu$ is the spin connection. The spin connection is determined by the condition
\[
\frac{\partial \gamma_\nu}{\partial x^\mu} - \Gamma^\lambda_{\nu\mu} \gamma_\lambda - \Gamma_\mu \gamma_\nu + \gamma_\nu \Gamma_\mu = 0.
\] (2.3)

We now introduce the constant Dirac matrices $\gamma^{(a)}$ defined by
\[
\gamma^{(a)} = e^{(a)}_\mu \gamma^\mu,
\] (2.4)
where $e^{(a)}_\mu$ is the orthogonal tetrad satisfying the relation
\[
g_{\mu\nu} = \eta_{ab} e^{(a)}_\mu e^{(b)}_\nu
\] (2.5)
($\eta_{ab} = \text{diag} (c^2, -1, -1, -1)$). Using these constant Dirac matrices, the spin connection is expressed as
\[
\Gamma_\mu = -\frac{1}{8} \left[ \gamma^{(a)}, \gamma^{(b)} \right] g^{\nu\lambda} e^{(a)}_\nu \nabla_\mu e^{(b)}_\lambda,
\] (2.6)
where square brackets denote the usual commutator.\(^*\)

2.2. Space-time

Here, we discuss the gravitational field arising from a rotating object. We now assume that the external field of the rotating object is described by the Kerr metric. If we restrict ourselves to the slowly rotating, weak gravitational field up to the first order in the angular velocity, which is related to the Kerr parameter $a$, and the Newtonian gravitational potential $\phi = -GM/r$, respectively, the line element is given by
\[
ds^2 \simeq \left(1 + 2 \frac{\phi}{c^2}\right) c^2 dt^2 + \frac{4GMa}{c^2 r^3} (x dy - y dx) dt
- \left(1 - 2 \frac{\phi}{c^2}\right) (dx^2 + dy^2 + dz^2),
\] (2.7)
where $a$ is expressed in terms of the mass $M$ and the angular momentum $J$ of the gravitational source:
\[
a \equiv \frac{J}{M}.
\] (2.8)

Assuming that the rotating object is a sphere of radius $R$ with uniform density, we have
\[
a \equiv \frac{J}{M} = \frac{2}{5} R^2 \omega,
\] (2.9)
where $\omega$ denotes the angular velocity of this object. (If the rotating object deviates from a sphere, or has an inhomogeneous density distribution, then the numerical factor $2/5$ might be changed by a factor of order unity.)

\(^*\) We have ignored a term proportional to the unit matrix.
Next, we turn our attention to time evolution of spinors. The covariant Dirac equation (2.1) has beautiful space-time symmetry. However, in order to investigate the time evolution of spinors, we must break the symmetrical form of this equation. For this purpose, we now use the (3+1) formalism. In the (3+1) formalism, the metric \(g_{\alpha\beta}\) is split as follows:

\[
\begin{align*}
g_{00} &= N^2 - \gamma_{ij} N^i N^j, \\
g_{0i} &= -\gamma_{ij} N^j \equiv -N_i, \\
g_{ij} &= -\gamma_{ij},
\end{align*}
\]

where \(N\) is the lapse function, \(N^i\) the shift vector, and \(\gamma_{ij}\) the spatial metric on the 3D hypersurface. Furthermore, we define \(\gamma^{ij}\) as the inverse matrix of \(\gamma_{ij}\). Using the metric (2.7) derived in the last section, we can write the lapse function, the shift vector and the spatial metric in the following way:

\[
\begin{align*}
N &= c \left(1 + \frac{\phi}{c^2}\right), \\
N^x &= \frac{4GM R^2}{5c^2 r^3} \omega y, \\
N^y &= -\frac{4GM R^2}{5c^2 r^3} \omega x, \\
N^z &= 0,
\end{align*}
\]

Furthermore, we define \(\gamma^{ij}\) as the inverse matrix of \(\gamma_{ij}\). Using the metric (2.7) derived in the last section, we can write the lapse function, the shift vector and the spatial metric in the following way:

