On the effects of turbulence on a screw dynamo

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In an experiment in the Institute of Continuous Media Mechanics in Perm (Russia) an non–stationary screw dynamo is intended to be realized with a helical flow of liquid sodium in a torus. The flow is necessarily turbulent, that is, may be considered as a mean flow and a superimposed turbulence. In this paper the induction processes of the turbulence are investigated within the framework of mean–field electrodynamics. They imply of course a part which leads to an enhanced dissipation of the mean magnetic field. As a consequence of the helical mean flow there are also helical structures in the turbulence. They lead to some kind of α–effect, which might basically support the screw dynamo. The peculiarity of this α–effect explains measurements made at a smaller version of the device envisaged for the dynamo experiment. The helical structures of the turbulence lead also to other effects, which in combination with a rotational shear are potentially capable of dynamo action. A part of them can basically support the screw dynamo. Under the conditions of the experiment all induction effects of the turbulence prove to be rather weak in comparison to that of the mean flow. Numerical solutions of the mean–field induction equation show that all the induction effects of the turbulence together let the screw dynamo threshold slightly, at most by one per cent, rise. The numerical results give also some insights into the action of the individual induction effects of the turbulence.

Keywords: Dynamo experiment, mean–field electrodynamics

1 Introduction

A screw–like, that is, helical motion of an electrically conducting fluid is capable of dynamo action. This has been first theoretically shown by Ponomarenko (1973). Later, in 1999, it was experimentally demonstrated in the liquid sodium facility of the Institute for Physics in Riga (Latvia) (Gailitis et al. 2000, 2001a,b, 2002a,b, 2003, 2004). It is also the background of another liquid sodium experiment which is under preparation in the Institute for Continuous Media Mechanics in Perm (Russia) (Denisov et al. 1999, Frick et al. 2001, 2002) (Denisov et al. 1999, Frick et al. 2001, 2002).

In this experiment a torus-shaped vessel filled with liquid sodium is rotated about its symmetry axis and then suddenly stopped. After the stop a flow of sodium occurs inside the vessel and decays in the course of time. Due to diverters, that is, proper arrangements of blades inside the vessel, the flow becomes helical. In this way the possibility of a dynamo of Ponomarenko type arises for a certain time interval. The large radius of the torus is about 0.40 m and the small one about 0.12 m. Rotation rates up to 50 rps are envisaged. Estimates show that magnetic Reynolds numbers sufficient for dynamo action are indeed feasible.

The critical magnetic Reynolds number is definitely higher than 10. For liquid sodium the hydrodynamic Reynolds number exceeds the magnetic Reynolds number by a factor of about $10^5$. Therefore in the case of a dynamo the fluid flow is necessarily turbulent, that is, may be considered as consisting of a main flow and a superimposed turbulence. The turbulence affects the dynamo in two ways. Firstly it influences the profile of the main flow. Secondly it leads to small–scale induction effects. In particular the turbulence will enlarge the effective magnetic diffusivity of the fluid, what raises the dynamo threshold. Likewise other small–scale induction effects will occur. As a consequence of helical features of the turbulence an α–effect or related effects are possible. Basically they could also support the screw dynamo or even open the possibility of another type of dynamo.

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In this paper we want to estimate these direct influences of turbulence on the electromagnetic field on the basis of mean–field electrodynamics. We will rely on a recent general calculation of the mean electromotive force due to turbulence (Rädler and Stepanov 2003), which was partially motivated by this experiment. In section 2 we describe the general concept of our investigation. In section 3 we discuss possible structures of the mean electromotive force which can be concluded from symmetry arguments, and in section 4 we explain results of more detailed calculations. On this basis we deliver in section 5 estimates for the influence of the turbulence on the dynamo. Finally in section 6 the main results are summarized.

2 The mean–field concept

Let us first recall some ideas of the mean–field concept which are important for the following (see, e.g., Krause and Rädler 1971, Moffatt 1978, Rädler 2000). We start from the induction equation that governs the behavior of the magnetic field \( B \) in an electrically conducting fluid,

\[
\partial_t B - \nabla \times (U \times B) - \eta \nabla^2 B = 0, \quad \nabla \cdot B = 0.
\]  

(1)

Here \( U \) is the velocity of the fluid and \( \eta \) its magnetic diffusivity, which is assumed to be constant.

The velocity \( U \) and, as a consequence, also the magnetic field \( B \) are assumed to consist of large–scale parts and small–scale turbulent fluctuations. We define the mean magnetic and velocity fields \( \overline{B} \) and \( \overline{U} \) as averages over space or time scales larger than those of the turbulence and put \( B = \overline{B} + b \) analogously to \( U = \overline{U} + u \). We assume that the Reynolds averaging rules apply. Taking the average of equation (1) we obtain the mean–field induction equation

\[
\partial_t \overline{B} - \nabla \times (\overline{U} \times \overline{B} + \mathcal{E}) - \eta \nabla^2 \overline{B} = 0, \quad \nabla \cdot \overline{B} = 0,
\]  

(2)

where \( \mathcal{E} \) is an electromotive force due to the fluctuations of magnetic field and velocity,

\[
\mathcal{E} = u \times \overline{B}.
\]  

(3)

The equation for \( b \) resulting from (1) and (2) allows us to conclude that \( \mathcal{E} \) is determined by \( \overline{U}, u \) and \( \overline{B} \). More precisely, \( \mathcal{E} \) at a given point in space and time depends not only on \( \overline{U}, u \) and \( \overline{B} \) at this point but also on their behavior in a certain neighborhood of this point. We adopt the frequently used assumption that \( \overline{B} \) varies only weakly in space and time so that \( \mathcal{E} \) in a given point depends on \( \overline{B} \) only via its components and their first spatial derivatives in this point. Then \( \mathcal{E} \) can be represented in the form

\[
\mathcal{E}_i = a_{ij} \overline{B}_j + b_{ijk} \partial \overline{B}_j / \partial x_k,
\]  

(4)

with tensors \( a_{ij} \) and \( b_{ijk} \) being averaged quantities determined by \( \overline{U} \) and \( u \). Here and in the following we refer to a Cartesian co–ordinate system \( (x_1, x_2, x_3) \) and adopt the summation convention. Relation (1) is equivalent to

\[
\mathcal{E} = -\alpha \circ \overline{B} - \gamma \times \overline{B} - \beta \circ (\nabla \times \overline{B}) - \delta \times (\nabla \times \overline{B}) - \kappa \circ (\nabla \overline{B})^{(s)}
\]  

