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On Quantization with an Indefinite Metric and the Lee Model.

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Summary. — The Lee model is discussed again in the case already considered by PAULI and KÄLLÉN *i.e.* when the characteristic equation of the eigen-values of the energy has only real and all different roots. It is proved that introducing a suitable redefinition of the ingoing and outgoing states it is possible to construct an unitary S -matrix having reasonable physical properties. The suggested procedure is in some way similar to a renormalization which eliminates the ghost effects.

1. — Introduction.

PAULI and KÄLLÉN ⁽¹⁾, studying the Lee model with a cut-off in the interaction, have pointed out that when the coupling constant is greater than a critical value depending from the cut-off, it is no more possible to use a positive definite metric for the quantization of the model and to renormalize with the conventional prescription.

The renormalization is possible only if one uses an indefinite metric. In this case, however, one obviously runs into difficulties with the physical interpretation.

PAULI and KÄLLÉN have examined the case in which the roots of the characteristic equation which determines the eigenvalues of the energy are all real and in the sub-sector $\Theta N, V$, all different.

After that, GLASER and KÄLLÉN ⁽²⁾ have taken into consideration the case

⁽¹⁾ G. KÄLLÉN and W. PAULI: *Mat. Fys. Medd. Dan. Vid. Selsk.*, **30**, no. 7 (1955).

⁽²⁾ V. GLASER and G. KÄLLÉN: *Nucl. Phys.*, **2**, 706 (1956).

of two complex conjugate roots and more recently HEISENBERG ⁽³⁾ has studied the case of two coincident roots. Lately, PAULI, and ASCOLI and MINARDI ⁽⁴⁾ have again considered the case of the complex conjugate roots.

However, nobody has treated again the case of real distinct roots, as it is generally considered completely settled by the first work of PAULI and KÄLLÉN.

Of course, the interest is not in the Lee model itself, but in the question whether it is possible still to give a physical interpretation in the case of a system for which it is necessary to use an indefinite metric without too drastic change to the usual rules of quantum mechanics. (This aspect of the problem has been particularly stressed by BOGOLJUBOV ⁽⁵⁾ and by ASCOLI and MINARDI).

The interest of the Lee model lies in the fact that one may try on it general prescriptions which may be invented in order to overcome the difficulties of interpretation of the indefinite metric. It is the purpose of this work to discuss some possibilities in this sense and to show that they may be even applied to the case « without hope » of the two real distinct roots.

Obviously if one wants to maintain the usual relationship between probabilities and state vectors, one cannot use directly the vectors in the « pseudo Hilbert space » with an indefinite metric. Quite generally one will have to establish an homomorphism between the vectors of the pseudo Hilbert space $\mathcal{H}_1$ (with an indefinite metric) and vectors of a Hilbert space $\mathcal{H}_2$ (with a definite positive metric). For the vectors of this Hilbert space it will be possible to apply the usual rules of interpretation.

All the difficulty is in establishing the appropriate homomorphism. Of course, there may be infinite ways to perform this task. In order to be a bit more definite, we shall suppose in general that the homomorphism in question will be a so-called « operator » homomorphism. That is to say it will conserve linear relationships between operators. But even so we have to restrict the field of possibilities with more hypotheses. We shall suppose that our system has an energy operator and that the eigenstates of the energy operator have either positive or negative norm. We shall therefore discard for the moment the case considered by HEISENBERG, in which there are eigenstates with norm equal to 0.

It is then immediately seen that, even disregarding the difficulties put in evidence by PAULI and KÄLLÉN, we cannot consider the possibility of using the full Hilbert space in the usual manner for physical interpretation if we still want to maintain that the (total) energy is an observable. In fact, if we

⁽³⁾ W. HEISENBERG: *Nucl. Phys.*, **4**, 532 (1957).

⁽⁴⁾ W. PAULI: *Proceedings of 1958 Annual International Conference on High Energy Physics at CERN* (1958), p. 127.

⁽⁵⁾ N. N. BOGOLJUBOV: *Proceedings of 1958 Annual International Conference on High Energy Physics at CERN* (1958), p. 129.

consider a vector which has a projection different from 0 on an eigenstate of the energy with a negative norm, and if we suppose that this vector represents a state and is, as usual, normalized to 1, we shall immediately find, applying the usual rules of interpretation, that the probability of finding an eigenvalue of the energy corresponding to any eigenstate with positive norm is greater than one. In other words, a measurement of the energy does present the same difficulties that have been pointed out by PAULI and KÄLLÉN for the « scattering experiments ».

This very obvious remark seems to suggest a very simple prescription: « All state vectors corresponding to a physical situation should have component equal to 0 in respect to all eigenstates of the energy of the interacting system with negative norm ». This prescription is obviously compatible with the equation of motion, at least in the case in which the equations of motion are the usual ones.

However, with this prescription alone, one runs again immediately into difficulties in defining the « ingoing » and « outgoing » states necessary for talking of the « scattering experiments ». In fact, as by hypothesis the eigenstates with positive norm of the energy do not form a complete set in the Hilbert space, they are not sufficient in general to express the « ingoing » and « outgoing » states if these states are defined in the usual way.

In order to define the ingoing and outgoing states, it is necessary to make use of the interpretation of the state function in the configuration space and namely to consider the behaviour of the state function amplitude for very great distances. We shall talk in this respect of « asymptotical localization ».

If it is possible to establish a 1-1 (isomorphic) correspondence C between the eigenstate of the energy with positive norm and the vectors of a Hilbert space $\mathcal{H}_2$ (of positive definite metric) in which the behaviour of the free particles is described, it might still be possible to define ingoing and outgoing states for our problem if the isomorphic correspondence C does satisfy the following properties:

- 1) It is a linear correspondence,
- 2) It leaves invariant scalar products (*).
- 3) The Hilbert space $\mathcal{H}_2$ is by definition the closure of the linear manifold formed by the vectors corresponding to the eigenvectors of the energy of positive norm (**). It is required that the Hilbert space $\mathcal{H}_2$ so defined be suitable for describing ingoing and outgoing states.

(*) This condition is not strictly necessary for all vectors. It will be sufficient that a similar condition holds only for vectors representing states in which the particles are « asymptotically localized ».

(**) The space so defined has obviously the property of « completeness ».

These conditions are not yet sufficient to allow the definition of an S -matrix and furthermore of an S -matrix having suitable physical properties. It is the particular aim of this paper to show that it is possible to find such correspondence allowing the definition of a sufficiently satisfactory S -matrix, at least in the case of the Lee model, with real distinct roots of the characteristic equation.

At this point we want to point out that the procedure which we have envisaged is perhaps less arbitrary than it might appear at first sight. In fact, for describing experiments, one has to make use essentially of the configuration space (*).

On the other hand, in a relativistic theory one cannot talk of localization of particles at all, but at most of localization of currents and fields. Even in the conventional theory it is not a trivial matter to overcome this difficulty and for this purpose it is necessary to use the concept of the renormalization, etc. But the quantization with the indefinite metric is just interesting because it probably allows to introduce « non-locality » in an essential way. It is therefore conceivable that in a theory with indefinite metrics the concept of « asymptotical localization » has to be reconsidered carefully again and redefined in a suitable manner. That is what is meant by the correspondence C .

For the purpose of this paper, a slightly modified version of the Lee model will be used, which allows to write down explicitly the solution of the eigenstate problem of the energy even in the sector $(\theta, \theta'N, \theta V)$. It will be shown in Appendix II that no essential feature of the Lee model is missed or modified in this way.

The study of this new version will be presented in the second and third paragraph. In the fourth paragraph we shall give an example of the isomorphic correspondence C valid for the modified version of the Lee model, and in the fifth paragraph we shall discuss the possibility of giving a general criterium for constructing such isomorphic correspondence.

2. - « One band » Lee model.

The model which we would like to study in this paragraph will not give rise to the usual difficulty of PAULI and KÄLLÉN for a rather trivial reason. However, it will not be useless to consider this model because it might throw some light on the mechanism of these difficulties and on the manner by which, following the general scheme of the introduction, they might be avoided.

(*) The use of ingoing and outgoing stationary states belonging to the continuum is a schematization which strictly speaking does not correspond by itself to any really possible physical situation, and even mathematically may be objectionable.

As usually, we start with the Hamiltonian of the Lee model in the following form (*)

$$(2.1) \quad H = H_0 + H_{\text{int}},$$

$$(2.1') \quad H_0 = m(N^2 \psi_V^* \psi_V + \psi_N^* \psi_N) + \sum_k \omega_k a_k^* a_k,$$

$$(2.1'') \quad H_{\text{int}} = -\frac{g}{\sqrt{2\Omega}} \left(\psi_V^* \psi_N \sum_i \frac{f_i}{\sqrt{\omega_i}} a_{ki} + \sum_i \frac{f_i}{\sqrt{\omega_i}} a_{ki}^* \psi_N^* \psi_N \right) - \Delta m N^2 \psi_V^* \psi_V.$$

The Hamiltonian is already expressed by means of the renormalized quantities. It is

$$(2.2) \quad \psi_V = \frac{1}{N} \psi_V \text{ not ren.},$$

$$(2.2') \quad \{\psi_V \psi_V^*\}_+ = \frac{1}{N^2},$$

$$(2.2'') \quad (0 | \psi_V V) = 1,$$

where, with the notation V) we indicate the state in which is present one *physical* V particle, and only one physical V particle. For the other notations, compare with PAULI and KÄLLÉN ⁽¹⁾. f_i indicates a «form factor». We shall investigate here a model in which f_i will be different from zero only in a narrow neighbourhood of a certain frequency ω . For the purpose of this paragraph we shall assume that f_i is constant and equal to one for $\omega_0 \leq \omega_i \leq \omega_0 + \Delta\omega$ and that it is zero everywhere else. For other purposes, we shall also investigate the case in which f_i is given by a Gaussian function.

For the reader's convenience, we shall recall here some of the known things about the Lee model and of the results of PAULI and KÄLLÉN which can obviously be applied even to the present model.

We shall use the following notation:

I_V) is a state in which there is a bare V only.

I_N) is a state in which there is a bare N only.

$I_V I_{k_i}$) is a state in which is contained one bare V particle and one θ particle of momentum k_i .

$I_N I_{k_i}$) is a state in which is contained one N particle and one θ particle of momentum k_i only.

etc.

(*) With ψ_V^* , ... we mean the adjoint of ψ_V , ...

Then we have that the state representing a physical V particle is given by

$$(2.3) \quad V) = C[I_V) + \sum \Phi_i \cdot I_N I_{k_i})]$$

(the Φ_i are easily determined but here we are not interested in writing the Φ_i explicitly).

As a consequence of (2.2'') PAULI and KÄLLÉN have proved that $C = N$. One gets also (with the condition that the mass of the physical V is equal to the mass of the N):

$$(2.4) \quad \Delta m = -\frac{g^2}{2N^2\Omega} \sum \frac{f_i^2}{\omega_i^2},$$

$$(2.5) \quad N^2 = 1 - \frac{g^2}{2\Omega} \sum \frac{f_i^2}{\omega_i^3}.$$

When

$$g^2 > \frac{2\Omega}{\sum (f_i^2/\omega_i^3)},$$

one gets $N^2 < 0$ and then, as PAULI and KÄLLÉN have pointed out, it is necessary to use an indefinite metric. The prescription is to normalize the state to -1 if it contains an odd number of bare V particles.