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N^y &= -\frac{4GM R^2}{5c^2 r^3} \omega x, \\
N^z &= 0,
\end{align*}
\]

where \(\gamma_{ij}\) is defined as

\[
\gamma_{ij} = \left(1 - 2 \frac{\phi}{c^2}\right) \delta_{ij}.
\]

Furthermore, we choose the tetrad as follows:

\[
\begin{align*}
e_{(0)}^{\mu} &= \left(\frac{1}{N}, -\frac{N^i}{N}\right), \\
e_{(k)}^{\mu} &= \left(0, e_{(k)}^i\right),
\end{align*}
\]

where the spatial triad \(e_{(k)}^i\) is defined as

\[
\gamma_{ij} e_{(k)}^i e_{(l)}^j = \delta_{kl}.
\]

From Eqs. (2.11)–(2.13), we derive

\[
\begin{align*}
e_{(0)}^0 &= 1 - \frac{\phi}{c^2}, \\
e_{(0)}^1 &= -\frac{4GM R^2}{5c^2 r^3} \omega y, \\
e_{(0)}^2 &= \frac{4GM R^2}{5c^2 r^3} \omega x, \\
e_{(0)}^3 &= 0,
\end{align*}
\]
Using our choice of the tetrad, the covariant Dirac matrices $\gamma^\alpha$ are written as

$$\gamma^0 = \gamma^{(a)}e^{0\,(a)} = \gamma^{(0)} \frac{c}{N},$$

$$\gamma^i = \gamma^{(a)}e^{i\,(a)} = - \gamma^{(0)} \frac{c}{N} N^i + \gamma^{(j)} e^{i\,(j)}.$$  

Hence the covariant Dirac equation (2.1) is written as follows:

$$i\hbar \frac{\partial}{\partial t} \Psi = H\Psi = \left[ \left( \gamma^{(0)} \gamma^{(j)} c N e^{i\,(j)} - N^i \right) (\overline{\psi} + i\hbar \Gamma_i) + i\hbar \Gamma_0 + \gamma^{(0)} mc^2 N \right] \Psi,$$

where $\overline{\psi}_i$ is the momentum operator in flat space-time. This equation describes the time evolution of spinors. If we adopt the Weyl representation as the constant Dirac matrices in this equation, then for massless particles in flat space-time we derive the well-known Weyl equations

$$i\hbar \frac{\partial}{\partial t} \psi = \pm c \sigma \cdot \overline{\psi} \psi,$$

where $\psi$ denotes two-component spinors.

### §3. Ultra-relativistic limit

We now restrict our attention to the ultra-relativistic limit, which means that the rest energy of the particle is much smaller than the kinetic energy in the observer’s frame. In particular, we expand the energy the particle itself has up to $O(m^2c^4/pc)$.

Here, we shall obtain the ultra-relativistic Hamiltonian up to the order of our interest by performing a unitary transformation similar to the Foldy-Wouthuysen-Tani (FWT) transformation [15, 16].

First, following the discussion of Ref. [14], we redefine the spinor and the Hamiltonian in the following way:

$$\Psi' = \gamma^{1/4} \Psi, \quad H' = \gamma^{1/4} H \gamma^{-1/4},$$

where $\gamma$ is the determinant of the spatial metric:

$$\gamma = \det (\gamma_{ij}).$$

Since the invariant scalar product is

$$(\psi, \varphi) \equiv \int \overline{\psi}\varphi \sqrt{\gamma} d^3 x,$$
under this redefinition the scalar product becomes the same form as in flat space-time:

\[ \langle \psi', \varphi' \rangle \equiv \int \overline{\psi'} \varphi' d^3 x \] (3.4)

It is sometimes convenient to adopt this definition of the scalar product.