(5)

(see Rädler 1980, 2000), where \( \alpha \) and \( \beta \) are symmetric tensors of the second rank, \( \gamma \) and \( \delta \) are vectors, and \( \kappa \) is a tensor of the third rank, all depending on \( \overline{U} \) and \( u \) only. \( (\nabla \overline{B})^{(s)} \) is the symmetric part of the gradient tensor of \( \overline{B} \), i.e. \( (\nabla \overline{B})^{(s)}_{jk} = \frac{1}{2}(\partial \overline{B}_j / \partial x_k + \partial \overline{B}_k / \partial x_j) \). Notations like \( \alpha \circ \overline{B} \) or \( \kappa \circ (\nabla \overline{B})^{(s)} \) are used in the sense of \( (\alpha \circ \overline{B})_i = \alpha_{ij} \overline{B}_j \) or \( (\kappa \circ (\nabla \overline{B})^{(s)})_i = \kappa_{ijk}(\nabla \overline{B})^{(s)}_{jk} \). The term with \( \alpha \) in (5) describes the \( \alpha \)–effect, which is in general anisotropic, that with \( \gamma \) a transport of mean magnetic flux by the turbulence. The terms with \( \beta \) and \( \delta \) can be interpreted by introducing a modified magnetic diffusivity, again in general anisotropic. They are usually accompanied by the \( \kappa \) term, which allows no simple interpretation.
3 The structure of the mean electromotive force $E$ in the presence of a screw motion

3.1 Mean motion

We consider the Ponomarenko dynamo here in its original cylindrical geometry. As observed in other investigations this provides a reasonable approximation for the situation in a torus even for aspect ratios as large as in the planned device [2000]. We assume that it is primarily a helical mean flow described by the mean velocity $\mathbf{U}$ which generates or maintains the mean magnetic field $\mathbf{B}$. Referring to a proper cylindrical co-ordinate system $(r, \varphi, z)$ we put

$$\mathbf{U} = (0, \varphi(r), z(r)), \quad \varphi = \Omega(r)r.$$  \hspace{1cm} (6)

As mentioned above we will study the influence of the turbulence, that is, of the velocity field $\mathbf{u}$, on the mean magnetic field $\mathbf{B}$ in the framework of mean-field electrodynamics.

For a more detailed discussion of the mean electromotive force $E$ caused by the turbulence and for its calculation we change temporarily from the frame of reference in which (6) applies to another one rotating about the axis $r = 0$ and moving along it. Let us consider $E$ in a given point at the surface $r = r_0$. The moving frame is then fixed such that the mean velocity of the fluid at the surface $r = r_0$ is equal to zero. The result for $E$ found in this rotating system, considered as a vector with the usual transformation properties, applies in the original frame of reference as well. In the moving frame the mean velocity $\mathbf{U}$ has, again in the cylindrical co-ordinate system introduced above, the form

$$\mathbf{U} = (0, (\Omega(r) - \Omega(r_0))r, \varphi(r) - \varphi(r_0)).$$  \hspace{1cm} (7)

In contrast to the original frame, in the rotating one a Coriolis force occurs, which is defined by an angular velocity $\Omega$, given by

$$\Omega = \Omega(r_0) \hat{z},$$  \hspace{1cm} (8)

where $\hat{z}$ means the unit vector in the $z$-direction of the cylindrical co-ordinate system.

3.2 Homogeneous turbulence

Consider now $E$ in the moving frame at a point with $\mathbf{U} = 0$. Then $E$ and so the quantities $a_{ij}$ and $b_{ijk}$, or $\alpha$, $\gamma$, $\beta$, $\delta$ and $\kappa$, depend on $\mathbf{U}$ and $\mathbf{u}$ in a certain neighborhood of this point. The dependence on $\mathbf{u}$ implies again a dependence on $\mathbf{U}$ and of course also a dependence on $\Omega$. We restrict our attention now to the case in which $\mathbf{U}$ varies only weakly in the relevant neighborhood and assume that its behavior can be there sufficiently precisely described by the gradient tensor $\nabla \mathbf{U}$, with respect to Cartesian co-ordinates defined by $(\nabla \mathbf{U})_{ij} = \partial \mathbf{U}_j / \partial x_i$. In addition we assume until further notice that the turbulent fluctuations $\mathbf{u}$ deviate from a homogeneous isotropic mirror-symmetric turbulence only as a consequence of the gradient $\nabla \mathbf{U}$ of $\mathbf{U}$ and the Coriolis force defined by $\Omega$. We may split $\nabla \mathbf{U}$ into its symmetric and antisymmetric parts. The first one is the rate of strain tensor $\mathbf{D}$, that is, $D_{ij} = \frac{1}{2}(\partial \mathbf{U}_i / \partial x_j + \partial \mathbf{U}_j / \partial x_i)$. It describes the deforming motion near the point considered. Due to the incompressibility of the fluid we have $\nabla \cdot \mathbf{U} = 0$ and therefore $D_{ii} = 0$. The second part, $\mathbf{A}$, given by $A_{ij} = \frac{1}{2}(\partial \mathbf{U}_i / \partial x_j - \partial \mathbf{U}_j / \partial x_i)$, corresponds to a rigid body rotation of the fluid near this point. We may represent it according to $A_{ij} = \frac{1}{2}\epsilon_{ijl}W_l$ by a vector $\mathbf{W} = \nabla \times \mathbf{U}$. Until further notice we do not use specifications of $\mathbf{D}$ and $\mathbf{W}$ according to (7). They will be introduced only later.

We now utilize the symmetry properties of the basic equations governing $\mathbf{B}$ and $\mathbf{U}$, or $\mathbf{B}$, $\mathbf{U}$, $\mathbf{b}$ and $\mathbf{u}$, from which $E$ and so the quantities $a_{ij}$ and $b_{ijk}$, or $\alpha$, $\gamma$, $\beta$, $\delta$ and $\kappa$, have to be derived. They allow two important conclusions (see also, e.g., [2005]). The first reads that these quantities cannot contain any other construction elements than the isotropic tensors $\delta_{lm}$ and $\epsilon_{lmn}$, the vectors $\Omega$ and $\mathbf{W}$ and the tensor $\mathbf{D}$. The second uses the distinction between polar and axial vectors as $\mathbf{U}$ and $\mathbf{B}$,
respectively, and between true and pseudo tensors. It reads that \( a_{ij} \) and \( b_{ijk} \), further \( \alpha \) and \( \kappa \) are pseudo tensors, \( \beta \) is a true tensor, \( \gamma \) a polar vector, and \( \delta \) an axial vector. It is then important that \( \Omega \) and \( W \) are axial vectors and \( D \) is a true tensor.