For the sector $(V, \theta N)$ one gets the following equation for the eigenvalue of the energy

$$(2.6) \quad h(z) = z \left(1 - \frac{g^2 z}{2\Omega} \sum \frac{f_i^2}{\omega_i^2(z - \omega_i)} \right) = 0.$$

It is supposed that the system is enclosed in the finite volume Ω ; the number of the values of ω_i for which $f_i \neq 0$ is therefore finite. Let it be n . Then $f_i \neq 0$ only for $i = 0, 1, \dots, n-1$. It is then easy to see that Eq. (2.6) has exactly $n-1$ roots between ω_0 and ω_{n-1} . It has indeed one root (and only one) between ω_{i-1} and ω_i , where i is going from 1 to $n-1$. If g is smaller than g_c there exists one root and only one which is larger than ω_{n-1} , and no root between 0 and ω_0 or smaller than 0.

If g is greater than g_c it exists one root and only one which is smaller than 0, and no root between 0 and ω_0 or greater than ω_{n-1} . In all cases, of course, there exists the root $z = 0$; therefore in all cases Eq. (2.6) has $n+1$ real roots (and only real roots). This result has to be expected for the completeness because in the sector $(V, \theta N)$ we have just $n+1$ independent states for the not interacting system, namely, one V bare particle state and $n(\theta, N)$ free particle states, one for each energy ω_i ($i = 0, \dots, n-1$) of the free θ particle.

So in the case of a finite volume in ordinary space and of a finite band in which the θ interacts with the heavy particles, the question of completeness becomes entirely trivial as the Hilbert space which one has to consider has a finite number of dimensions (*). This circumstance makes entirely obvious that already in the section $(V, \theta N)$ there is some trouble coming from the indefinite metric (if $g > g_c$) if one likes to express, for instance, all the conventionally possible states in which initially a N and a θ particle are present, because it will not be possible to do that, without using also the eigenstate V_- of the energy with negative eigenvalue.

Now PAULI and KÄLLÉN have shown that in case of $g > g_c$ the eigenstate of the energy corresponding to the negative root of Eq. (2.6) has a negative norm. On the other hand, as already remarked, in the introduction, one cannot allow states with negative norm energy eigenstates components, if the energy is still to be considered observable. It is therefore apparent that already in the section $(V, \theta N)$ one has to put some restriction on the *physically* possible state of the θ particle if one likes to avoid troubles with the indefinite metric. Under this respect the $(V, \theta N)$ section is entirely similar to the higher section as for instance the $(\theta V, \theta \theta' N)$ section.

The essential difference between the $(V, \theta N)$ section and the higher section is that the restrictions which one has to impose on the possible states do not affect the even conventionally defined scattering states in the $(V, \theta N)$ sector, but they frequently do affect the conventionally defined scattering states on the higher sectors.

We have now to take into consideration the sector $(\theta V, \theta \theta' N)$. The eigenstates of the energy for such a sector will be represented by

$$(2.7) \quad \varphi = \sum N \Phi_i I_V I_{k_i} + \sum \Phi_{st} I_N I_{ks} I_{kl}.$$

This state has to satisfy the equations

$$(2.7') \quad ZN\Phi_i = (\omega_i - \Delta m)N\Phi_i - \frac{g}{\sqrt{2\Omega}} \cdot \frac{2}{N} \cdot \sum_r \Phi_{ir} \frac{f_r}{\sqrt{\omega_r}},$$

$$(2.7'') \quad (Z - \omega_r - \omega_i)\Phi_{ri} = -\frac{g}{\sqrt{2\Omega}} \cdot \frac{1}{2} \left(\Phi_i \frac{f_r}{\sqrt{\omega_r}} + \Phi_r \frac{f_i}{\sqrt{\omega_i}} \right).$$

(*) Of course we might also consider the other values of the momentum of the θ -particle for which one has no interaction and which, of course, are infinite in number. However, these states do not enter in any essential way in our consideration (in this case there is a completely trivial 1-1 correspondence between the state of the not-interacting system and the state of the interacting system).

We shall study explicitly the complete system of solutions of Eqs. (2.7') and (2.7''). This is actually possible with our model if one expands in powers of the parameter $\Delta\omega/\omega_0$. The system of the equations which we have to solve is equivalent to

$$(2.7''') \quad h(z - \omega_i)\Phi_i = \frac{g^2}{2\Omega\sqrt{\omega_i}} \sum_r \frac{\Phi_r}{\sqrt{\omega_r}} \cdot \frac{1}{z - \omega_i - \omega_r}.$$

Let us put

$$(2.8') \quad \begin{cases} \omega_{i+1} = \omega_0(1 + l\varepsilon), \\ z = \omega_0(1 + \xi). \end{cases}$$

and

$$(2.8'') \quad \gamma_i = (1 - \xi) \frac{2\Omega\omega_0^2}{g^2} \cdot h(z - \omega_i).$$

We now remark that

$$\frac{g^2}{2\Omega} \cdot \frac{1}{\omega_0^2} \approx \frac{1 - N^2}{n} \approx 0 \left(\frac{1}{n} \right).$$

We have then to distinguish 3 cases:

$$(A) \quad |1 - \xi| \gtrsim \frac{1}{n}; \quad z - \omega \neq z_i + 0(\Delta\omega),$$

where z_i is any root of Eq. (2.6).

$$(B) \quad |1 - \xi| \gtrsim \frac{1}{n}; \quad z - \omega_0 = z_i + 0(\varepsilon\omega_0),$$

$$(C) \quad |1 - \xi| \ll \frac{1}{n}.$$

Case C) is verified when

$$z - \omega_0 = \omega_0 + 0(\varepsilon\omega_0)$$

and it represents the scattering of θ, θ' on a N particle. It never gives rise to any difficulty in our model. Therefore we shall not consider it in detail. We shall have to consider only the cases A) and B). In both cases $|1 - \xi| \gtrsim 1/n$ and therefore we might develop in $\Delta\omega/\omega_0$ and, neglecting terms

$0(\Delta\omega^2/\omega_0^2)$ one gets (cf. (2.7''')):

$$\sum_0^{n-1} \left\{ 1 - \left(\frac{l}{2} + \frac{1}{1-\xi} \right) (l+s)c + \left[\frac{(s+l)^2}{2(l-\xi)} + \frac{1}{2} (sl^2 + sl^2 + 2sl) + \frac{(1+\xi)^2}{(1-\xi)^2} \right] \varepsilon^2 \right\} \Phi_l = -\gamma_1 \Phi_s.$$

We put now

$$(2.9) \quad \chi_l = \gamma_l \Phi_l, \quad c = \frac{1}{2} + \frac{1}{2(1-\xi)} + \frac{1}{(1-\xi)^2}.$$

Then, with obvious manipulation, we get easily from the preceding equation:

$$(2.10) \quad \begin{cases} \chi_s - 2\chi_{s+1} + \chi_{s+2} + 2c\varepsilon^2 \sum \Phi_l = 0 & s = 0, \dots, n-3, \\ \chi_0 - \chi_1 + \sum_l \left(\frac{1}{2} + \frac{1}{1-\xi} \varepsilon - \left[\frac{2l+1}{2(1-\xi)} + \frac{2l+1}{(1-\xi)^2} + \frac{2l+3}{8} \right] \varepsilon^2 \right) \Phi_l = 0, \\ \chi_0 + \sum_l \left(1 - \left(\frac{l}{2} + \frac{l}{1-\xi} \right) \varepsilon + \frac{l^2 \varepsilon^2}{(1-\xi)^2} + \frac{l^2 \varepsilon}{2(1-\xi)} + \frac{3}{8} l^2 \varepsilon^2 \right) \Phi_l = 0. \end{cases}$$

From the last equation we get

$$(2.10') \quad \sum \Phi_l = -\chi_0 + 0 \left(\frac{\Delta\omega}{\omega_0} \right),$$

and therefore, neglecting again terms of the order $0(\Delta\omega^3/\omega_0^3)$ the first $n-2$ equations (2.10) can be written in the following form

$$(\chi_{l+2} - \chi_{l+1}) - (\chi_{l+1} - \chi_l) = 2c\varepsilon^2 \chi_0.$$

and the system (2.10) is then equivalent to:

$$(2.10'') \quad \begin{cases} \chi_l = \chi_0 (1 + cl(l-1)\varepsilon^2) + l(\chi_1 - \chi_0), \\ \chi_1 - \chi_0 = \left(\frac{1}{2} + \frac{1}{1-\xi} \right) \varepsilon \sum \Phi_l - \varepsilon^2 \sum \left[\frac{2l+1}{2(1-\xi)} + \dots \right] \Phi_l. \end{cases}$$

Now, we consider in detail case A). In case A) $z - \omega_l$ differs from the roots of Eq. (2.6) for quantities which are great compared to $\Delta\omega$. Therefore we may develop $1/\gamma_l$ and we may write

$$(2.11) \quad \frac{1}{\gamma_l} = \frac{1}{\gamma_0} + 0 \left(\frac{\Delta\omega}{\omega} \right).$$

Keeping into account Eqs. (2.9) and (2.11) one gets, with the usual approximation from Eq. (2.10')

$$n/\gamma_0 = -1.$$

The eigenvalue equation in the case (A) for which Eq. (2.11) is valid, is given (neglecting terms $0(\Delta\omega/\omega)$) by

$$(2.12) \quad \gamma_0(z - \omega_0) = -\frac{1}{n}.$$

It is easy to recognize that Eq. (2.12), together with the condition $(z - \omega_0) \neq z_- + 0(\Delta\omega)$, can be satisfied by two values of $(z - \omega_0)$ only. These values are both negative but greater than z_- . To the smaller of these two values does correspond a negative norm and to the greater one, a positive norm. It remains now to consider simply case (B). In case (B) we have two possibilities:

$$(B_1) \quad z - \omega_0 = z_- + 0(\Delta\omega)$$

(z_- is the energy eigenvalue of the «ghost state» V_-);

$$(B_2) \quad (z - \omega_0) = 0(\Delta\omega)$$

($z - \omega_0 = \omega_0$ is the case (C)).

Let us consider the case (B₁). Let us put

$$(2.13) \quad \xi = \frac{z_-}{\omega} + \eta\varepsilon = \xi_- + \eta\varepsilon.$$

Then:

$$h(z - \omega_l) = h(z_- + \omega_0(\eta - l)\varepsilon) = h'_{z=z_-} \cdot (\eta - l)\omega_0\varepsilon + \frac{1}{2}h''_{z=z_-} \cdot (\eta - l)^2\omega_0^2\varepsilon^2 + \dots$$

and

$$\gamma_l = \bar{\gamma}\varepsilon(\eta - l)(1 + \alpha(\eta - l)\varepsilon) + \dots,$$

where

$$\bar{\gamma} = (1 - \xi_- - \eta\varepsilon) \frac{2\Omega}{g^2} \omega_0^3 h'_{z=z_-},$$

$$\alpha = \frac{1}{2} \left(\frac{h''}{h'} \right)_{z=z_-}.$$

Then

$$\frac{1}{\gamma_l} = -\frac{1}{\bar{\gamma}\varepsilon} \cdot \frac{1}{l - \eta} - \frac{\alpha}{\bar{\gamma}} + 0\left(\frac{\Delta\omega}{\omega_0}\right).$$

We may now use this expression, together with Eqs. (2.9) and (2.10''), and we get easily that the equation for the eigenvalues of the energy in case (B₁) might be written in the following form

$$(2.14) \quad \left(-\frac{1}{\bar{\nu}} + 0 \left(\frac{\Delta\omega}{\omega_0} \right) \right) \sum_l \frac{1}{l - \eta} = 0 \left(\frac{\Delta\omega}{\omega_0} \right).$$

Eq. (2.14) in 0-approximation becomes

$$(2.14') \quad \sum_l \frac{1}{l - \eta} = 0.$$

Eq. (2.14') has exactly $n - 1$ roots. There are therefore $n - 1$ eigenvalues of the energy corresponding to the case (B₁). We shall discuss briefly later the eigenfunction of the energy for case (B₁).

