Minimal entanglement generating set for general
pure multipartite states

Hoshang Heydari
Institute of Quantum Science, Nihon University,
1-8 Kanda-Surugadai, Chiyoda-ku, Tokyo 101-8308, Japan

March 17, 2006

Abstract
We propose a minimal entanglement generating set for a general multipartite state based on concurrences classes. In particular, we construct concurrence for general three-partite states based on a redefinition of our concurrence classes. This shows that a measure of entanglement for three-partite states can be decomposed into two different classes that are constructed by two different local operators. Then based on this construction we introduce the minimal entanglement generating set for three-partite states. Moreover, we generalize our result to general multipartite states.

1 Introduction
Quantification and classification of multipartite quantum entangled states is still an ongoing research activity in the emerging field of quantum information and quantum communication with many applications. There are many entanglement measures for bipartite states and among these entanglement measures the concurrence, that gives an analytic formula for the entanglement of formation [1] is widely known. In recent years there have been some proposals to generalize this measure to general bipartite and multipartite states [2, 3, 4, 5, 6]. Quantifying entanglement of multipartite states has been discussed in [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Recently, we have also defined concurrence classes for multi-qubit mixed states [19] based on an orthogonal complement of a positive operator valued measure (POVM) on quantum phase. Moreover, we have constructed different concurrence classes for general pure multipartite states in [20]. Classification of multipartite states has been discussed in [21, 22, 23, 24]. For example, F. Verstraete et al. [21] have considered a single copy of a pure four-partite state of qubits and investigated its behavior under stochastic local quantum operation and classical communication (SLOCC), which gave a classification of all different classes of pure states of four qubits. A. Osterloh and J. Siewert [22] have constructed entanglement measures for pure states of multipartite qubit systems. The key element of their approach is an antilinear operator that they called comb. For qubits, the combs are invariant under the action of the special linear group. They have also discussed inequivalent types of genuine four-qubit entanglement, and found three types of entanglement for these states. This result coincides with our
classification, where we have three types of concurrence classes for four-qubit states. A. Miyake \cite{miyake2006quantum}, has also discussed classification of multipartite states in entanglement classes based on the hyper-determinant. He has shown that two states belong to the same class if they are interconvertible under SLOCC. Finally, A. M. Wang \cite{wang2002classification} has proposed two classes of the generalized concurrence vectors of the multipartite systems consisting of qubits. Our classification is similar to Wang’s classification of multipartite state. However, the advantage of our method is that our POVM can distinguish these concurrence classes without prior information about inequivalence of these classes under local quantum operation and classical communication (LOCC).

Let us denote a general, multipartite quantum system with m subsystems by $Q = Q_m(N_1, N_2, \ldots, N_m) = Q_1 Q_2 \cdots Q_m$, consisting of a state $|\Psi\rangle = \sum_{k_1=1}^{N_1} \cdots \sum_{k_m=1}^{N_m} \alpha_{k_1, k_2, \ldots, k_m} |k_1, k_2, \ldots, k_m\rangle$ and, let $\rho_Q = \sum_{n=1}^{N_1} p_n |\Psi_n\rangle \langle \Psi_n|$, for all $0 \leq p_n \leq 1$ and $\sum_{n=1}^{N_1} p_n = 1$, denote a density operator acting on the Hilbert space $\mathcal{H}_Q = \mathcal{H}_{Q_1} \otimes \mathcal{H}_{Q_2} \otimes \cdots \otimes \mathcal{H}_{Q_m}$, where the dimension of the $j$th Hilbert space is given by $N_j = \dim(\mathcal{H}_{Q_j})$. Moreover, let us introduce a complex conjugation operator $\tilde{C}_m$ that acts on a general multipartite state $|\Psi\rangle$ as $\tilde{C}_m |\Psi\rangle = \sum_{k_1=1}^{N_1} \cdots \sum_{k_m=1}^{N_m} \alpha_{k_1, k_2, \ldots, k_m}^* |k_1, k_2, \ldots, k_m\rangle$. We are going to use this notation throughout this paper, i.e., we denote a mixed pair of qubits by $Q_2(2, 2)$. The density operator $\rho_Q$ is said to be fully separable, which we will denote by $\rho_Q^{sep}$, with respect to the Hilbert space decomposition, if it can be written as $\rho_Q^{sep} = \sum_{n=1}^{N_1} p_n \bigotimes_{j=1}^{m} \rho_{Q_j}^n$, $\sum_{n=1}^{N_1} p_n = 1$, for some positive integer $N$, where $p_n$ are positive real numbers and $\rho_{Q_j}^n$ denotes a density operator on Hilbert space $\mathcal{H}_{Q_j}$. If $\rho_Q^m$ represents a pure state, then the quantum system is fully separable if $\rho_Q^m$ can be written as $\rho_Q^m = \bigotimes_{j=1}^{m} \rho_{Q_j}$, where $\rho_{Q_j}$ is a density operator on $\mathcal{H}_{Q_j}$. If a state is not separable, then it is called an entangled state. Some of the generic entangled states are called Bell states and EPR states.