Let us consider first \( \alpha_{ij} \) and \( \gamma_i \). The mentioned construction elements \( \delta_{lm}, \epsilon_{lm}, \Omega, W \) and \( D \) allow us neither to build a pseudo tensor of the second rank nor a polar vector. That is, we have

\[
\alpha_{ij} = 0, \quad \gamma_i = 0.
\] (9)

We stress that this applies independently of any assumptions concerning linearity in \( \Omega, W \) and \( D \) like those we will introduce below.

In contrast to this there are several non–zero contributions to \( \beta_{ij}, \delta_i \) and \( \kappa_{ijk} \). For the sake of simplicity we give here only those of which they are linear in \( \Omega, W \) and \( D \),

\[
\beta_{ij} = \beta^{(0)} \delta_{ij} + \beta^{(D)} D_{ij}, \quad \delta_i = \delta^{(0)} \Omega_i + \delta^{(W)} W_i, \\
\kappa_{ijk} = \frac{1}{2} \kappa^{(0)} (\Omega_j \delta_{ik} + \Omega_k \delta_{ij}) + \frac{1}{2} \kappa^{(W)} (W_j \delta_{ik} + W_k \delta_{ij}) + \kappa^{(D)} (\epsilon_{ijl} D_{kl} + \epsilon_{ikl} D_{jl}).
\] (10)

The coefficients \( \beta^{(0)}, \beta^{(D)}, \delta^{(0)}, \ldots, \kappa^{(D)} \) are determined by \( u \) but do not depend on \( \Omega, W \) and \( D \). Because of \( \nabla \cdot \mathbf{B} = 0 \), terms of \( \kappa_{ijk} \) containing \( \delta_{jk} \) would not contribute to \( \mathcal{E} \) and have therefore been dropped.

As a consequence of (9) and (10) we have

\[
\mathcal{E} = -\beta^{(0)} \nabla \times \mathbf{B} - \beta^{(D)} D \circ (\nabla \times \mathbf{B}) \\
- (\delta^{(0)} \Omega + \delta^{(W)} W) \times (\nabla \times \mathbf{B}) \\
- (\kappa^{(0)} \Omega + \kappa^{(W)} W) \circ (\nabla \mathbf{B})^{(s)} - \kappa^{(D)} \kappa(D) \circ (\nabla \mathbf{B})^{(s)},
\] (11)

where \( \kappa(D) \) is a tensor of the third rank defined by \( \hat{\kappa}_{ijk} = \epsilon_{ijl} D_{kl} + \epsilon_{ikl} D_{jl} \).

The \( \beta^{(0)} \) and \( \beta^{(D)} \) terms in (11) make that the mean–field diffusivity deviates from the original, that is, molecular magnetic diffusivity of the fluid. Due to the \( \beta^{(D)} \) term the mean–field diffusivity becomes anisotropic. The \( \delta^{(0)} \) and \( \delta^{(W)} \) terms, too, can be discussed as contributions to the mean–field diffusivity. They lead to skew–symmetric contributions to the diffusivity tensor. In another context the effect described by the \( \delta^{(0)} \) term has been called “\( \Omega \times J \)–effect”. It has been shown that this effect in combination with a differential rotation, here a dependence of \( \Omega \) or \( r \), is able to establish a dynamo [Rädler 1969, Roberts 1972, Moffatt and Proctor 1982, Rädler et al. 2003]. The \( \delta^{(W)} \) term describes an effect analogous to the \( \Omega \times J \)–effect, which has been revealed only recently [Rogachevskii and Kleedorir 2003, Rädler and Stepanov 2005]. It occurs even in the absence of the Coriolis force as consequence of a shear in the mean motion. The possibility of a dynamo due to this effect is still under debate, see section 5.3. We refrain from discussing the \( \kappa^{(0)}, \kappa^{(W)} \), and \( \kappa^{(D)} \) terms here.

### 3.3 Inhomogeneous turbulence

Let us now relax the assumption that the original turbulence is homogeneous and isotropic. We admit an inhomogeneity and an anisotropy due to a gradient of the turbulence intensity and introduce a vector \( g \) in the direction of this gradient by putting \( \nabla u^2 = gu^2 \). Then we have to add \( g \), which is a polar vector, to the above–mentioned construction elements of \( \alpha_{ij}, \gamma_i, \beta_{ij}, \delta_i \) and \( \kappa_{ijk} \). As a consequence \( \alpha \) and \( \gamma \) can well be non-zero. For the sake of simplicity we assume that the influence of \( g \) on \( \alpha, \beta, \gamma, \delta \) and \( \kappa \) is so weak that they are at most of first order in \( g \). We have then

\[
\alpha_{ij} = \alpha^{(0)}_{ij}(g \cdot \Omega) \delta_{ij} + \alpha^{(0)}_{ij}(g \Omega_j + g_j \Omega_i) + \alpha^{(W)}_{ij}(g \cdot W) \delta_{ij} + \alpha^{(W)}_{ij}(g_i W_j + g_j W_i) \\
+ \alpha^{(D)}(\epsilon_{ilm} D_{jl} + \epsilon_{jlm} D_{il}) g_m
\] (12)
whereas \( \Omega \) remains unchanged.