Case (B₂) is completely analogous to case (B₁). Here

$$z - \omega_0 = 0(\Delta\omega)$$

and using the same kind of development as in case (B₁) we get that there are again $n - 1$ eigenvalues of the energy which satisfy the requirements of case (B₂).

We may discuss now the eigenfunctions. Let us write down the form of the eigenfunction for case (B₂). If we put

$$\eta_m = m + \eta'_m; \quad v < \eta'_m < 1; \quad m = 0, \dots, n - 2.$$

where η_m is one of the roots of Eq. (2.14'), and if we remember Eqs. (2.7), (2.7') and (2.7''), we find out easily that these $n - 1$ eigenfunctions have the form:

$$(2.15) \quad \varphi^{(m)} = c \sum_l \frac{1}{m - l + \eta'_m} \left[I_v I_{k_l} \right] + \frac{g}{\sqrt{2\Omega}} \cdot \frac{1}{\omega_0} \cdot \sum_r^{n-1} I_R I_{k_l} I_{k_r} \Big] + 0 \left(\frac{\Delta\omega}{\omega_0} \right).$$

It is not difficult to show (determining explicitly the state V) in our case that Eq. (2.15) may even be written in the following form

$$(2.15') \quad \varphi^{(m)} = \sum_l \frac{c}{m - l + \eta'_m} a_{kl}^* V + 0 \left(\frac{\Delta\omega}{\omega_0} \right).$$

Neglecting the small term $0(\Delta\omega/\omega_0)$ the interpretation of Eq. (2.15') is particularly simple and clear. Obviously the eigenfunctions of the energy represented by (2.15') are standing θ particle waves in presence of the physical

V) particle. It is pretty obvious from (2.15') that the norm of these eigenstates is positive.

Now, in case (B_1) , we have instead standing θ particle waves in presence of a «ghost state» V_- and therefore the eigenstates of the energy corresponding to case (B_1) have a negative norm.

So, summarizing the results obtained until now, we might say that in the cases (A) and (B) together, we have $2n$ eigenstates (n with negative norm and n with positive norm). It is very easy to recognize that in case (C) we have

$$(n-1)(n-2)/2 + (n-1) = n(n-1)/2$$

eigenstates of the energy all with positive norm.

Therefore the total number of eigenstates which we have found is equal to $2n + n(n-1)/2$ which is exactly the number of the corresponding states in the interaction free-case. We have so a check that we have not forgotten any state in our discussion. On the other hand, it is clear that for the completeness we should have to make use of the n states with negative norm. However, as already said, this circumstance does not produce any trouble in the scattering. The reason is due to the fact that in our model there is a gap between the energy of the states with negative norm and the energy of the states with positive norm. This circumstances, merely for the conservation of the energy, will prevent to get any negative norm state in the outgoing wave, if there is not such a state in the ingoing wave.

We might express the same thing even in a different form. Although the eigenstates of the energy of positive norm do not form a complete set, we are able to represent the ingoing and outgoing states by means of the eigenstates of the energy of positive norm only.

Now this is completely trivial in the example which we have discussed because it is automatically made possible by the conservation of the energy. However, it is relevant for the purpose of this paper to consider how actually this thing happens to be verified in the present case.

The state of the system will be in general represented by the superposition:

$$(2.16) \quad \sum_m \exp[-iE_m t] c_m \varphi^{(m)} = \sum_{m,l} \frac{c_m \exp[-iE_m t]}{m-l+\eta_m} a_{k_l}^* V).$$

We shall have then to consider (cf. also Appendix I) the expression

$$\sum_m \sum_l \frac{c_m}{m-l+\eta_m} \exp[-iE_m t] \exp[i\mathbf{k}_l \cdot \mathbf{x}],$$

for very great value of $r=|\mathbf{x}|$. But remembering Eq. (2.14') and supposing

that m is not in the neighbourhood either of 0 or of n one gets very easily that

$$(2.16') \quad \sum_l \frac{\exp[ik_l r]}{m-l+\eta'_m} \sim -\exp[ik_m r] \left(\pi i + \log \frac{\omega_0 + \Delta\omega - E_m}{\omega_0 - E_m} \right) + o\left(\frac{1}{r}\right),$$

which is not different from the conventional expression. There is a difference only if m is in the neighbourhood of 0 or n . In these cases one gets also a term which is proportional to $\log r \exp[ik_m r]$. Even this state does not give rise to any inconvenience, provided one takes in the definition of the wave packet (2.16) an interval ΔE_m which is much greater than $\varepsilon\omega_0$. But such condition has to be satisfied anyway if one wants to describe ingoing and outgoing states in a finite volume and, on the other hand, does not put any limit to the precision by which the momentum of the ingoing and outgoing states may be defined, provided, of course, that one takes a sufficiently large volume Ω (and provided it is not asked that the precision is actually infinite). So although the eigenstates of energy of positive norm do not form a complete set and in spite of the fact that we are considering only a Hilbert space of a finite number of dimensions, the eigenstates of the energy of positive norm are in this case perfectly sufficient to describe in a satisfactory manner ingoing and outgoing waves.

Similar considerations will hold by means of a suitable redefinition, even in the case in which it is not automatically granted by the conservation of the energy. Such a case, which is a simple generalization of the model considered in this paragraph, will be discussed in the next paragraph.

3. - Double band model.

We shall consider again in this paragraph a Lee model described by Eqs. (2.1), (2.1'), (2.1''), (2.2), (2.2') and (2.2''). This time, however, the form factor will be described by

$$(3.1) \quad \begin{cases} f_i = \exp\left[-\frac{(\omega_i - \omega_1)^2}{\Delta\omega^2}\right] + \exp\left[-\frac{(\omega_i - \omega_2)^2}{\Delta\omega^2}\right], \\ \omega_1 < \omega_2; \quad \frac{\Delta\omega}{\omega_1} \ll 1; \quad \frac{\Delta\omega}{\omega_2 - \omega_1} \ll 1. \end{cases}$$

We shall suppose that the constant g is greater than g_c and therefore we shall have to use an indefinite metric. Now let z_- be the energy of the ghost state in the section (V, 0N). We suppose

$$(3.2) \quad \alpha \ll 1, \quad \text{with} \quad \alpha = -\frac{z_-}{\omega_1}.$$

We now make the hypothesis that the coupling constant g is such that the relationships

$$(3.3) \quad z_- = \omega_1 - \omega_2 - \beta\epsilon\omega_1,$$

$$(3.3') \quad 0 < \beta < 1,$$

are satisfied. This hypothesis is made in order that the conservation of the energy may not grant the automatic possibility of describing scattering states in the section $(\theta V, \theta\theta'N)$ by means of eigenstates of the energy of positive norm only.

In fact, it will be energetically possible that a θ particle of energy ω_1 colliding with a V particle, is scattered with an energy ω_2 leaving the heavy particle in the ghost state V_- . (Eq. (3.3')) is then imposed in order not to have a too sharp resonance for this process. This, however, is not a restrictive condition at all).

Now let us put

$$(3.4) \quad \begin{cases} \omega_1 l = \omega_1(1 + l\epsilon) \\ \omega_2 l = \omega_2 + \omega_1 l\epsilon = \omega_1(1 + \alpha - \beta\epsilon + l\epsilon) \end{cases}$$

$$(3.4') \quad f_{1i} = \exp\left[-\frac{(\omega_{1i} - \omega_1)^2}{\Delta\omega^2}\right]; \quad f_{2i} = \exp\left[-\frac{(\omega_{2i} - \omega_2)^2}{\Delta\omega^2}\right], \quad f_{1i} = f_{2i} = \bar{f}_i,$$

and

$$(3.5) \quad z = \omega_1(1 + \xi).$$

Of course Eqs. (2.3), (2.4), (2.5) and (2.6) will formally be satisfied even for the case of this paragraph. We might then perform an analysis completely similar to that of the previous paragraph also for the present model. It is then possible to recognize that the interesting case in this section $(\theta V, \theta\theta'N)$ is verified when

$$(3.5') \quad \xi = \frac{\Delta\omega}{\omega_1}.$$

In this case we can write the following equations:

$$(3.6) \quad h(z - \omega_{1i}) = \omega_1(\xi - l\epsilon) \left(1 + \frac{g^2\omega_1}{2\Omega} (\xi - l\epsilon) \sum \frac{f_i^2}{\omega_i^2} + \dots\right),$$

$$(3.6') \quad h(z - \omega_{1i}) = \omega_1(\xi - l\epsilon + \beta\epsilon)h'_{z=\omega_-} + \frac{\omega_1^2}{2} (\xi - l\epsilon + \beta\epsilon)^2 h''_{z=\omega_-}.$$

We shall now put

$$(3.7) \quad \begin{cases} \gamma_{1l} = \frac{2\Omega\omega_1^2}{g^2\bar{f}_l^2} h(z - \omega_{1l}) = \frac{2\Omega\omega_1^3}{g^2\bar{f}_l} (\xi - l\varepsilon + \dots), \\ \gamma_{2l} = \frac{2\Omega\omega_1^3}{g^2\bar{f}_l^2} h'_{z=z_-}(\xi + \beta\varepsilon - l\varepsilon + \dots), \end{cases}$$

$$\psi_{1l} = \bar{f}_l \Phi_{1l}; \quad \psi_{2l} = \bar{f}_l \Phi_{2l}.$$

Then we get

$$(3.7'') \quad \begin{cases} \gamma_{1l}\psi_{2l} + \sum_r \psi_{1r} \left(\frac{1}{1-\xi} - \frac{(l+r)\varepsilon}{(1-\xi)^2} + \dots \right) + \sum_s \psi_{2s} \left(\frac{1}{1+\alpha-\xi} + \dots \right) = 0, \\ \gamma_{2l}\psi_{2l} + \sum_r \psi_{1r} \left(\frac{1}{1+\alpha-\xi} + \dots \right) + \sum_s \psi_{2s} \left(\frac{1}{1+2\alpha-\xi} + \dots \right) = 0. \end{cases}$$

Using the same procedure as in the last paragraph, we get:

$$(3.8) \quad \psi_{1l} = \frac{c_1 \bar{f}_l^2}{\xi - l\varepsilon} + \dots; \quad \psi_{2l} = \frac{c_2 \bar{f}_l^2}{\xi - l\varepsilon + \beta\varepsilon}.$$