Next, we perform a unitary transformation to derive the ultra-relativistic Hamiltonian which is the “even” operator up to the order of our interest. From this, we have

\[ \tilde{H}' = U H' U^\dagger \]

\[ = \begin{pmatrix} H_R & 0 \\ 0 & H_L \end{pmatrix} + \left[ O \left( \frac{m^3 c^6}{p^2 c^2} \right) \text{ or } O \left( \frac{\phi^2}{c^4}, \omega^2 \right) \right]. \] (3.5)

The Dirac spinor is also divided into each of two-component spinors:

\[ \tilde{\Psi}' = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \] (3.6)

where the subscript \( R \) and \( L \) denote the right-handed and the left-handed component, respectively.

We pay attention to the left-handed component. Then we find that the equation for the left-handed component is given by

\[ i \hbar \frac{\partial}{\partial t} \psi_L = H_L \psi_L \]

\[ = - \left\{ 1 + \frac{1}{c^2} \left( \phi + \frac{\overline{\nabla} \cdot \phi}{p} \overline{p} + \frac{2GM}{r^3} \frac{L \cdot S}{p} \right) \right\} \frac{c \overline{p} \cdot \sigma \cdot \overline{p}}{p} \]

\[ - \frac{1}{c^2} \left( \frac{4GM R^2}{5r^3} \omega \cdot (L + S) + \frac{6GM R^2}{5r^5} S \cdot [r \times (r \times \omega)] \right) \]

\[ + \left\{ 1 + \frac{1}{4c^2} \left( \phi - \frac{1}{p^2} \phi p^2 + \frac{1}{p^2} \overline{p} \cdot \phi \overline{p} - \overline{p} \cdot \phi \overline{p} \right) \right\} \frac{m^2 c^2}{2p} \frac{\sigma \cdot \overline{p}}{p} \]

\[ + \frac{1}{8} m^2 c^2 \frac{1}{c^2} \left( A \frac{1}{p^2} - 2 \frac{\sigma \cdot \overline{p}}{p^2} A \frac{\sigma \cdot \overline{p}}{p^2} + \frac{1}{p^2} A \right) \] \( \psi_L, \) (3.7)

where

\[ A = \frac{4GM R^2}{5r^3} \omega \cdot (L + S) + \frac{6GM R^2}{5r^5} S \cdot [r \times (r \times \omega)]. \] (3.8)

The details of the calculations are given in Appendix \[A\]. From this, we find how the spin-orbit coupling, the coupling between the spin and the rotation of the gravitational source, or the coupling between the total angular momentum and the rotation is coupled to the non-vanishing mass.

In radial propagation, the orbital angular momentum vanishes. Therefore, in this case, the spin effects coupled only to the rotation appear. If we set \( \omega = 0 \), then there is no spin effect in radial propagation. This consequence is consistent with the previous work \[11\].
§4. An application

In this section, we consider an application of the two-component equation derived in the last section to neutrino oscillations in the presence of the gravitational field. In neutrino oscillations (see, e.g., Refs. 17, 18, 19) for the analysis in flat space-time), the most important one is the phase difference of the two different mass eigenstates. Hence we now concentrate on the phase shift of the particle.

4.1. Neutrino oscillation in Kerr space-time

We shall derive the phase shift directly from the two-component equation derived in the last section. Furthermore, for simplicity, we consider the radial propagation (r-direction), in which the spin-orbit coupling vanishes.

We now regard terms arising from the non-vanishing mass and the gravitational field as perturbations. Then the equation (3.7) for the left-handed component is considered as

\[ i\hbar \frac{\partial}{\partial t} \psi_L = (H_{0L} + \Delta H_L) \psi_L, \]  

where \( H_{0L} \) denotes the unperturbed Hamiltonian \( H_{0L} = -c\sigma \cdot \vec{p} \), and \( \Delta H_L \) the corrections arising from the non-vanishing mass and the gravitational field.