The general form of \( \mathcal{E} \) reads now

\[
\mathcal{E} = -\alpha_1^{(\Omega)} (g \cdot \Omega) \mathbf{B} - \alpha_2^{(\Omega)} ((\Omega \cdot \mathbf{B}) g + (g \cdot \mathbf{B}) \Omega) \\
- \alpha_1^{(W)} (g \cdot \mathbf{W}) \mathbf{B} - \alpha_2^{(W)} ((\mathbf{W} \cdot \mathbf{B}) g + (g \cdot \mathbf{B}) \mathbf{W}) \\
- \alpha_1^{(D)} \hat{\alpha}(g, D) \circ \mathbf{B} \\
- (\gamma^{(\Omega)} g + \gamma^{(W)} g \times \Omega + \gamma^{(W)} \mathbf{g} \times \mathbf{W} + \gamma^{(D)} g \circ \mathbf{D}) \times \mathbf{B} \\
- \beta^{(\Omega)} \nabla \times \mathbf{B} - \beta^{(D)} D \circ (\nabla \times \mathbf{B}) \\
- (\delta^{(\Omega)} \Omega + \delta^{(W)} \mathbf{W}) \times (\nabla \times \mathbf{B}) \\
- (\kappa^{(\Omega)} + \kappa^{(W)} D) \circ (\nabla \mathbf{B})^{(s)} - \kappa^{(D)} \hat{\kappa}(D) \circ (\nabla \mathbf{B})^{(s)},
\]

where \( \hat{\alpha}(g, D) \) is a symmetric tensor defined by \( \hat{\alpha}_{ij} = (\epsilon_{ilm} D_{lj} + \epsilon_{jlm} D_{li}) g_m \). In the following the induction effects described by the individual contributions to \( \mathcal{E} \) are called \( \alpha^{(\Omega)} \)-effect, \( \alpha^{(W)} \)-effect, \( \cdots \kappa^{(D)} \)-effect or, more summarizing, \( \alpha \)-effects, \( \beta \)-effects, \( \cdots \kappa \)-effects.

The occurrence of an \( \alpha \)-effect as a consequence of the gradient of the turbulence intensity and the Coriolis force, that is, of \( g \) and \( \Omega \), is known for a long time. The possibility of an \( \alpha \)-effect due to a combination of \( g \) and \( \mathbf{W} \), or of \( g \) and \( \mathbf{D} \), has however been revealed only recently (Rogachevskii and Kleeorin 2003; Rädler and Stepanov 2005). Likewise the \( \gamma \)-effect and its modification by the Coriolis force, that is, \( \Omega \), is known for a long time, but modifications by \( \mathbf{W} \) and \( \mathbf{D} \) have been found only in the last–mentioned investigations.

### 3.4 \( \alpha \)-effect

We still remain in the co–moving frame defined above, in which \( \mathbf{U} \) is given by \( \mathbf{T} \), and consider a point with \( r = r_0 \). As a consequence of \( \mathbf{T} \) both \( \mathbf{W} \) and \( \mathbf{D} \) take specific forms. The non–zero components of \( \mathbf{W} \) are \( W_\varphi = -dU_z/dr \) and \( W_z = rd\Omega/dr \). As can be concluded with the help of the relations in the Appendix the only non–zero elements of \( \mathbf{D} \) are \( D_{r\varphi} = D_{\varphi r} = \frac{\varphi}{\varphi} d\Omega/dr \) and \( D_{rz} = D_{zr} = \frac{1}{2} dU_z/dr \). As for \( \Omega \) we refer to \( \mathbf{T} \).

In view of the Perm dynamo experiment the simplest assumption on \( g \) is that it is a radial vector. In this case the tensor \( \alpha \) has a remarkable peculiarity. The terms with \( \alpha_1^{(\Omega)} \) and \( \alpha_1^{(W)} \) disappear. In the cylindrical co–ordinate system \( \alpha \) has the form

\[
\alpha = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & 0 \end{pmatrix}
\]

\[
\alpha_1 = -\left( \alpha_2^{(\omega)} - \frac{1}{2} \alpha^{(D)} \right) g \frac{dU_\varphi}{dr}, \quad \alpha_2 = \left( \alpha_2^{(\omega)} - \frac{1}{2} \alpha^{(D)} \right) g \frac{dU_\varphi}{dr} + \alpha_2^{(\Omega)} g \Omega,
\]

where \( U_\varphi \) and \( U_z \) are, of course, measured in the rotating frame of reference. All diagonal elements of \( \alpha \) are equal to zero. Therefore a mean magnetic field in \( z \) direction does not generate any mean electromotive force in \( \varphi \) or \( z \) direction. The last statement remains true even if we relax the assumption on linearity with respect to \( g \).

In this context experiments with torus–shaped devices similar to but smaller than that prepared for the dynamo experiment are of interest. In an experiment with water instead of sodium clearly helical
small-scale motions have been observed [Frick et al. 2002]. This suggests the existence of an $\alpha$–effect in the conducting fluid. In the case of a complete $\alpha$–effect an azimuthal mean magnetic field leads to a mean electromotive force with a non–zero azimuthal component. However, our above result concerning $\alpha$, with the $z$ direction interpreted as the azimuthal one, does not predict such a component. The existence of helical motions alone is not sufficient for the occurrence of a complete $\alpha$–effect. Another experiment with gallium instead of sodium was carried out, too [Noskov et al. 2004]. It clearly confirmed that there is indeed no azimuthal electromotive force or electric current due to an azimuthal magnetic field.

4 Specific results for the mean electromotive force $\mathcal{E}$

Our considerations on the structure of the electromotive force $\mathcal{E}$ did not provide us with relations between the coefficients $\alpha^{(\Omega)}_1$, $\alpha^{(\Omega)}_2$, $\cdots$, $\kappa^{(D)}$ and the parameters of the turbulent flow. We rely here on recent calculations of these coefficients presented in detail in a paper by Rädler and Stepanov (2005). They were carried out just under the restrictions introduced above, namely that $\mathcal{E}$ is only weakly influenced by the Coriolis force and the gradient of the mean velocity so that it is linear in $\Omega$, $W$ and $D$. In addition, again in the sense of restrictions already made, it was assumed that all mean quantities determining $\mathcal{E}$ vary only weakly in space and time so that it is also linear in the operator $\nabla$ acting on quantities like $\overline{u^2}$ or $\overline{B}$, and contains no time derivatives of them. Finally the second–order correlation approximation was used. In this approximation the velocity correlation tensor $v_{ij}$ of the second rank,

$$v_{ij}(\mathbf{R}, T; \mathbf{r}, t) = u_i(\mathbf{R} + \mathbf{r}/2, T + t/2)u_j(\mathbf{R} - \mathbf{r}/2, T - t/2),$$  \hspace{1cm} (15)$$
plays a central part. The calculations have partially been carried out in the Fourier space, and the Fourier–transformed correlation tensor $\tilde{v}_{ij}$ has been used, defined such that

$$v_{ij}(\mathbf{R}, T; \mathbf{r}, t) = \int \int \tilde{v}_{ij}(\mathbf{R}, T; k, \omega) \exp(i(\mathbf{k} \cdot \mathbf{r}) - \omega t) \, d^3k \, d\omega .$$  \hspace{1cm} (16)$$