It is then easy to see that, neglecting terms of second order in $\Delta\omega/\omega_1$, the equations which can be used for determining the eigenvalues of the energy are:

$$(3.7''') \quad \begin{cases} \sum \psi_{1r} + \frac{1}{1+\alpha} \sum \psi_{2l} = \sum r\varepsilon\psi_{1r} + \sum \frac{r\varepsilon\psi_{2r}}{(1+\alpha)^2} - \gamma_{10}\psi_{10}, \\ \sum \psi_{1l} + \frac{1+\alpha}{1+2\alpha} \sum \psi_{2l} = \frac{1}{1+\alpha} \sum r\varepsilon\psi_{1r} + \frac{1+\alpha}{(1+r\alpha)^2} \sum s\varepsilon\psi_{2s} - \gamma_{20}\psi_{20}. \end{cases}$$

Now, let us put:

$$(3.9) \quad \sum \frac{\bar{f}_l^2}{\xi - l\varepsilon} = \sigma_1(\xi),$$

$$(3.9') \quad \sum \frac{\bar{f}_l^2}{\xi - l\varepsilon + \beta\varepsilon} = \sigma_2(\xi),$$

$$(3.9'') \quad \sum \bar{f}_l^2 = \varrho.$$

Comparing with (3.1), (3.2), and (3.3) and with Eq. (2.6), we get easily:

$$(3.9''') \quad \varrho = \frac{1}{\varepsilon} \frac{\Delta\omega}{\omega_1} \approx \frac{1}{\alpha} \frac{\omega_1^3}{g^2} \Omega.$$

Putting:

$$\sum \varphi_{1i}(\xi) = c_1 \sigma_1(\xi),$$

$$\sum \varphi_{2i}(\xi) = c_2 \sigma_2(\xi).$$

Eq. (3.8) gives (neglecting $\beta\varepsilon$ with respect to ξ):

$$\sum \varepsilon l \psi_{1i} = c_1 [\xi \sigma_1(\xi) - \varrho],$$

$$\sum \varepsilon l \psi_{2i} = c_2 [\xi \sigma_2(\xi) - \varrho],$$

and keeping into account (3.9'') and that

$$\gamma_{10} = \frac{2\omega_1^3}{g^2} \Omega \xi, \quad \gamma_{20} = \frac{2\omega_1^3}{g^2} \Omega h'_{z-z_0} \xi,$$

we get

$$(3.10) \quad \left. \begin{matrix} \gamma_{10} \\ \gamma_{20} \end{matrix} \right\} \sim \alpha \varrho \cdot 0 \left(\frac{\Delta \omega}{\omega_1} \right).$$

Therefore the system of equations of the order 0 (neglecting terms $0(\Delta\omega/\omega_1)$) which determine the eigenvalue of the energy is then

$$(3.11) \quad \begin{cases} c_1 \sigma_1 + \frac{1}{1+\alpha} c_2 \sigma_2 = -\varrho \left[c_1 + \frac{1}{(1+\alpha)^2} c_2 \right], \\ c_1 \sigma_1 + \frac{1+\alpha}{1+2\alpha} c_2 \sigma_2 = -\varrho \left[\frac{c_1}{1+\alpha} + \frac{1+\alpha}{(1+2\alpha)^2} c_2 \right]. \end{cases}$$

We shall see later that when

$$\sigma_1 \approx \varrho,$$

then

$$(3.11') \quad \frac{\sigma_1}{\sigma_2} \approx \frac{\Delta \omega}{\omega_1}.$$

Eqs. (3.11) keeping into account (3.11') are then easily transformed into

$$(3.11'') \quad \left(\frac{(1+\alpha)^2}{1+2\alpha} - 1 \right) \sigma_1 = \varrho \left(\frac{1}{1+\alpha} - \frac{(1+\alpha)^2}{1+2\alpha} \right) \approx \frac{1}{2} \left(\frac{1}{1+\alpha} - \frac{(1+\alpha)^2}{1+2\alpha} \right) \frac{\Delta \omega}{\omega_1},$$

and remembering that

$$\sigma_1 = \sum \frac{\bar{f}_i^2}{\xi - l\varepsilon} = \frac{1}{2} \sum \frac{\bar{f}_i^2}{m - l + \eta},$$

we get the following energy eigenvalue characteristic equation in the approximation zero:

$$(3.11'') \quad \sum \frac{\tilde{f}_l^2}{m-l+\eta} = 0; \quad 0 < \eta < 1.$$

Subtracting then the two equations (11), and neglecting terms of the order $\Delta\omega/\omega_1$, one gets

$$(3.12) \quad C_2 \sigma_2 \left(\frac{1}{1+\alpha} - \frac{1+\alpha}{1+2\alpha} \right) = -\varrho C_1 \left(1 - \frac{1}{1+\alpha} \right),$$

and that shows that when (3.11') is satisfied

$$(3.12') \quad \frac{C_2}{C_1} \approx 0 \left(\frac{\varrho}{\sigma_2} \right) \approx 0 \left(\frac{\sigma_1}{\sigma_2} \right) \approx 0 \left(\frac{\Delta\omega}{\omega_1} \right),$$

which means that C_2 is negligible with respect to C_1 in the approximation zero.

Now we have really to show that (3.11') is satisfied. For this purpose we have simply to prove that when the condition of (3.11'') is satisfied, i.e. when (3.11'') is satisfied, then no matter which m is,

$$(3.13) \quad \sum \frac{\tilde{f}_l^2}{m+\eta_m+\beta-l} \neq 0.$$

This again is equivalent to showing that

$$\sum_l \left(\frac{\tilde{f}_{m+l}^2}{l-\eta_m-\beta} - \frac{\tilde{f}_{m+l}^2}{l-\eta_m} \right) \neq 0.$$

Now we have to consider two cases.

1st case:

$$\eta_m + \beta < 1,$$

then:

$$\sum_l \left(\frac{\tilde{f}_{m+l}^2}{l-\eta_m-\beta} - \frac{\tilde{f}_{m+l}^2}{l-\eta_m} \right) = \beta \sum_{-\infty}^{+\infty} \frac{\tilde{f}_{m+l}^2}{(l-\eta_m-\beta)(l-\eta_m)}.$$

Now the sum which appears in the second member has all the terms greater than 0 because $l-\eta-\beta$ and $l-\eta$ always have the same sign. We have therefore proved (3.13) in the first case.

2nd case

$$\eta_m + \beta = 1 + \beta'; \quad 0 < \beta' < \eta_m,$$

In this case:

$$\sum \left(\frac{\bar{f}_{m+l}^2}{l-\eta_m-\beta} - \frac{\bar{f}_{m+l}^2}{l-\eta_m} \right) = -(\eta_m - \beta') \sum \frac{\bar{f}_{m+l}^2}{(l-\beta')(l-\eta)} + \frac{\sum \bar{f}_{m+l}^2 - \bar{f}_{m+l}^2}{l-1-\beta'}.$$

But

$$\sum \frac{\bar{f}_{m+l}^2 - \bar{f}_{m+l-1}^2}{l-1-\beta'} = 0(\varepsilon\sigma_2), \quad \eta_m - \beta' = 1 - \beta \neq 0,$$

therefore (3.13) holds. We may remark that if $\eta + \beta \rightarrow 1$, $\sum \bar{f}^2/(l-\eta-\beta) \rightarrow \pm \infty$; in this case (3.13) holds *a fortiori*.

We remark also that $\sum \bar{f}_{l+m}^2/(l-\beta')(l-\eta)$ is always necessarily of the order of unity, even if m is very big and consequently $\bar{f}_m^2$ is very small because of the eigenvalue equation (3.11'). (η_m must be so small that there are terms of 0(1) in the sum).

Therefore (3.11') is completely proved.

It does exist of course an analogous solution in which, however,

$$(3.12'') \quad \frac{C_1}{C_2} \approx 0 \left(\frac{\Delta\omega}{\omega_1} \right),$$

and the characteristic equation of the eigenvalue energy is in this case

$$(3.11''') \quad \sum \frac{\bar{f}_l^2}{m + \eta'_m + \beta - l} = 0.$$

In the first case when (3.12') and (3.11''') are valid, it is easy to recognize that the eigenstate of the energy may be written under the form

$$(3.14) \quad C_1 \sum \frac{f_l}{m + \eta_m - l} a_i^* V + C_2 \sum \frac{\bar{f}_l}{m + \eta_m + \beta - l} a_k^* V_- ,$$

plus « other terms ».

The « other terms » which appear in Eq. (3.14) are of the type scattering of « θ' and θ on N^- » and therefore they are of positive norm and cannot give rise to any trouble. It is easy to see that the state represented by Eq. (3.14) is of positive norm. In the approximation 0 in which one neglects all terms of the order $\Delta\omega/\omega_1$ it would simply represent a standing θ wave in presence of a particle. Therefore even in the present case in the approximation 0 we are exactly in the same situation in which we were with the model of the previous paragraph, and, neglecting terms of the order of $\Delta\omega/\omega_1$ as will be seen in more detail in the next paragraph, it would be possible to represent

ingoing or outgoing θ particles packets in presence of a V particle, without making use of the eigenstate of the energy of negative norm. Unfortunately this is not any more the situation when one is taking into account the term of the order $\Delta\omega/\omega_1$

$$O_2 \sum \frac{\bar{f}_i}{m + \eta_m + \beta - i} a_i^* V_- ,$$

which is present in Eq. (3.14). In fact, a superposition of eigenstates of energy of positive norm of the form (3.14) will always contain not only a θ wave packet in presence of V , but also a component in which a ghost V_- appears. In order to eliminate this ghost V_- , for instance in the ingoing wave, to obtain a conventional expression of a θ ingoing in presence of a V particle only, one should make use of eigenstates of the energy of negative norm which are obtained when (3.12'') and (3.11''') are valid instead of (3.12') and (3.11'') and then, of course, the ghost would appear in the outgoing waves.

We shall see in the next paragraph how this difficulty may be overcome, following the general prescription which we have outlined in the introduction.

4. - Construction of an unitary S matrix for the double band model.

The purpose of this paragraph is to show that it is possible to find a correspondence C between the eigenstates of positive norm of the energy of the Lee model considered in the previous paragraph, and the vectors of another Hilbert space suitable for describing incoming and outgoing states, and to show that by means of this correspondence it is possible to define an S -matrix with reasonable physical properties.

For this purpose we remind that there is no difficulty for the eigenstates of the Hamiltonian representing scattering of two θ particles with and N particle, because in our model the energy of those eigenstates does not overlap with the energy of the states of negative norm. As already pointed out, the only states which in our model will give rise to trouble are the eigenstates represented by Eq. (3.14).