Here we assume that the spinor \( \psi_L \) is given by

\[ \psi_L(x,t) = e^{i\Phi(t)} \psi_{0L}(x,t), \]  

where \( \psi_{0L} \) satisfies the equation

\[ i\hbar \frac{\partial}{\partial t} \psi_{0L} = H_{0L} \psi_{0L}. \]  

Substituting Eq. (4.2) into Eq. (4.1) and using Eq. (4.3), we obtain

\[ \Phi = -\frac{1}{\hbar} \int^t \Delta H_L dt. \]  

In order to derive the gravitationally induced phases practically, we assume that corresponding to the left-handed component, \( \psi_{0L} \) satisfies the relation

\[ \sigma \cdot \vec{p} \psi_{0L}(x,t) = -\psi_{0L}(x,t). \]  

Furthermore, we here replace the q-numbers in \( \Delta H_L \) with the c-numbers. This is a kind of semi-classical approximation. From this, except for the spin effects, we derive the phase

\[ \Phi = -\frac{1}{\hbar} \int^t_{t_A} \left( 2\sigma \cdot \vec{p} \frac{\psi_{0L}(\vec{x},t)}{c^3} + \frac{m^2 c^3}{2p} \right) dt, \]  

where we have considered the case that the neutrino is produced at a space-time point \( A(t_A, r_A) \), and detected at a space-time point \( B(t_B, r_B) \). We now concentrate on the term related to \( m^2 \), because neutrino oscillations take place as a result of mass square difference. Let the two different mass eigenstates have common momentum \( p \).
and propagate along a same path. Then the relative phase $\Delta \Phi_{ij}$ of the two different mass eigenstates, $|\nu_i\rangle$ and $|\nu_j\rangle$, is given by

$$\Delta \Phi_{ij} = \frac{\Delta m_{ij}^2 c^3}{2 \hbar} \int_{r_A}^{r_B} \frac{1}{p} \, dt.$$  \hspace{1cm} (4.7)

Next, we discuss the spin-rotation coupling. In a similar way, we replace the q-numbers in terms concerning the spin of the particle in Eq. (3.7) with the c-numbers again. From this kind of semi-classical approximation, we find that the spin-rotation term coupled to $m^2$ vanishes. However, the spin-rotation effects in higher order terms may survive. This fact means that there is no influence of the spin-rotation coupling on neutrino oscillations at least up to the order related to $m^2$.

Consequently, in a radially propagating case, we obtain the phase difference (4.7) between the two different mass eigenstates as a final result.

### 4.2. Comparison with previous works

Finally, we compare the result obtained above with the previous works. For this purpose, we consider the Schwarzschild limit (i.e., $\omega \to 0$), which leads to the result (4.7) again.

First, we assume the “background” neutrino trajectory as the radial null geodesics:

$$0 = ds^2 = \left(1 + 2 \frac{\phi}{c^2}\right) c^2 dt^2 - \left(1 - 2 \frac{\phi}{c^2}\right) dr^2.$$  \hspace{1cm} (4.8)

Then we obtain

$$dt \simeq \frac{1}{c} \left(1 - 2 \frac{\phi}{c^2}\right) dr.$$  \hspace{1cm} (4.9)

Hence, if we transform the integral (4.7) with respect to $t$ to that with respect to $r$, then the relative phase $\Delta \Phi_{ij}$ is given by

$$\Delta \Phi_{ij} = \frac{\Delta m_{ij}^2 c^3}{2 \hbar} \int_{r_A}^{r_B} \frac{1}{pc} \left(1 - 2 \frac{\phi}{c^2}\right) dr.$$  \hspace{1cm} (4.10)

The second term in the round brackets corresponds to the gravitational correction as indicated by Ahluwalia and Burgard\textsuperscript{3}) Indeed, under the assumption that the tetrad component of the radial momentum $p_\gamma = e_\gamma^\mu p_\mu$ is constant along the trajectory, we can obtain the same expression as that in Ref.\textsuperscript{3}). (Note that Ahluwalia and Burgard\textsuperscript{3}) assume $p_\gamma c$ as the energy of the neutrino.)