The “original” turbulence, which occurs in the limit of vanishing Coriolis force and vanishing gradient of the mean motion, is assumed to deviate from a homogeneous isotropic and mirror–symmetric one only by a gradient of its intensity. The corresponding Fourier–transformed correlation tensor $\tilde{v}_{ij}$ was assumed to have the form

$$\tilde{v}_{ij}(\mathbf{R}, T; k, \omega) = \frac{1}{2} \left( \delta_{ij} - \frac{k_i k_j}{k^2} + \frac{i}{2k^2} (k_i \nabla_j - k_j \nabla_i) \right) W(\mathbf{R}, T; k, \omega),$$  \hspace{1cm} (17)$$

with some function $W$ describing the kinetic energy distribution in the Fourier space. It was chosen such that

$$\overline{u(\mathbf{R} + \mathbf{r}/2, T + t/2) \cdot u(\mathbf{R} - \mathbf{r}/2, T - t/2)} = \overline{u^2}(\mathbf{R}, T) \exp(-r^2/2\lambda_c^2 - t/|\tau_c|),$$  \hspace{1cm} (18)$$

where $\overline{u^2}$ describes the turbulence intensity in the limit of vanishing Coriolis force and mean velocity gradient, and $\lambda_c$ and $\tau_c$ are correlation length and time in this limit.

The results for the coefficients $\alpha^{(\Omega)}_1$, $\alpha^{(\Omega)}_2$, $\cdots$, $\kappa^{(D)}$ can be represented with the help of the Prandtl number $P_m$ of the fluid, the magnetic Reynolds number $R_u$ of the turbulent motion and a dimensionless parameter $q$,

$$P_m = \nu/\eta, \hspace{1cm} R_u = u\lambda_c/\eta, \hspace{1cm} q = \lambda_c^2/\eta\tau_c,$$  \hspace{1cm} (19)$$

where $u = \sqrt{\overline{u^2}}$. Clearly $R_u$, like $u$, in general depends on position. The quantity $q$ is the ratio of the magnetic diffusion time $\lambda_c^2/\eta$ to the correlation time $\tau_c$. We speak of low–conductivity limit if $q \rightarrow 0$, and of high–conductivity limit if $q \rightarrow \infty$. 

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As for the general results concerning \( \alpha_1(\Omega) \), \( \alpha_2(\Omega) \), \( \ldots \), \( \kappa(D) \) we refer to the mentioned paper, in particular relations (55)–(58) and figures 1 and 2. These results cover arbitrary \( P_m \) and \( q \). We may write \( \alpha_1(\Omega) \), \( \alpha_2(\Omega) \), \( \ldots \), \( \kappa(D) \) except \( \gamma(0) \) and \( \beta(0) \) in the form

\[
\alpha_1(\Omega) = \tilde{\alpha}_1(P_m, q) R_u^2 \lambda_c^2, \quad \alpha_2(\Omega) = \tilde{\alpha}_2(P_m, q) R_u^2 \lambda_c^2, \quad \ldots, \quad \kappa(D) = \tilde{\kappa}(D)(P_m, q) R_u^2 \lambda_c^2, \tag{20}
\]

and

\[
\gamma(0) = \tilde{\gamma}(0)(q) R_u^2 \eta, \quad \beta(0) = \tilde{\beta}(0)(q) R_u^2 \eta. \tag{21}
\]

The \( \tilde{\alpha}_1(\Omega) \), \( \tilde{\alpha}_2(\Omega) \), \( \ldots \), \( \tilde{\kappa}(D) \) are dimensionless functions of \( P_m \) and \( q \). Only \( \tilde{\gamma}(0) \), \( \tilde{\beta}(0) \) and \( \tilde{\delta}(W) \) are independent of \( P_m \). In the case of \( \tilde{\gamma}(0) \) and \( \tilde{\beta}(0) \) this independence appears as natural when considering the derivation of these results, in the case of \( \tilde{\delta}(W) \) it occurs as an accidental compensation of different influences. Further we have \( \tilde{\gamma}(0) = \frac{1}{2} \tilde{\beta}(0) \) and \( \tilde{\gamma}(0) = \tilde{\delta}(W) \). As will become clear in the following section here only results for specific \( P_m \) and \( q \) are of interest. Table 1 gives some values of the \( \tilde{\alpha}_1(\Omega) \), \( \tilde{\alpha}_2(\Omega) \), \( \ldots \), \( \tilde{\kappa}(D) \) for such \( P_m \) and \( q \).

### 5 The effect of turbulence in the Perm dynamo device

#### 5.1 Estimates of the turbulence effects

Let us make a few estimates of the turbulence effects to be expected in the Perm dynamo device on the basis of the general form (13) of the mean electromotive force \( \mathcal{E} \), the relations (20) and (21), and numerical values represented in table 1. As for the properties of the liquid sodium we put \( \eta = 0.1 \text{m}^2/\text{s} \) and \( P_m = 10^{-5} \). As a typical value \( U \) of the mean velocity \( \overline{U} \) we use \( U = 50 \text{m/s} \). For the velocity \( u \) of the turbulent motion, the correlation length \( \lambda_c \) and the correlation time \( \tau_c \) we assume \( u = 1 \text{m/s} \), \( \lambda_c = 0.01 \text{m} \) and \( \tau_c = 0.05 \text{s} \). This implies \( R_u = 0.1 \) and \( q = 0.02 \). Further \( |\mathbf{g}| \) is estimated by \( 1/R \), where \( R \) means the radius of the fluid cylinder. Likewise \( |\mathbf{W}|, |\mathbf{D}| \) and \( |\mathbf{B}| \) are estimated by \( U/R \), and spatial derivatives of \( \overline{\mathbf{B}} \) by \( \overline{\mathbf{B}}/R \). We put \( R = 0.12 \text{m} \).