In the approximation 0 (with respect to an expansion in $\Delta\omega/\omega_1$) the states (3.14) may be represented by:

$$(4.1) \quad \gamma^{(m)} = \sum_i \beta_{mi} a_i^* V ,$$

where

$$(4.1') \quad \beta_{mi} = \frac{C_m \bar{f}_i}{m + \eta_m - i} .$$

We now remark that the set of states

$$(4.1'') \quad \theta^{(m)} = \sum_i \beta_{mi} a_i^* 0$$

is exactly an orthogonal set. In fact:

$$\begin{aligned} \sum_i \beta_{ni}^* \beta_{mi} &= C_{in}^* C_{im} \sum_i \frac{\bar{f}_i^2}{(n + \eta_n - l)(m + \eta_m - l)} = \\ &= \frac{C_{in}^* C_{im}}{m - n + \eta_m - \eta_n} \left(\sum_i \frac{\bar{f}_i^2}{n + \eta_n - l} - \sum_i \frac{\bar{f}_i^2}{m + \eta_m - l} \right), \end{aligned}$$

but $\sum \bar{f}_i^2/(n + \eta_n - l)$ and $\sum \bar{f}_i^2/(m + \eta_m - l)$ are both equal to zero because of Eq. (3.11'''), and therefore the scalar product $(\theta^{(n)} | \theta^{(m)})$ vanishes. (The denominator $m - n + \eta_m - \eta_n$, is obviously different from 0 for $m \neq n$ because η_m and η_n are both positive numbers less than 1.)

In order to construct the correspondence C we first put in correspondence the state V representing a physical V particle with a state V of $\mathcal{H}_2$ representing a not-interacting V particle. (It should be noted that the state V_0) is *not* the state 1_v) representing a bare V particle. The state V_0 is of course taken with a positive norm.) Then we put in correspondence the states $\gamma^{(m)}$ to the states

$$(4.2) \quad \beta_m = \sum_i \beta_{mi} a_i^* V_0$$

of $\mathcal{H}_2$. It has to be noted that no inconvenience does arise from the substitution $V) - V_0$ if we limit ourselves to consider in the second Hilbert space only « asymptotical » θ packets, i.e., θ packets which do not overlap in the ordinary space with the physical V particle (cf. Appendix).

Now, as V_0 is not interacting with θ the set of vectors β_m is orthogonal as a consequence of the fact that the vectors defined by (4.1'') are orthogonal, as already proved. Therefore the correspondence which we are now defining is satisfying the requirement (4.2) of the introduction for the correspondence C .

The extension of the correspondence C to the other eigenstates of positive norm is trivial and not interesting. Let us call K the unitary operator which performs the correspondence C between Hilbert space 1 and Hilbert space 2. We remark that K does not depend on the time. We have now to verify that the operator K defined in this manner allows the construction of a suitable S -matrix. For this purpose let us consider the two operators:

$$(4.3) \quad U = \exp[-iHt],$$

$$(4.3') \quad U_0 = \exp[-iH_0t],$$

where H is the total Hamiltonian and H_0 is the non-interacting Hamiltonian. U will operate only on the subspace of the eigenstates of H of positive norm. In this subspace U is unitary (not pseudounitary).

We have now to consider

$$(4.4) \quad U_1(t', t) = U_0^{-1}(t') K U(t' - t) K^{-1} U_0(t).$$

Then, if the limit

$$(4.5) \quad S = \lim_{t' \rightarrow \infty, t \rightarrow \infty} U_1(t', t),$$

does exist, this will be by definition the S -matrix.

We have now to prove that the limit (4.5) does really exist and that the S -matrix (4.5) is a physically reasonable S -matrix. For this purpose let us put

$$(4.4') \quad U^{(0)} = K U K^{-1}.$$

Obviously $U^{(0)}$ is obtained from U replacing in place of the eigenvectors (4.1), the vectors (4.2). Then

$$(4.4'') \quad U_1(t', t) = U_0^{-1}(t') U^{(0)}(t' - t) U_0(t).$$

Now keeping into account very well known properties of the Hilbert space (*), the proof of the existence of the limit (4.5) is then equivalent to the proof of the existence of the limit (**):

$$(4.5') \quad \lim_{t \rightarrow \infty, t' > t} U_1(t', t) = 1.$$

We shall prove that a relation of the type (4.5') holds if the operator U_1 is applied to wave packets

$$(4.6) \quad \psi = \sum_m c_m \exp[-iE_m t] \beta_m,$$

where the coefficient c_m in the limit of the very large volume may be prolonged in an analytic function $c(m)$ of the energy which is limited in a neighbourhood as small as we like—but finite—of the real axis (**).

(*) « Completeness » of the Hilbert space $\mathcal{H}_2$ (see the remark p. 395 in the introduction).

(**) Of course the existence of the limit $\lim_{t' \rightarrow -\infty, t < t'} U_1(t', t) = 1$ is also necessary but follows in the same way as (4.5').

(**) Such a hypothesis is merely made in order to define conveniently the « very future » and « very past » time which appears in the limit (4.5). It will be noted that with the passage of the time the coefficient c_m will go over to $c_m \exp[-iE_m t]$ and, due to the exponential factor, for t sufficiently great in absolute value, the analytical prolongation of the coefficient will become in modulus bigger than any prefixed number in any finite neighbourhood of the real axis.

Furthermore, we suppose that c_m near 0 may be put equal to mG_m with G_m which may be prolonged in an analytical function limited in a neighbourhood of 0.

Now let us put

$$(4.7) \quad \alpha_k = a_k^* V_0$$

then

$$(4.2') \quad \beta_m = \sum_k \beta_{mk} \alpha_k.$$

It follows (cf. (4.6))

$$U_1(t', t'')\psi = \sum_k \sum_m \exp[-iE_m t'] c_m \beta_{mk} \exp[i\omega_k(t' - t)] \alpha_k.$$

In order to verify (4.5') we have not to verify that:

$$(4.5'') \quad \lim_{t \rightarrow \infty} \sum_k \left| \sum_m (c_m \beta_{mk} \exp[-iE_m t' + i\omega_k(t' - t)] - c_m \beta_{mk} \exp[-iE_m t]) \right|^2 = 0.$$

For this purpose we remark that we may put in the limit $\Omega \rightarrow \infty$

$$\begin{aligned} \sum_m c_m \beta_{mk} \exp[-iE_m t] (\exp[-i(E_m - \omega_k)(t' - t)] - 1) &= \\ &= c_k B_k (\exp[-i(E_k - \omega_k)(t' - t)] - 1) + \\ &+ \int_C c_m \beta_{mk} \exp[-iE_m t] (\exp[-i(E_m - \omega_k)t' - t] - 1) dm, \end{aligned}$$

where we have performed a transformation to the continuous variable m and the contour C has a indenting under the pole $m = k$ of β_{mk} .

In fact, remembering (4.1') we have to calculate an expression like $\sum g_m/(m + \eta_m - k)$, where g_m is regular in a neighbourhood of k . This expression for $\Omega \rightarrow \infty$ may always be written as

$$\int_0^{k - \varepsilon'_k} g_m \frac{dm}{m - k} + \int_{k + \varepsilon''_k}^{\infty} g_m \frac{dm}{m - k} + \text{terms } o\left(\frac{1}{\Omega^{\frac{1}{2}}}\right),$$

where ε'_k and ε''_k do depend only on η_m and not on g_m . But:

$$\int_0^{k - \varepsilon'_k} \frac{g_m dm}{m - k} + \int_{k + \varepsilon''_k}^{\infty} \frac{g_m dm}{m - k} = P \int_0^{\infty} \frac{g_m dm}{m - k} + g_k \lg \frac{\varepsilon''_k}{\varepsilon'_k} = \int_0^{\infty} \frac{g_m dm}{m - k} + g_k \left(\lg \frac{\varepsilon''_k}{\varepsilon'_k} - i\pi \right),$$

and therefore our statement is verified.

B_k is a function of k which may even diverge for $k \rightarrow \infty$. We however, always assume that $c_k B_k \rightarrow 0$ for $k \rightarrow 0$ with a sufficient velocity. This is always true, if we consider suitable wave packets for which the spread in momentum is small. On the other hand, if $\Omega = L^3$ for $\Omega \rightarrow \infty$ we may always have that

$$\frac{t' - t}{L} \rightarrow 0;$$

this condition is of course necessary in order to be able to talk of scattering in a box at all, and is not a limitation on t' but a condition on L . Therefore

$$\lim_{\Omega \rightarrow \infty} \exp[i(E_k - \omega_k)(t' - t)] - 1 \rightarrow 0.$$

We have then to consider only:

$$\int_0^{\infty} dm c_m \beta_{mk} \exp[-iE_m t] (\exp[-i(E_m - \omega_k)(t' - t)] - 1).$$

Now let us put

$$(4.8) \quad \zeta_{\bar{m}, \tau} = \sum_{\bar{m}}^{\infty} c_m \exp[-iE_m \tau] \beta_m.$$

We may always choose $\bar{m}$ in such a way that

$$(4.8') \quad |\zeta_{\bar{m}, \tau}|^2 < \varepsilon^2,$$

ε being any positive number. Then

$$\sum_k \left| \sum_{\bar{m}}^{\infty} c_m \beta_{mk} \exp[-iE_m \tau] \exp[-i\omega_k \tau'] \right|^2 < \varepsilon^2,$$

because the previous expression is the modulus square of the vector $\zeta_{\bar{m}, \tau}$ (this modulus has been written by means of the components of $\zeta_{\bar{m}, \tau}$ with respect to the axis $\exp[i\omega \tau'] \cdot \alpha_k$). This is true no matter what τ and τ' are because $|\zeta_{\bar{m}, \tau}|^2$ is independent of τ and τ' . We put now:

$$\psi_{1\bar{m}} = \sum_k \sum_0^{\bar{m}} c_m \beta_{mk} \exp[-iE_m t] \exp[-i(E_m - \omega_k)t'] \alpha_k,$$

$$\zeta_{1\bar{m}} = \sum_k \sum_{\bar{m}}^{\infty} c_m \beta_{mk} \exp[-iE_m t] \exp[-i(E_m - \omega_k)\tau'] \alpha_k,$$

$$\psi_{2\bar{m}} = \sum_k \sum_0^{\bar{m}} c_m \beta_{mk} \exp[-iE_m t] \alpha_k,$$

$$\zeta_{2\bar{m}} = \sum_k \sum_{\bar{m}}^{\infty} c_m \beta_{mk} \exp[-iE_m t] \alpha_k.$$

We have to determine:

$$P^2 = |\psi_{1\bar{m}} + \zeta_{1\bar{m}} - \varphi_{2\bar{m}} - \zeta_{2\bar{m}}|^2$$

and then this quantity satisfies the following inequality

$$(4.8'') \quad P^2 \leq |\psi_{1\bar{m}} - \varphi_{2\bar{m}}|^2 + 2|\psi_{1\bar{m}} - \varphi_{2\bar{m}}| \cdot |\zeta_{1\bar{m}} - \zeta_{2\bar{m}}| + \\ + |\zeta_{1\bar{m}} - \zeta_{2\bar{m}}|^2 \leq (\psi_{1\bar{m}} - \varphi_{2\bar{m}})^2 + 8\varepsilon + 4\varepsilon^2,$$

(because $|\psi_{1\bar{m}}|^2 \leq 1$ and $|\varphi_{2\bar{m}}|^2 \leq 1$).

In order to prove that $P^2 \rightarrow 0$ when $t \rightarrow \infty$ it will be then sufficient to prove that $|\psi_{1\bar{m}} - \varphi_{2\bar{m}}|^2 \rightarrow 0$ for any finite $\bar{m}$.

Of course (4.8'') holds *a fortiori* even if we do not extend the sum with respect to the index k to all the interval of variability of k and it holds therefore even if we break the whole sum with respect to k into two parts for each part of the sum separately.