Next, let us see whether our result (4.7) reproduces the other form of the results\textsuperscript{5, 7, 8}). From the mass shell condition $g_{\mu\nu} p_\mu p_\nu = m^2 c^2$, $p$ is related with the energy $E(\equiv p_t c)$ in the following way:

$$pc = \left(1 - 2 \frac{\phi}{c^2}\right) E + \left[O(m^2) \text{ or } O(\phi^2)\right].$$  \hspace{1cm} (4.11)

Under the assumption that $E$ is constant along the trajectory, we finally obtain

$$\Delta \Phi_{ij} = \frac{\Delta m_{ij}^2 c^3}{2 \hbar} \int_{r_A}^{r_B} \frac{dr}{E}. $$
\[ = \frac{\Delta m_i^2 c^3}{2 \hbar E} (r_B - r_A), \]  

which is the same result obtained in the previous works.\cite{3, 5, 7} In this sense, our result Eq. (4.7) contains both of the previous expressions. Our analysis here clearly shows that the controversy in appearance between the result of Ahluwalia and Burgard\cite{3} and that of others\cite{5, 7, 8} simply comes from the different assumptions of constancy along the neutrino trajectory.

§5. Summary and conclusion

We have studied the general relativistic effects of gravity on spin-1/2 particles with non-vanishing mass. In particular, we have considered the particles propagating in the Kerr geometry in the slowly rotating, weak field approximation. By performing a unitary transformation similar to the FWT transformation, we have obtained the two-component Weyl equations with the corrections arising from the non-vanishing mass and the gravitational field from the covariant Dirac equation. The Hamiltonian clearly shows how the spin-orbit coupling, the spin-rotation coupling or the coupling between the total angular momentum and the rotation is coupled to the non-vanishing mass.

Furthermore, we have discussed an application of the two-component equations to neutrino oscillations in the presence of the gravitational field, and derived the phase difference of the two different mass eigenstates in radial propagation. It is worthwhile mentioning that our result contains both of the previous expressions. It is manifestly shown that the controversy in appearance between the result of Ahluwalia and Burgard\cite{3} and that of others\cite{5, 7, 8} simply comes from the different assumptions of constancy along the neutrino trajectory. Moreover, as seen in the transformation of the integral variable, the gravitationally induced neutrino oscillation phases arise from the modification of the propagating distance. Indeed, we found that the gravitational correction term comes out in this variable transformation.

We have not applied our approach to a non-radially propagating case in detail in this paper. However, it is of interest whether the spin-orbit coupling affects the neutrino oscillations. It will be the subject of further investigation.

Although it seems to be difficult to provide the verification of these effects with current experimental detectability, we think that the investigation of the topics in which both quantum effects and gravitational effects come into play is important. Progress in technology may make the verification of the effects possible.

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A.1. Components of spin connection

The spin connection is given by Eq. (2.6):

$$\Gamma_\mu = -\frac{1}{8} \left[ \gamma^{(a)}, \gamma^{(b)} \right] g_{\lambda\sigma} \epsilon^{(a)}_\lambda \nabla_\mu \epsilon^{(b)}_\sigma$$  \hspace{1cm} (A.1)

It is convenient to introduce the following $4 \times 4$ matrices similar to the Pauli spin matrices:

$$\rho_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$  \hspace{1cm} (A.2)

where $I$ is the $2 \times 2$ unit matrix. These matrices satisfy the relations

$$\rho_i \rho_j = \delta_{ij} + i\varepsilon_{ijk} \rho_k,$$  \hspace{1cm} (A.3)

where $\varepsilon_{ijk}$ is the Levi-Civita antisymmetric tensor ($\varepsilon_{123} = +1$). If we adopt the Weyl representation as the constant Dirac matrices, then we have

$$\gamma^{(0)} = \frac{1}{c} \rho_1, \quad \gamma^{(i)} = -i \rho_2 \sigma_i,$$  \hspace{1cm} (A.4)

where $\sigma$ are the Pauli spin matrices.