Relation (13) for \( \mathcal{E} \) ignores any time variation of \( \overline{\mathbf{B}} \). Since we are in the limit of small \( q \) this is justified as long as the characteristic time of this variation is large compared to \( \lambda_c^2/\eta \), with the above data \( 10^{-3} \text{s} \). The calculations described below show that this characteristic time is not smaller than \( 10^{-2} \) so that this condition is satisfied.

Consider first the \( \beta \)-effects, which may be interpreted by introducing a mean–field version of the magnetic diffusivity instead of the molecular diffusivity \( \eta \). This mean–field version is in general a tensor, \( \eta_{mn} \overline{v}_n = \frac{1}{2} \eta_{mn} \overline{\mathbf{v}}_n = \frac{1}{2} \eta_{mn} \overline{v}_n \overline{v}_n \) or \( \eta_{mn} \overline{v}_n = \frac{1}{2} \eta_{mn} \overline{\mathbf{v}}_n \overline{\mathbf{v}}_n \).
(\eta + \beta(0))\delta_{ij} + \beta(D)D_{ij}. The contribution of the \beta(0)–effect to the mean–field diffusivity, measured in units of \eta, is \beta(0)/\eta = \tilde{\beta}(0)R^2_u. With the above data it takes a value of about 10^{-3}. The corresponding contribution of the \beta(D)–effect is of the order of \beta(D)U/\eta R = \tilde{\beta}(D)(UR/\eta)(\lambda_c/R)^2R^2_u, with the above data about 1.3 \cdot 10^{-2}.

It is natural to compare the \gamma–effects, which describe a transport of mean magnetic flux, with that of the mean velocity \overline{U}. As for the \gamma(0)–effect we find that \gamma(0)g/|\overline{U}| is of the order of \gamma(0)/UR = \tilde{\gamma}(0)(\eta/UR)R^2_u, which takes a value of about 10^{-5}. In analogous estimates for the \gamma(W) and \gamma(D)–effects the ratios corresponding to \gamma(0)/UR are \gamma(W)/R^2, \gamma(W)/R^2 and \gamma(D)/R^2, all being independent of U. The first two are smaller than the last one, which is about 4 \cdot 10^{-4}.

We may also compare the \alpha, \delta and \kappa–effects with that of the mean velocity \overline{U}. They are then characterized by the ratios \alpha_1(\Omega)/R^2, \alpha_2(\Omega)/R^2, \alpha_1(W)/R^2, \delta(\Omega)/R^2, \delta(\Omega)/R^2, \delta(\Omega)/R^2, \delta(\Omega)/R^2, \delta(\Omega)/R^2, \kappa(\Omega)/R^2 \cdot \delta(\Omega)/R^2, again being independent of U. As for the \alpha_1(\Omega)/R^2, \alpha_2(\Omega)/R^2, \delta(\Omega)/R^2, \delta(\Omega)/R^2, the first one is about 6 \cdot 10^{-4} but all others are smaller. We recall that the \alpha_1(W) or the \alpha_1(\Omega)–effects vanish if g is orthogonal to \Omega or W, respectively. Further \delta(\Omega)/R^2 is about 7 \cdot 10^{-6} but \delta(W)/R^2 smaller. Among the \kappa(\Omega) \cdot \kappa(\Omega)/R^2 the last one is about 2 \cdot 10^{-4} but the others are smaller.

We may conclude from these estimates that the influence of turbulence on the screw dynamo in the Perm experiment should be rather small. There is hardly the possibility of another type of dynamo, which would become possible by turbulence effects.

5.2 Dynamo model

Some more detailed numerical investigations concerning the effects of turbulence on the screw dynamo have been carried out, mainly with the intention to find out in which sense they influence its excitation condition. Again g was considered as a radial vector such that \nabla u^2 = g u^2.

Let us first give equation (2) governing \overline{B} inside the cylinder and the supplementing relations in a non–dimensional form. In that sense we put

\[ \Omega = \Omega_0\tilde{\Omega}, \quad \overline{U}_z = U_0\tilde{U}_z, \quad u^2 = u_0^2\chi. \]  

(22)

Here \Omega_0 and U_0 are constants describing typical values of \Omega and \overline{U}_z, e.g., being their values at r = 0, and \tilde{\Omega} and \tilde{U}_z are dimensionless functions of r. Further u_0 means the value of u at r = 0, and \chi is again a dimensionless function of r. In the following we measure all lengths in units of the radius R of the cylinder and the time in units of R^2/\eta. Then (2) can be written in the form

\[ \partial_t\overline{B} - \nabla \times (R_U\tilde{U} \times \overline{B} + R^2_u (\tilde{F}(\overline{B}) + \xi R_U \tilde{G}(\overline{B}))) - \nabla^2\overline{B} = 0, \quad \nabla \cdot \overline{B} = 0 \]  

(23)

with the parameters

\[ R_U = \sqrt{\Omega_0^2R^2 + U_0^2 R/\eta}, \quad R^2_u = u_0\lambda_c/\eta, \quad \xi = (\lambda_c/R)^2. \]  

(24)

and

\[ \tilde{U} = (0, \zeta_\Omega \tilde{\Omega}(r) r, \zeta_U \tilde{U}_z) \]  

(25)

\[ \tilde{F} = -\tilde{\gamma}(0)\chi'\hat{r} \times \overline{B} - \tilde{\beta}(0)\chi \nabla \times \overline{B} \]  

(26)

\[ \tilde{G} = -\tilde{\alpha}_2^{(\Omega)}\chi'\zeta_\Omega \tilde{\Omega}(\overline{B}_z\hat{r} + \overline{B}_r\hat{z}) - (\tilde{\alpha}_2^{(W)} - \tilde{\alpha}_2^{(D)})\chi'(\zeta_\Omega r\tilde{\Omega}'(\overline{B}_z\hat{r} + \overline{B}_r\hat{z}) - \zeta_U \tilde{U}_z' (\overline{B}_r\hat{r} + \overline{B}_r\hat{r})) \]
The unit vectors in the cylindrical coordinate system we refer to the cylindrical coordinate system we refer to the Appendix. Of course, equation (23) has to be completed by boundary conditions.

We will look for solutions of the above equations for $\mathbf{B}$ having the form

$$
\mathbf{B} = \overline{\mathbf{B}} \exp(i(m\varphi + k z) + \lambda t)
$$

with a positive integer $m$, a real non-zero wavenumber $k$ and a complex growth rate $\lambda$. Note that by Cowling’s theorem solutions with $m = 0$ have to be excluded as long as $R_{u0} = 0$.