Now let us suppose that $\bar{m}$ is such that Eq. (4.8'') holds. We shall break then the sum with respect to the index k into two parts. We shall indeed consider:

$$P_1^2 = \int_0^{2\bar{m}} dk \left| \int_{C_1} dm c_m \beta_{mk} \exp[-iE_m t] (\exp[-i(E_m - \omega_k)(t' - t) - 1]) \right|^2,$$

and

$$P_2^2 = \int_{2\bar{m}}^{\infty} dk \left| \int_0^{\bar{m}} dm c_m \beta_{mk} \exp[-iE_m t] (\exp[-i(E_m - \omega_k)(t' - t) - 1]) \right|^2,$$

where C_1 is going (with the indenting under $m = k$) from 0 to $3\bar{m}$.

For reasons already discussed, it will be enough to show that P_1^2 and P_2^2 are going to 0 when t is going to infinity for any $\bar{m}$. For this purpose, beginning with P_1^2 we change the contour of integration C_1 in such a way that now it will be composed of three rectilinear segments, one going from 0 to $-i\Delta m$, the other going from $-i\Delta m$ to $3\bar{m} - i\Delta m$ and the third one from $3\bar{m} - i\Delta m$ to $3\bar{m}$.

Now in the first interval $0 - i\Delta m$ we have $c_m < mG_0$ (*) and therefore the absolute value of the integral in the first interval is less than $2C_1(\exp[\Delta m^2/\Delta\omega^2])G_0\Delta m$; i.e., of a quantity as small as we like if we take Δm small enough.

(*) Remember the hypothesis about c_m .

In the third interval

$$|\beta_{mk}| \leq C_1 \frac{\exp[(4\Delta m \bar{m} + \Delta m^2)/\Delta \omega^2]}{\bar{m}} = \frac{\bar{C}_1}{\bar{m}}.$$

Therefore the absolute value of the integral in the third interval becomes smaller than $\bar{C}_1 M(\Delta m/\bar{m})$, where $M(*)$ is a fixed number and is greater than the upper limit of the function c_x in the neighbourhood of the real axis between the two parallels to the real axis at a distance $\geq$ than Δm from the real axis. Therefore again this value may become as small as we like, provided we take Δm sufficiently small.

Now, on the other hand, the absolute value of the integral on the second interval will be smaller than

$$\bar{c}_1 M \exp\left[-\frac{\Delta m^2}{2\mu} t\right] \cdot \frac{3\bar{m}}{\Delta m}, \quad (*)$$

because

$$|\beta_{mk}| \leq \frac{\bar{c}_1}{\Delta m}.$$

It will also become as small as we like if we take t sufficiently great independently of $\bar{m}$ and Δm . So the absolute value of the first integrals with respect to m will become smaller than any prefixed ε_1^2 and P_1^2 will be then smaller than $2\varepsilon_1^2$.

Now we discuss P_2^2 . We evaluate the integral with respect to m along the contour $0 \rightarrow -i\Delta m$; $-i\Delta m \rightarrow \bar{m} - i\Delta m$; $\bar{m} - i\Delta m \rightarrow \bar{m}$. It is then easy to see that in the first and third intervals the integral will be in absolute value less than $\bar{C}_1 M(\Delta m/(k - \bar{m}))$ and in the second interval smaller than

$$\bar{C}_1 M \frac{\exp[-(\Delta m^2/2\mu)t]}{k - \bar{m}}.$$

Therefore again we may have

$$P_2^2 < \int_{2\bar{m}}^{\infty} dk \frac{\varepsilon_2^2}{|k - \bar{m}|^2} = \frac{\varepsilon_2^2}{\bar{m}},$$

with ε_2 being any positive number.

We have proved the relation (4.5'') and therefore that the limit (4.5) does exist. We may now add some remarks. The proof of the existence of the

(*) For $\Delta m \leq 2\mu$ where μ is the mass of the θ -particle.

limit (4.5) is in itself a proof that the θV wave packet will behave in the limit of an extremely great box, and for sufficiently far past time or far future time in a physically reasonable manner. In fact, we have proved that for such times this wave packet behaves as nearly as a free wave packet as we like. In order to complete the proof that the formalism which we are suggesting here gives, at least for the case of the model which we are discussing, physical results which are as reasonable as possible, we have simply to show that no unreasonable things happen for times which are not in the very far past or in the very far future.

For this purpose we remark that the representation in the configuration space of the vectors β_m has a correct asymptotic behaviour (for $r \rightarrow \infty$). That means that it behaves like

$$\beta' \frac{\exp[-imr]}{r} + \beta'' \frac{\exp[imr]}{r}.$$

The proof that for not very far time, the representation of ψ in configuration space will vanish for very high value of r (which is all we need to know) is then entirely analogous and even simpler than the proof of the limit (4.5). The only difference is that this time one has to play with the variable r instead of playing with the variable t . This remark achieves the proof that in our case a reasonable correspondence C between the $\mathcal{H}_1$ and $\mathcal{H}_2$ does exist.

Of course, it is pretty obvious that from the existence of one correspondence C it follows that an infinite number of such correspondences exist. Therefore the method is not unambiguous. We shall discuss in the conclusion the probable meaning of this ambiguity.

5. - Extension of the method to a more general model.

We have now to see further that the possibility of constructing a suitable correspondence C is not an accidental case for the particular model which we have considered. In fact, we shall now briefly outline a procedure which may be applied quite in general to a Lee model without any particular restriction on the function f_i . For this purpose let us consider a Lee model with a quite general form factor f_i . We shall suppose for the moment only that the form factor f_i will vanish for arguments greater than a certain argument, which may be as big as we like.

Furthermore of course we shall suppose that f_i is limited every where. Then the critical value g_c of the coupling constant will be finite. We remark now that there is a second value g'_c greater than the g_c such that, if the coupling

constant satisfies the inequalities

$$(5.1) \quad g_c < g < g'_c,$$

there is a gap between the spectrum of the energy of the eigenstates of the total Hamiltonian with positive norm and the spectrum of the energy of the eigenstate of the Hamiltonian with negative norm (we consider only the states belonging to the sectors V , $N\theta$ and the sector $\theta V N\theta \theta'$). In fact, the energy of the ghost states V_- approaches $-\infty$ when g approaches g_c from the above. Remembering that by hypothesis f_i vanishes for values of the argument greater than a certain value, it is then easy to verify the previous statement.

Then, in this case, for values of the coupling constant satisfying the inequalities (5.1), there is no difficulty whatsoever; the correspondence C may just be the identity because we are essentially in the same case considered in Section 2 and conservation of the energy alone will grant the unitarity of the S -matrix. But, we want of course to consider the case in which g is not only greater than g_c but also of g'_c . In this case C cannot be any more the identity and we have to find out a procedure for constructing C in a suitable manner. For this purpose we shall start from a value g of the coupling constant satisfying the inequality (5.1) and we shall write the total Hamiltonian for the value $g' > g'_c$ in the following way:

$$(5.2) \quad H = H_0 + H_1 + H_2,$$

where

$$(5.2') \quad H_1 = -\frac{g}{\sqrt{2\Omega}} \frac{N_{g'}}{N_g} \left(\psi_V^* \psi_N \sum \frac{f_i}{\sqrt{\omega_i}} a_{k_i} + \sum \frac{f_i}{\sqrt{\omega_i}} a_{k_i}^* \psi_N^* \psi_V \right) - \Delta m_g N_g^2 \psi_V^* \psi_V,$$

$$(5.2'') \quad H_2 = -\frac{\Delta g}{\sqrt{2\Omega}} \left(\psi_V^* \psi_N \sum \frac{f_i}{\sqrt{\omega_i}} a_{k_i} + \sum \frac{f_i}{\sqrt{\omega_i}} a_{k_i}^* \psi_N^* \psi_V \right) - (\Delta m_{g'} - \Delta m_g) N_g^2 \psi_V^* \psi_V,$$

$N_{g'}$ and N_g are the renormalization constant for g' and g . The operator ψ_V is renormalized relatively to g' and

$$\Delta g = g' - \frac{N_{g'}}{N_g} \cdot g.$$

The mass renormalization constants Δm_g and $\Delta m_{g'}$ will be fixed in such a way that the mass of the physical V particle does not change from passing from the value g to the value g' .

Now we remark that in general if the Hamiltonian can be written under

the form (2), if we put

$$U = \exp [-iHt]$$

$$U_1 = \exp [-i(H_0 + H_1)t]$$

$$U_0 = \exp [-iH_0t]$$

and if

$$(5.3) \quad S_0 = \lim_{t' \rightarrow \infty, t \rightarrow -\infty} U_0^{-1}(t') U_1(t' - t) U_0(t),$$

$$(5.3') \quad S_1 = \lim_{t' \rightarrow \infty, t \rightarrow -\infty} U_1^{-1}(t') U(t' - t) U_1(t),$$

are existing, then

$$(5.3'') \quad S = \lim_{t' \rightarrow \infty, t \rightarrow -\infty} U_0^{-1}(t') U(t' - t) U_0(t),$$

is existing too.

This may be proved remarking that if S_0 does exist it exists also

$$\Sigma_0 = \lim_{t \rightarrow \infty} U_0^{-1}(t) U_1(t),$$

$$\Sigma'_0 = \lim_{t \rightarrow -\infty} U_1(t) U_0(t).$$

In fact, as we have already remarked, a necessary and sufficient condition for the existence of S_0 is that:

$$\lim_{t' > t, t \rightarrow \infty} U_0^{-1}(t') U_1(t' - t) U_0(t) = 1,$$

(and the similar condition for $t' \rightarrow -\infty$).

But the preceding relation is also a necessary and sufficient condition for the existence of Σ'_0 . Indeed it is sufficient to remark that:

$$U_0^{-1}(t') U_1(t') = U_0^{-1}(t') U_1(t' - t) U_1(t) = U_0^{-1}(t') U_1(t' - t) U_0^{-1}(t) U_0(t) U(t).$$

Then of course necessary and sufficient conditions for the existence of S and S_1 are respectively the existence of the limits:

$$\Sigma = \lim_{t \rightarrow \infty} U_0^{-1}(t) U(t), \quad (\text{and } \Sigma'),$$

$$\Sigma_1 = \lim_{t \rightarrow \infty} U_1^{-1}(t) U(t), \quad (\text{and } \Sigma'_1),$$

but

$$U_0^{-1}(t) U(t) = U_0^{-1}(t) U_1(t) U_1^{-1}(t) U(t).$$

This result, remembering that all the operators under consideration are bounded, is sufficient to prove the existence of S . In our case we identify H_1 with (5.2) and then we may suppose safely the existence of the operator S_0 . The only thing which we have to do is therefore to construct the operator S_1 which has the meaning of an S -matrix defined taking as a free Hamiltonian not H_0 but $H_0 + H_1$.