Using the above quantities, the components of spin connection, up to the order of our interest, are given by

$$i\hbar \Gamma_0 = \frac{1}{2c} \rho_3 \sigma \cdot (\overline{\rho} \phi)$$

$$+ \frac{1}{c^2} \left[ \frac{4GMR^2}{5r^3} \omega \cdot S + \frac{6GMR^2}{5r^5} S \cdot [r \times (r \times \omega)] \right],$$  \hspace{1cm} (A.5)

$$i\hbar \Gamma_1 = -\frac{\hbar}{2c^2} (\phi, 2 \sigma_3 - \phi, 3 \sigma_2)$$

$$+ i \frac{\hbar}{c} \rho_3 \frac{3GMR^2}{5r^5} \omega \left[ -2xy \sigma_1 + (x^2 - y^2) \sigma_2 - yz \sigma_3 \right],$$  \hspace{1cm} (A.6)

$$i\hbar \Gamma_2 = -\frac{\hbar}{2c^2} (\phi, 3 \sigma_1 - \phi, 1 \sigma_3)$$

$$+ i \frac{\hbar}{c} \rho_3 \frac{3GMR^2}{5r^5} \omega \left[ (x^2 - y^2) \sigma_1 + 2xy \sigma_2 + zx \sigma_3 \right],$$  \hspace{1cm} (A.7)

$$i\hbar \Gamma_3 = -\frac{\hbar}{2c^2} (\phi, 1 \sigma_2 - \phi, 2 \sigma_1)$$

$$+ i \frac{\hbar}{c} \rho_3 \frac{3GMR^2}{5r^5} \omega \left[ -yz \sigma_1 + zx \sigma_2 \right].$$  \hspace{1cm} (A.8)

A.2. Unitary transformation

The Hamiltonian defined in Eq. (2.21) is given by

$$H = \rho_3 c \sigma \cdot \overline{p} + \rho_3 \left[ -\frac{1}{2} c \sigma \cdot \left( \overline{\rho} \frac{\phi}{c^2} \right) + \frac{2}{c^2} \phi c \sigma \cdot \overline{p} \right]$$
Moreover, the Hamiltonian redefined by Eq. (3.1) is then

\[ H' = \rho_3 c \sigma \cdot \mathbf{p} + \rho_3 \left( c \sigma \cdot \mathbf{p} \frac{\phi}{c^2} + \frac{\phi}{c^2} c \sigma \cdot \mathbf{p} \right) + \frac{1}{c^2} A + \rho_1 mc^2 \frac{\phi}{c^2} \]

where terms proportional to \( \rho_1 \) or \( \rho_2 \) are “odd”, whereas those proportional to \( \rho_3 \) are “even”.

Next, by performing a unitary transformation similar to the FWT transformation, let us derive the ultra-relativistic Hamiltonian for the left-handed component. We here divide the unitary transformation into several steps. First, we use the unitary operator

\[ U_1 = \exp \left( i \rho_2 \frac{1}{2} mc^2 \frac{c \sigma \cdot \mathbf{p}}{c^2 \mathbf{p}^2} \right), \quad (A.11) \]

which is introduced to eliminate the odd term \( \rho_1 mc^2 \). Using the useful formula

\[ e^{iS} He^{-iS} = H + i [S, H] + \frac{i^2}{2!} [S, [S, H]] + \frac{i^3}{3!} [S, [S, [S, H]]] + \cdots \quad (A.12) \]

and the relation \( \text{(A.8)} \), we obtain the transformed Hamiltonian

\[ U_1 H' U_1^\dagger \]

\[ = \rho_3 c \sigma \cdot \mathbf{p} + \rho_3 \left( c \sigma \cdot \mathbf{p} \frac{\phi}{c^2} + \frac{\phi}{c^2} c \sigma \cdot \mathbf{p} \right) + \frac{1}{c^2} A + \rho_1 mc^2 \frac{\phi}{c^2} \]