As for $\tilde{\Omega}$ and $\tilde{U}_z$ we put

$$
\tilde{\Omega}(r) = \tilde{U}_z(r) = \frac{\cosh(\rho) - \cosh(\rho r)}{\cosh(\rho) - 1}
$$

with some parameter $\rho$. Interpreted in view of $\tilde{U}_z$, for $\rho = 1$ the flow has a Poiseuille profile, for $\rho \to \infty$ a piston profile. The outcomes of the water experiments made to simulate the flow in the dynamo device fit best to $\rho = 18$. In view of $\chi$, which is defined by (22), we adopt a relation for the r.m.s. value $u$ of the turbulent velocity given by Schlichting (1964),

$$
u(2.5 \ln \frac{uR}{\nu} + 1.75) = U_z.
$$

On this basis $\chi$ has been calculated with $\nu$ such that $Pm = 10^{-5}$. Profiles of $\tilde{\Omega}$ and $\tilde{U}_z$ and of $\chi$ and $\chi'$ for several $\rho$ are depicted in figure 1.

All calculations reported in the following have been carried out with $Pm = 10^{-5}$ and $q = 0.02$. The corresponding numerical values of $\tilde{\alpha}_2^{(0)}$, $\tilde{\alpha}_2^{(W)}$, $\tilde{\kappa}^{(D)}$ are given in table 1.

As for the surroundings of the fluid cylinder two cases are considered. In the first one all surroundings are assumed to be free space. Since only non-zero $k$ are admitted the total electric current through a
cross-section of the cylinder is equal to zero. Therefore it is to be required that $\mathcal{B}$ inside the cylinder continues as a single-valued potential field in outer space. In the second case it is assumed that the fluid cylinder is muffled by a rigid electrically conducting shell, in the experimental device made of copper, and there is free space outside this shell. Again $\mathcal{B}$ has to continue as a single–valued potential field in this free space. The thickness of the shell, measured in units of $R$, is denoted by $d$, and the ratio of the electrical conductivity of the shell to that of the fluid by $s$. The first case may be considered as the special case $d = 0$ of the second one. The experimental device corresponds to $d = 0.15$ and $s = 5$.

The equations (23)–(27) together with the ansatz (29) have been reduced to a system of ordinary differential equation which, together with the boundary conditions, defines an eigenvalue problem with the growth rate $\lambda$ as eigenvalue parameter. After discretization of the equations this problem has been solved with the QR algorithm. In the following we restrict our attention to solutions with $\Re(\lambda) = 0$. The corresponding values of $R_U$ are denoted as critical values.

### 5.3 Influences of the turbulence on the excitation threshold

Let us first give critical values of $R_U$ for the ideal situation without turbulence, that is, $R_{u0} = 0$. For any given $\xi$ and $\rho$ they still depend on $m$ and $k$. We restrict ourselves to $m = 1$. All solutions of the equations considered with the same parameters but other $m$ have higher critical values of $R_U$. We denote the minimal critical $R_U$ for given $\xi$ and $\rho$ but varying $k$ by $R_U^*$ and the corresponding $k$ by $k^*$. Figure 2 shows $R_U^*$ and $k^*$ as functions of $\xi$ for several cases concerning $d$ and $\rho$. It shows in particular that a highly conducting shell around the fluid cylinder markedly favors the screw dynamo; (see also Avalos-Zuniga et al. 2003, Avalos-Zuniga and Plunian 2005).

We admit now turbulence. Since in the experiment $\xi$ will probably be close to unity we put now throughout $\xi = 1$. As figure 3 demonstrates the turbulence generally enlarges the dynamo threshold. Consider the most realistic case $d = 0.15$, $s = 0.5$ and $\rho = 18$. Assuming that $R_{u0}$ is of the order of $10^{-1}$ and $\xi$ of the order $10^{-2}$ we conclude that $R_U^*$ grows by at most 0.5 percent. We point out, however, that all our calculations have been done with $P_m = 10^{-5}$ and $q = 0.02$. The data of table 1 suggest that the influence of the turbulence will be markedly larger if larger values of $q$ have to be assumed.

Several investigations have been made in order to find out which specific induction effects of the turbulence such as $\alpha$–effects, $\gamma$–effects etc. hamper the screw dynamo or possibly support it. In that sense only parts of the mean electromotive force $\mathcal{E}$ have been taken into account. Both cases with $\rho = 1$ and more realistic ones with $\rho = 18$ have been studied. In the latter one parts of the profiles of $\alpha$, $U_z$ and $\chi$ are markedly steeper.

Let us consider first the $\alpha$–effects. Note that the $\alpha_1^{(\Omega)}$ and $\alpha_1^{(W)}$–effects are absent since $g$ is orthogonal to both $\Omega$ and $W$. The influence of the $\alpha_2^{(\Omega)}$, $\alpha_2^{(W)}$ and $\alpha^{(D)}$–effects on the screw dynamo depends crucially on $\rho$. In cases with $\rho = 1$ it was found that they act against the screw dynamo. Figure 4 shows results for a more realistic case with $\rho = 18$, in which these effects clearly support the screw dynamo.
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Consider next the \( \gamma \)-effects. In all investigated cases, both with \( \rho = 1 \) and also with higher \( \rho \), they hamper the screw dynamo. Figure 4 illustrates this for the above case with \( \rho = 18 \).

Proceeding now to the \( \beta \)-effects we note first that the \( \beta^{(0)} \)-effect may be interpreted in the sense of the mean-field magnetic diffusivity \( \eta + \beta^{(0)} \) different from the molecular diffusivity \( \eta \). Since \( \beta^{(0)} \) is always positive it raises the threshold of the screw dynamo. The \( \beta^{(D)} \)-effect makes the mean-field magnetic diffusivity anisotropic. It is then described by the tensor \((\eta + \beta^{(0)}) \delta_{ij} + \beta^{(D)} D_{ij}\). In all investigated cases, with \( \rho = 1 \) and also with higher \( \rho \), the \( \beta^{(D)} \)-effect for not too small \( \xi \) always dominates the \( \beta^{(0)} \)-effect and clearly supports the screw dynamo. This implies of course that the mentioned tensor is not positive definite; otherwise it had to act against the screw dynamo. Results for the above case with \( \rho = 18 \) are shown in figure 5a.