For this purpose, we try to correct the interaction H_2 by a counter term H_c in such a way that the scalar product of any eigenstate of positive norm of $H_0 + H_1$ by any eigenstate of negative norm of $H + H_c$ be equal to 0. Let us suppose that we have determined some H_c satisfying this condition. Then we may construct the correspondence C in the following way. First we put in 1-1 correspondence the eigenstate of positive norm β'_i of the Hamiltonian H with the eigenstate of positive norm β'_{ic} of the Hamiltonian $H + H_c$. Then we put in correspondence V_g (physical eigenstate of one V particle corresponding to g) with V_{gc} and furthermore, to any state vector which behaves asymptotically (for a very far past or future) like

$$(5.4) \quad \psi = \sum \varphi_i a_i^* V + \sum \varphi_{ik} a_k^* a_i^* N$$

(where φ_i and φ_{ik} define wave packets which in the limiting case of very far past or future and $\Omega \rightarrow \infty$ do not have any spatial superposition with V) (*) we put in correspondence

$$(5.4') \quad \psi' = \sum \varphi_i a_i^* V_g + \sum \varphi_{ik} a_k^* a_i^* N.$$

Finally we express ψ' by means of the eigenstate of positive norm of $H + H_c$. This is actually possible given our hypothesis about H_c and given the fact that ψ' may be expressed by means of the eigenstate of positive norm of the Hamiltonian $H_0 + H_1$. So if

$$(5.4'') \quad \psi' = \sum \psi_i \beta'_{ci},$$

to ψ in $\mathcal{H}_2$ will correspond

$$(5.4''') \quad \psi'' = \sum \psi_i \beta'_i.$$

The correspondence C is so defined in such a way to conserve scalar products as required because both sets β'_i and β'_{ic} which define the correspondence are orthogonal sets, and because of properties of asymptotical localized states (cf. Appendix).

(*) Cfr. Appendix.

Now we have simply to indicate how to determine H_c and to show that the correspondence C which we have constructed in this manner satisfies the condition of allowing the definition of the S -matrix S_1 .

We shall indicate with β_i the eigenstate of positive norm of $H_0 + H_1$ with γ_i the eigenstate of negative norm of $H_0 + H_1$, and with γ'_{ci} the eigenstate of negative norm of $H + H_c$. The condition which we have imposed is

$$(5.5) \quad (\beta_i | \gamma'_{cl}) = 0$$

for all i and l .

We shall prove, now, that the necessary and sufficient condition for the relation (5.5) is

$$(5.5') \quad (\beta'_{ci} | H_2 + H_c | \gamma'_{cl}) = 0.$$

Let E_n and E'_n be the eigenvalue of $H_0 + H_1$ and $H + H_c$ respectively. Then:

$$\begin{aligned} E_n \beta_n &= (H_0 + H_1) \beta_n = (H + H_c - (H_2 + H_c)) \beta_n = \\ &= \sum E'_m \cdot \beta'_{cm} (\beta'_{cm} | \beta_n) + \sum E'_l \gamma'_{cl} (\gamma'_{cl} | \beta_n) - \\ &= \sum (H_2 + H_c) \cdot \beta'_{cm} (\beta'_{cm} | \beta_n) - \sum (H_2 + H_c) \gamma'_{cl} (\gamma'_{cl} | \beta_n) \end{aligned}$$

It follows:

$$\begin{aligned} E_n (\gamma'_{cl} | \beta_n) &= E'_l (\gamma'_{cl} | \beta_n) - \sum_m (\gamma'_{cl} | H_2 + H_c | \beta'_{cm}) (\beta'_{cm} | \beta_n) - \\ &= \sum_p \gamma'_{cl} | H_2 + H_c | \gamma'_{cp} (\gamma'_{cp} | \beta_n) \end{aligned}$$

and if (5.5') is satisfied,

$$(5.6) \quad (E_n - E'_l) (\gamma'_{cl} | \beta_n) = - \sum_p (\gamma'_{cl} | H_2 + H_c | \gamma'_{cp}) (\gamma'_{cp} | \beta_n).$$

In an analogous way we get:

$$(5.6') \quad (E_n - E'_n) (\beta'_{im} | \beta_n) = - \sum_q (\beta'_{im} | H_2 + H_c | \beta'_{cq}) (\beta'_{cq} | \beta_n)$$

and a similar equation for $(\gamma'_{cl} | \gamma_n)$.

It follows that if (5.5') is satisfied, first the equations for $(\beta'_{cm} | \beta_n)$, $(\gamma'_{cl} | \beta_n)$ and $(\gamma'_{cl} | \gamma_n)$ are not only independent but they are not connected. Second the Eq. (5.6) has the solution 0 as we want. The condition (5.5') is therefore sufficient for (5.5); it is clear that it will be also necessary if we ask that (5.5) holds identically with g' .

Now we like to make a remark. In general when there is crossing of levels E'_n , E_m due to some interaction the perturbation treatment of the interaction

will not be allowed. The reason of this fact is due to the expression $1/(E'_n - E_m)$ which appears in the perturbation expansion (of the « Wigner type ») and this expression gives rise to singularities due to the crossing. However, this will not be any more the case if all the terms of the expansion containing a denominator with crossing are identically equal to 0 because of the vanishing of the related matrix elements of the interaction. Now we are just in a similar situation, if the condition (5.5') holds. In fact, we shall have crossing of energy levels due to the fact that the energy of the ghost state is increasing when we increase the coupling constant. Therefore the eigenvalue of some of the states of negative norm will cross the eigenvalue of some other state of positive norm when we give a variation to g' . However, following our preceding remark, this fact will not produce any harm in a perturbation expansion of the transformation function $(\beta'_{cm}|\beta_n)$. In fact, this transformation function, owing to the condition (5.5') will satisfy the equation

$$(5.6'') \quad (\beta'_{cm}|\beta_n) = -\frac{(\beta'_{cm}|H_2 + H_c|\beta_n)}{E_n - E'_m}(\beta'_{cn}|\beta_n) - \sum_{p \neq n} \frac{(\beta'_{cm}|H_2 + H_c|\beta'_{cp})}{E_n - E'_m}(\beta'_{cp}|\beta_n),$$

and never crossing denominators will appear in the expansion obtained by iteration. One has to notice that we always renormalize the mass of the V physical state in such a way that it is independent of the value of g' . Of course, as the transformation between β_n and β'_{cn} is a unitary transformation, the possibility of the expansion obtained by iteration of (5.6'') is sufficient to grant us that it is possible to pass from $H_0 + H_1$ to $H + H_c$ by means of a perturbation expansion.

We can then use this circumstance first in order to determine H_c by means of an expansion in powers of Δg . We shall write

$$(5.7) \quad \begin{cases} H_c = \Delta g W, \\ W = W_0 + \Delta g W_1 + \dots, \\ H_2 = \Delta g H'. \end{cases}$$

We have now to determine by successive steps W_0, W_1 etc. Let us determine W_0 ; we get:

$$\begin{aligned} (\beta'_{cn}|H' + W|\gamma'_i) &= (\beta_n|H'\gamma_i) + (\beta_n|W_0\gamma_i) + \\ &+ \Delta g \left\{ (\beta_n|W_1\gamma_i) + \left(\sum_r \delta_r \frac{(\beta_n|H' + W_0|\delta_r)}{E_{\beta_n} - E_{\delta_r}} (W + H')k_i \right) + \right. \\ &\quad \left. + \left(\beta_n|H' + W|\delta_r \frac{(\delta_r|H' + W|\gamma_i)}{E_{\gamma_i} - E_{\delta_r}} \right) \right\}, \end{aligned}$$

where δ_r is any β) or γ) state. Therefore we may put:

$$(5.7') \quad \begin{cases} (\beta_r | W_0 | \beta_n) = 0; & (\gamma_l | W_0 | \gamma) = 0 \\ (\beta_r | W_0 | \gamma_s) = -(\beta | H' | \gamma_s) \end{cases}$$

and W_0 will be completely determined by (5.7') (*).

Proceeding in this way, we may determine step by step the operators W_1, W_2 , etc. For instance one gets that W_1 may be put equal to 0.

It remains now to control whether this procedure allows us to define the S -matrix S_1 . For this purpose we recall the proof of the existence of the S -matrix given in the previous Section in the case of the « double-band model ». One may easily recognize that we have made use essentially only of the fact that the transformation function between the eigenstate of KUK^{-1} and the eigenstate of U_0 in the limit of $\Omega \rightarrow \infty$ could be put under the form

$$(5.8) \quad \beta_{m,k} = B_k \delta(E_m - E_k) + \frac{\bar{\beta}_{m,k}}{E_m - E_k - i\varepsilon},$$

there $\bar{\beta}_{m,k}$ may be prolonged in an analytical function limited in a neighbourhood of the real axis interval on which $\bar{\beta}_{m,k}$ is directly defined.

Now, if we keep in mind our condition about the self energy of the V particle (which has to remain unchanged), we see that if we use a perturbation expansion, the transformation function $(\beta_m | \beta_{cn})$ between the states β_m and β'_{cn} may again be put under the form (5.8).

If we consider only the first order of the perturbation expansion, then the functions $\beta_{m,n}$ are merely the matrix elements of the interaction H_2 between the states β_m and β_n . Now, the decisive point to notice is that our procedure of determining the counter terms H_c merely suppresses the matrix elements of the interaction between β_m and γ'_{cl} and has no influence whatsoever on the matrix elements between β_m and β_n . This procedure, therefore, does not modify in no way the possibility of prolongating the function $\bar{\beta}_{m,n}$ in a function having the wanted analytical properties. This is essentially due to the fact that the eigenstate with positive and the eigenstate with negative norm of the Hamiltonian $H_0 + H_1$ do not go the one into the other in any continuous way. It may be further remarked that as our procedure may be obviously iterated the proof reached for first order perturbation expansion may

(*) There may be a correction in the self-energy term of the V -particle which has no importance for our purpose.

be immediately extended, at least so far as the transformation function may be prolonged as analytical function of the coupling constant g without going out of the real axis.

The same remark about the behaviour of the wave packets applies in this case as in the case of the «double-band» model.

Before closing this Section, we would like to add a few remarks. The first is concerning the fact that we have to make an essential use of a kind of cut-off in the form factor f_i . This fact, of course, would be rather annoying in a relativistic theory, but has no importance in the present case. Anyhow, the relevant remark is that we may construct a theory even with values of the coupling constant greater than the critical values if we take a cut-off. But as the cut-off may be as large as we like, this fact does not prevent to think of the possibility of the theory without a cut-off and with a coupling constant different from 0 as a limit of the case with a finite cut-off when the cut-off is going to infinity. The second remark is about the possibility of extension to a relativistic case. This possibility is very far from being obvious. However, we would like to point out that the rule of putting $=0$ all the matrix elements of the interaction between states of positive norm and states of negative norm seems in a certain way not too dangerous with respect to possibilities of a relativistic extension, if, granted the existence of eigenstates of the energy of positive norm and the eigenstate of the energy of negative norm, even in the relativistic case, one thinks that this subdivision has to be relativistic invariant.

Third, we notice that the «asymptotically localized states» do not form a complete set even in $\mathcal{H}_s$ and consequently the proposed procedure could give rise to new «bound states». This, however, does not seem to be the case in a simple Lee model.

6. - Conclusion.

We have shown that for the Lee model it is possible, with an appropriate redefinition of the in-going and out-going states, to construct a unitary S -matrix, even in the case in which the characteristic equation for the eigenvalues of the energy in the sector V , θN has only real distinct roots. However, against our procedure the objection may be obviously raised that it is not an unique prescription and therefore it is arbitrary. This is without any doubt so; but, we think that we have to expect this arbitrariness when we introduce the indefinite metrics. In fact, the introduction of the indefinite metrics does not allow any more to maintain the usual interpretation of states of quantum mechanics and to have a localized description of the system. Therefore, es-

essentially less detailed information about systems is possible than is in quantum mechanics with a positive definite metrics.