\[ - \rho_1 \frac{1}{2} mc^2 \left[ c \sigma \cdot \mathbf{p} \left( c \sigma \cdot \mathbf{p} \frac{\phi}{c^2} + \frac{\phi}{c^2} c \sigma \cdot \mathbf{p} \right) + \frac{\phi}{c^2} c \sigma \cdot \mathbf{p} \right] \]

\[ + i \rho_2 \frac{1}{2} mc^2 \left( c \sigma \cdot \mathbf{p} A - A \frac{c \sigma \cdot \mathbf{p}}{c^2 \mathbf{p}^2} \right) + \rho_3 \frac{1}{2} mc^2 c^4 \frac{c \sigma \cdot \mathbf{p}}{c^2 \mathbf{p}^2} \]

\[ - \rho_3 \frac{1}{8} mc^4 \left( A \frac{1}{c^2 \mathbf{p}^2} - \frac{2 c \sigma \cdot \mathbf{p} A}{c^2 \mathbf{p}^2} + \frac{1}{c^2 \mathbf{p}^2} \right), \quad (A.13) \]

where \( A \) is given by Eq. \( \text{(A.8)} \):

\[ A = \frac{4GM R^2}{5c^3} \omega \cdot (L + S) + \frac{6GM R^2}{5c^5} S \cdot [r \times (r \times \omega)]. \quad (A.14) \]

Second, in order to eliminate the second line in Eq. (A.13), we use the unitary operator

\[ U_2 = \exp \left[ -i \rho_2 \frac{1}{2} mc^2 \left( \frac{c \sigma \cdot \mathbf{p}}{c^2 \mathbf{p}^2} + \frac{\phi}{c^2} c \sigma \cdot \mathbf{p} \right) \right]. \quad (A.15) \]
Finally, we use the two unitary operators $U_3 = e^{iS_3}$ and $U_4 = e^{iS_4}$ where $S_3$ and $S_4$ satisfy, respectively, the relations

$$i [S_3, \rho_3 \sigma \cdot \vec{p}] = -\rho_1 mc^2 \frac{\phi}{c^2},$$

(A.17)

$$i [S_4, \rho_3 \sigma \cdot \vec{p}] = -i \rho_2 \frac{1}{2} mc^2 \left( \frac{\sigma \cdot \vec{p} A}{c^2 p^2} - \frac{A \sigma \cdot \vec{p}}{c^2} \right).$$

(A.18)

We here assume the existence of these unitary operators, which make the remaining odd terms vanish. (We need not find the concrete forms of these unitary operators, because extra terms arising from these unitary transformations are higher order terms.) Using these unitary operators, we obtain the transformed Hamiltonian $UHU^\dagger$ which is even up to the order of our interest:

$$UHU^\dagger = \rho_3 \sigma \cdot \vec{p} + \rho_3 \left( \frac{\sigma \cdot \vec{p}}{c^2} \frac{\phi}{c^2} + \frac{\phi}{c^2} \sigma \cdot \vec{p} \right) + \frac{1}{c^2} A$$

$$+ \rho_2 \frac{1}{2} m^2 c^4 \frac{\sigma \cdot \vec{p}}{c^2 p^2}$$

$$- \rho_3 \frac{1}{8} m^2 c^4 \left[ \frac{1}{c^2 p^2} \frac{\phi}{c^2} \sigma \cdot \vec{p} + \frac{\phi}{c^2} \sigma \cdot \vec{p} \frac{1}{c^2} \right]$$

$$- \rho_3 \frac{1}{8} m^2 c^4 \left( \frac{A}{c^2} \frac{1}{c^2 p^2} - \frac{2}{c^2 p^2} \frac{A \sigma \cdot \vec{p}}{c^2} + \frac{1}{c^2} A \right),$$

(A.19)

where $U$ is given by $U = U_4 U_3 U_2 U_1$. From this, we can derive Eq. (3.7) by simple calculation.

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