Both the \( \delta^{(\Omega)} \) and \( \delta^{(W)} \)-effects and also the \( \kappa^{(\Omega)} \)-effect always support the screw dynamo. The \( \kappa^{(W)} \) and \( \kappa^{(D)} \)-effects and also all \( \kappa \)-effects together act against it. This is partially reflected in figures 5b and 6a applying again to the above case with \( \rho = 18 \).

We recall here that \( \delta^{(\Omega)} \) and \( \kappa^{(\Omega)} \)-effects together with a sufficiently strong rotational shear open the possibility of a mean-field dynamo different from the screw dynamo, often called \( \Omega \times J \) dynamo (Rädler 1969, 1980, 1986, Roberts 1972, Moffatt and Proctor 1982). The positive influence of the \( \delta^{(\Omega)} \) and \( \kappa^{(\Omega)} \)-effects on the screw dynamo, demonstrated by figure 6b, can possibly be interpreted in the sense of this dynamo mechanism.

In contrast to the combination of the \( \delta^{(\Omega)} \) and \( \kappa^{(\Omega)} \)-effects that of the \( \delta^{(W)} \) and \( \kappa^{(W)} \)-effects, as figure 7a shows, does not support the screw dynamo. For the \( W \times J \) dynamo, which has recently been proposed (Rogachevskii and Kleeorin 2003), the \( \delta^{(W)} \), \( \kappa^{(W)} \), \( \beta^{(D)} \) and \( \kappa^{(D)} \)-effects are important. However, it turned out that at least the simple model in plane geometry, which was used to explain this proposal, fails to work as a dynamo as long as \( \delta^{(W)} \), \( \kappa^{(W)} \), \( \beta^{(D)} \) and \( \kappa^{(D)} \) are determined in the second–order correlation
approximation  \cite{Raderl and Stepanov 2005, Ruediger and Kitchatinov 2006}. In this context it is of interest that the combination of the $\delta^{(W)}$, $\kappa^{(W)}$, $\beta^{(D)}$, and $\kappa^{(D)}$-effects, as figure 7b shows, does not support the screw dynamo. This is also in so far quite remarkable as we already know that the $\beta^{(D)}$-effect supports it. It is the $\kappa^{(W)}$ and $\kappa^{(D)}$-effects which have a strong opposite influence.

Our results show that the total influence of the induction effects of the turbulence on the threshold of the screw dynamo, which we have estimated above, is weaker than the influences of some of the individual effects. Obviously a part of them compensates each other.
6 Summary

The electromagnetic processes in Perm dynamo experiment are influenced by turbulent motions accompanying the main flow. Their influence on the dynamo process has been investigated within the framework of mean-field electrodynamics. The critical magnetic Reynolds number $R^*_U$ of the main flow, which defines the onset of the screw dynamo, grows with the magnetic Reynolds number $R_{u0}$ of the turbulence and the also with the parameter $q$, that is, the ratio of the decay time of a magnetic field extended over one correlation length to the correlation time. Under the assumptions on these quantities made above, which we consider as realistic, $R^*_U$ changes hardly by more than 0.5 percent. If they are by any reason larger, $R^*_U$ becomes larger, too.

The electromagnetic effects of the turbulence are analyzed in terms of $\alpha$, $\gamma$, $\beta$, $\delta$ and $\kappa$–effects. Despite helical structures in the turbulence there is no complete $\alpha$–effect. The mean electromotive force due to this $\alpha$–effect has not always a component in the direction of the mean magnetic field. This fact explains the measurements made at an experimental setup simulating the dynamo device in a smaller scale. Despite this peculiarity the $\alpha$–effect supports the screw dynamo slightly. The $\gamma$–effect acts always against the screw dynamo. The two contributions to the $\beta$–effect, the $\beta^{(0)}$ and the $\beta^{(D)}$–effect, act in opposite ways. The $\beta^{(0)}$–effect leads to a mean-field diffusivity larger than the molecular one and so raises the screw dynamo threshold. The $\beta^{(D)}$–effect makes the mean–field diffusivity anisotropic and strongly supports the screw dynamo. Likewise the $\delta$–effects support the screw dynamo whereas the $\kappa$–effects act against it. It is known that the $\delta^{(W)}$ and $\kappa^{(W)}$–effects in combination with a proper, sufficiently strong rotational shear may work as a dynamo, sometimes called $\Omega \times J$ dynamo. They also support the screw dynamo. Analogous to the $\Omega \times J$ dynamo the possibility of a $W \times J$ dynamo has been discussed in the literature, for which the $\delta^{(W)}$, $\kappa^{(W)}$, $\beta^{(D)}$ and $\kappa^{(D)}$–effects are important, but the question of its existence remained open so far. Interestingly enough it was found that the combination of these effects does not support the screw dynamo.

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REFERENCES

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Appendix

The symmetric part $(\nabla V)^{(s)}$ of the gradient tensor of a vector field $V$, in a Cartesian co-ordinate system defined by $(\nabla V)^{(s)}_{ij} = \frac{1}{2}(\partial V_i/\partial x_j + \partial V_j/\partial x_i)$, has in a cylindrical co-ordinate system the components

$$(\nabla V)^{(s)}_{rr} = \frac{\partial V_r}{\partial r}, \quad (\nabla V)^{(s)}_{\varphi r} = (\nabla V)^{(s)}_{\varphi \varphi} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial V_r}{\partial \varphi} + \frac{\partial V_\varphi}{\partial r} - \frac{V_\varphi}{r} \right)$$

$$(\nabla V)^{(s)}_{r\varphi} = (\nabla V)^{(s)}_{\varphi r} = \frac{1}{2} \left( \frac{\partial V_r}{\partial \varphi} + \frac{\partial V_\varphi}{\partial r} \right), \quad (\nabla V)^{(s)}_{\varphi\varphi} = \frac{1}{r} \left( \frac{\partial V_\varphi}{\partial \varphi} + V_r \right)$$

$$(\nabla V)^{(s)}_{\varphi z} = (\nabla V)^{(s)}_{z\varphi} = \frac{1}{2} \left( \frac{\partial V_\varphi}{\partial z} + \frac{1}{r} \frac{\partial V_z}{\partial \varphi} \right), \quad (\nabla V)^{(s)}_{zz} = \frac{\partial V_z}{\partial z}. $$