Consequently, in order to have a definite theory we have to establish which are the observables in the case of the indefinite metrics and in which manner these observables may be used in order to define the states. Without answering this question, it is obvious that the interpretation of the theory does remain arbitrary. However, it is sufficient to fix already some one of the observables to reduce greatly the arbitrariness of the theory.

In our example this is evident; in general, we had to establish some homomorphic correspondence between pseudo Hilbert space $\mathcal{H}_1$ and Hilbert space $\mathcal{H}_2$. Having fixed the energy as one of the observables already reduces enormously the possible homomorphism. As we have remarked in the introduction, the possible homomorphisms are reduced, indeed, to isomorphism between the subspace of $\mathcal{H}_1$, formed by the closure of the linear manifold of the eigenstate of the energy of positive norm, and $\mathcal{H}_2$. However, the energy only and the requirement that the asymptotical motion of the wave packets has to be described in exactly a similar manner as the motion of free wave packets are not sufficient to fix this isomorphism, and for that purpose it would be necessary to specify other observables of the system.

It is quite obvious that a further research would be hardly worth-while for answering this question for the Lee model. This model is too far from reality, and therefore as far as the Lee model is concerned, we have probably to be satisfied with the results which we have reached in the present investigation.

In conclusion, we have shown the possibility of an approach to the problem of the physical interpretation of a quantum theory with an indefinite metric, which is perhaps in some aspect new and which, as far as we know, is not much worst than the previous approach.

Whether it is any better it is a question which presently I cannot answer.

APPENDIX

We would like, in this appendix, to indicate how the notion of the localization of the θ -particle may be defined in the Lee model. Let us start with a trivial case in which we have not any «heavy» particles present, and let us consider the sector in which we have only one θ -particle. Then we consider the operator

$$(A.1) \quad H_r = \int f^2(\mathbf{x}) [\pi_0^2 + (\text{grad } \psi_0)^2 + \mu^2 \psi_0^2] d\mathbf{x}.$$

The function f^2 defines a dominion in space (the «region f^2 ») in which it is different from zero.

The operators ψ_0 and π_0 are the operators of the θ field and its canonical conjugate operator. It is easy to prove that the operator (A.1) first has no negative eigenvalue (for this purpose it is sufficient to show that the expectation value of H_f is always positive or 0 for any state belonging to the sector 1 θ). Second: supposing that f^2 is different from zero only in a limited domain in the limit $\Omega \rightarrow \infty$ it is possible to construct states belonging to the sector 1 θ for which the expectation value for the operator H_f is less than any given quantity.

This means that in this case the operator H_f has eigenvalues which are smaller than any prefixed positive quantity.

In the case in which the volume Ω where we quantize the system is not actually infinite if the domain in which $f^2 \neq 0$ is much smaller than Ω , it is possible to show that one can always construct states in such a way that the expectation value of the operator H_f is 0 ($\exp[-\Omega^{\frac{1}{2}}\mu]$).

Given then some f^2 for which it does exist some eigenvalue of the operator H_f less than ε we may consider the subspace of all the state functions belonging to the sector 1 θ and which are normal to the manifold of the eigenstate of the operator H_f having an eigenvalue $\leq \varepsilon$. This subspace will be called the subspace of states in which the θ -particle is localized out of the region defined by f^2 with the precision ε .

Of course, this notion may be extended even to a case in which the function f^2 is equal to 0 only in a limited domain. If the domain contains a sphere of radius R the precision may be of order $\exp[-R\mu]$. Having defined the notion of localization with a precision ε out of a region defined by a function f^2 , we have at the same time defined the notion of the localization inside the region defined by $1 - f^2$.

Coming back to the case in which f^2 is different from 0 only in a limited domain, R_{f^2} , we may define the notion of an «asymptotical localization» out of R_{f^2} , if the precision of the localization out of R_{f^2} is going to zero.

Of course the notion of asymptotical localization may be extended immediately to the case in which f^2 is not exactly 0 out of a finite domain, but is going to 0 with a sufficient rapidity with increasing distance from a fixed point.

Now we may extend these definitions to the case in which we have a θ -particle in presence of a «heavy» particle. The case in which the «heavy» particle is an N particle is trivial. We shall consider therefore briefly the case of a θ -particle in presence of a physical V-particle. Let us consider the vector:

$$(A.2) \quad \psi) = \sum_k \chi_k a_k^* V),$$

where

$$(A.2') \quad V) = (\psi_v + N \sum \Phi_i a_i^*) \cdot 0),$$

is the state in which we have a physical V-particle.

Now we notice that the function:

$$(A.3) \quad f_V^2 = |\sum \Phi_i \exp[ik_i \cdot x]|^2,$$

is going to 0 exponentially when $|x|$ is going to infinite and we may therefore define the asymptotical localized state for the 1θ sector with respect to the function f_V^2 following the definition which we have given. It is immediately verified that if the vector $\chi \cdot 0$, where

$$(A.4) \quad \chi = \sum \chi_k a_k^*,$$

is localized out of R_{r_V} with the precision ε then

$$(A.5) \quad (0 | [(\psi_V + N \sum \Phi_i a_i^*), \chi] - 0) = 0(\varepsilon).$$

It follows also that if $\chi_1 \cdot 0$ and $\chi_2 \cdot 0$ are two orthogonal states localized out of R_{r_V} with the precision ε ,

$$(A.6) \quad (V \chi_1 | \chi_2 V) = 0(\varepsilon).$$

The relations (A.5) and (A.6) allow us to talk in the case of asymptotical localization out of R_{r_V} of states representing a particle θ in the state $\chi \cdot 0$ and one physical particle V without any ambiguity.

This is not always the case; for instance this is not the case for the states $a_k^* V$. We cannot say that this state does represent a particle θ of momentum k and a physical V -particle. In fact, the operator a_k^* does not commute with the operator $\psi_V + N \sum \Phi_i a_i^*$. It is true that the commutator is of the order $1/\Omega^{\frac{1}{2}}$ where Ω is as usual the volume of integration, and therefore it goes to 0 when the volume is going to infinity. However, it does not go to zero with a sufficient rapidity. This may be easily seen in the following manner. Let us consider the set of states

$$(A.7) \quad \varphi_k = a_k^* V,$$

and let us suppose that ψ_1 and ψ_2 are any two states which may be represented by the following expression

$$(A.7') \quad \psi_1 = \sum \psi_k^{(1)} \varphi_k, \quad \psi_2 = \sum \psi_k^{(2)} \varphi_k.$$

If we could consider even in the limit of $\Omega \rightarrow \infty$ the set (A.7) as an orthogonal set, then it should be $(\psi_1 | \psi_2) = \sum \psi_k^{(1)*} \psi_k^{(2)}$. Instead we have exactly:

$$(A.8) \quad (\psi_1 | \psi_2) = \sum \psi_k^{(1)*} \psi_k^{(2)} + \frac{g^2}{2\Omega} \sum_i \frac{f_i \psi_{k_i}^{(1)*}}{\omega_i} \sum_l \frac{f_l \psi_{k_l}^{(2)}}{\omega_l}.$$

Now $\psi_k \sim 1/\Omega^{\frac{1}{2}}$ and therefore, remembering that $\sum f_i \sim \Omega$, it follows that $1/\Omega^{\frac{1}{2}} \sum f_i \psi_{ki}/\omega_i$ is in general different from 0, even in the case of $\Omega \rightarrow \infty$. Therefore the states $a_{k_1}^* V$ and $a_{k_2}^* V$ cannot be considered orthogonal, not even in the limit $\Omega \rightarrow \infty$ and consequently it cannot be said with a clear meaning that they do represent a θ -particle with a momentum k_1 , and respectively k_2 , together with a physical V-particle. The case of asymptotically localized packet is different because the precision ε may go to zero *exponentially* with $\Omega^{\frac{1}{2}} \rightarrow \infty$.

We would like now to make some remark about the procedure followed in Sects. 4 and 5. Let us consider, for instance, the procedure of Sect. 5.

There we were considering two different values of the coupling constant g and g' and the corresponding states of the physical V-particle V_g and $V_{g'}$. It is essential to remark that if $\chi \cdot 0$ is asymptotically localized out of $R_{f_{V_g}}$, it is also asymptotically localized out of $R_{f_{V_{g'}}}$. There is not therefore any difficulty whatsoever from a kinematical point of view when one represents a physical situation in which there is a physical V-particle and a θ -particle in the asymptotical localized state $\chi \cdot 0$ by means of $\chi \cdot V_g$ even in the case in which the coupling constant has a value g' , following the prescription of Sect. 5.

We would like to add a last remark. Let us consider a «one band» model in which, however, the form factor is given by a Gaussian. It is then easy to verify that a Lee model of this kind is not worst, about «locality», than the usual one. Now, it is easy to find that the standing waves in the sector θV corresponding to the case of a θ -particle in presence of a V-particle are of the form

$$(A.9) \quad \psi^{(m)} = \sum_i \psi_i^{(m)} a_i^* \left[N I I \nabla 00 \right) + \frac{g}{\sqrt{2\Omega}} \sum_r \frac{f_r}{\omega_r} \cdot I_N I_r \right] + \\ + \frac{g}{\sqrt{2\Omega}} \sum_{i,r} \psi_i^{(m)} \frac{z_m - \omega_i}{\omega_r^2} f_r \cdot I_N I_i I_r ,$$

with

$$(A.9') \quad \psi_i^{(m)} = c_m f_i \left[\frac{1}{(m - l + \eta_m)} + c'_m \right] .$$

It has to be noted that this is the form in the case in which $g < g_c$. Now in an ordinary case of a system having forces with a finite range, one should expect that in case of two particles, the standing waves in the limiting case of $\Omega \rightarrow \infty$ should be simply a product of standing waves of the first particle by standing waves of the second particle. This is not the case with our model because the last term of the second member of (A.9) is not going to 0 when $\Omega \rightarrow \infty$. This is a rather strange behaviour and it may originate the suspect that even when $g < g_c$ a θ -particle is never dynamically independent from a V-particle, also when it is localized at a very far distance from the V-particle. This, however, is not the case (when $g < g_c$) and this may be verified applying to the standing wave represented by Eq. (A.9) any operator χ^* where $\chi \cdot 0$ represents an asymptotically localized θ -particle state. It is then possible to verify that the last term of the second member of Eq. (A.9) does not give

any contribution. That means that when the θ -particle is really localized at a very far distance of the V-particle, the V-particle is left unchanged. Therefore, one may be fairly sure that the trouble which arises in the Lee model when $g > g_c$ depends really from the negative metric and not from any peculiarity of the Lee model itself.

RIASSUNTO

Nel presente lavoro si discute di nuovo il modello di Lee nel caso già considerato da PAULI e KÄLLÉN, cioè nel caso in cui le radici dell'equazione caratteristica dell'energia sono tutte reali e distinte. Si mostra che è possibile ridefinendo gli stati entranti e uscenti costruire una matrice S unitaria e con proprietà fisiche ragionevoli. Il procedimento suggerito è in qualche modo simile a un procedimento di rinormalizzazione ed elimina gli effetti dei fantasmi.