## 2 Concurrence classes for general pure multipartite states

In this section, we will review the construction of concurrence classes for general pure multipartite states $Q_m^m(N_1, \ldots, N_m)$ \cite{nielsen2000quantum, horodecki1996quantum}. In order to simplify our presentation, we will use $\Lambda_m = k_1, l_1; \ldots; k_m, l_m$ as an abstract multi-index notation. The unique structure of our POVM enables us to distinguish different classes of multipartite states, which are inequivalent under LOCC operations. In the $m$-partite case, the off-diagonal elements of the matrix corresponding to

$$
\Delta_Q(\varphi_{Q_1; k_1, l_1}, \ldots, \varphi_{Q_m; k_m, l_m}) = \Delta_{Q_1}(\varphi_{Q_1; k_1, l_1}) \otimes \cdots \otimes \Delta_{Q_m}(\varphi_{Q_m; k_m, l_m}),
$$

have phases that are sums or differences of phases originating from two and $m$ subsystems. That is, in the latter case the phases of $\Delta_Q(\varphi_{Q_1; k_1, l_1}, \ldots, \varphi_{Q_m; k_m, l_m})$ take the form $\langle \varphi_{Q_1; k_1, l_1} \pm \varphi_{Q_2; k_2, l_2} \pm \cdots \pm \varphi_{Q_m; k_m, l_m} \rangle$ and identification of these joint phases makes our classification possible. Thus, we can define linear operators for the $W_m$ class based on our POVM which are sum and difference of phases of two subsystems, i.e., $(\varphi_{Q_1; k_1, l_1} \pm \varphi_{Q_2; k_2, l_2} \pm \cdots \pm \varphi_{Q_m; k_m, l_m})$. That is, for the $W_m$
class we have
\[
\Delta_{Q_{r_1 \cdots r_2}}^{\text{EPR}_m}(N_{r_1}, N_{r_2}) = \mathcal{I}_{N_1} \otimes \cdots \otimes \Delta_{Q_{r_1}} (\varphi_{Q_{r_1 \cdots r_2}^{\pm}; k_{r_1}, l_{r_1}}) \\
\quad \quad \quad \otimes \cdots \otimes \Delta_{Q_{r_2}} (\varphi_{Q_{r_2}^{\pm}; k_{r_2}, l_{r_2}}) \otimes \cdots \otimes \mathcal{I}_{N_m}.
\] (2)

Let \(C(m, k) = \binom{m}{k}\) denote the binomial coefficient. Then there is \(C(m, 2)\) linear operators for the EPR\(^m\) classes and the set of these operators gives the \(W^m\) class concurrence.

For the GHZ\(^m\) class, we define linear operators based on our POVM which are sum and difference of phases of \(m\)-subsystems, i.e., \((\varphi_{Q_{r_1}; k_{r_1}, l_{r_1}} \pm \varphi_{Q_{r_2}; k_{r_2}, l_{r_2}} \pm \cdots \pm \varphi_{Q_m; k_m, l_m})\). That is, for the GHZ\(^m\) class we have
\[
\Delta_{Q_{r_1 \cdots r_2}}^{\text{GHZ}_m}(N_{r_1}, N_{r_2}) = \Delta_{Q_{r_1}} (\varphi_{Q_{r_1}^{\pm}; k_{r_1}, l_{r_1}}) \otimes \cdots \otimes \Delta_{Q_{r_1}} (\varphi_{Q_{r_2}^{\pm}; k_{r_2}, l_{r_2}}) \\
\quad \quad \quad \otimes \cdots \otimes \Delta_{Q_{r_2}} (\varphi_{Q_{r_2}^{\pm}; k_{r_2}, l_{r_2}}) \otimes \cdots \otimes \Delta_{Q_{r_2}} (\varphi_{Q_{r_2}^{\pm}; k_{r_2}, l_{r_2}}).
\] (3)

where by choosing \(\varphi_{Q_{r_1}; k_{r_1}, l_{r_1}} = \pi\) for all \(k_j < l_j, j = 1, 2, \ldots, m\), we get an operator which has the structure of Pauli operator \(\sigma_x\) embedded in a higher-dimensional Hilbert space and coincides with \(\sigma_x\) for a single-qubit. There are \(C(m, 2)\) linear operators for the GHZ\(^m\) class and the set of these operators gives the GHZ\(^m\) class concurrence.

Moreover, we define the linear operators for the GHZ\(^{m-1}\) class of \(m\)-partite states based on our POVM which are sum and difference of phases of \(m - 1\)-subsystems, i.e., \((\varphi_{Q_{r_1}; k_{r_1}, l_{r_1}} \pm \varphi_{Q_{r_2}; k_{r_2}, l_{r_2}} \pm \cdots \pm \varphi_{Q_{r_{m-1}}; k_{r_{m-1}}, l_{r_{m-1}}} \pm \varphi_{Q_{r_{m-1}}; k_{r_{m-1}}, l_{r_{m-1}}}\). That is, for the GHZ\(^{m-1}\) class we have
\[
\Delta_{Q_{r_1 \cdots r_2 - r_3}}^{\text{GHZ}_{m-1}}(N_{r_1}, N_{r_2}) = \Delta_{Q_{r_1}} (\varphi_{Q_{r_1}^{\pm}; k_{r_1}, l_{r_1}}) \otimes \Delta_{Q_{r_2}} (\varphi_{Q_{r_2}^{\pm}; k_{r_2}, l_{r_2}}) \\
\quad \quad \quad \otimes \Delta_{Q_{r_3}} (\varphi_{Q_{r_3}^{\pm}; k_{r_3}, l_{r_3}}) \otimes \cdots \otimes \\
\quad \quad \quad \Delta_{Q_{r_m}} (\varphi_{Q_{r_m}^{\pm}; k_{r_m}, l_{r_m}}) \otimes \mathcal{I}_{N_m},
\] (4)

where \(1 \leq r_1 < r_2 < \cdots < r_{m-1} < m\). Now, based on these operators we can construct concurrence classes for general multiparticle states \([19, 20]\). It is now more clear to us that to get an entanglement monotone, at least for multi-qubit states, we to need to add these concurrence classes and define a concurrence for the whole quantum system. However, we will modify our construction of concurrence classes in the following section such that it will coincide with the generalized concurrence. Then based on this result, we will propose the MEGS for general three-partite states and multipartite states.

### 3 Minimal entanglement generating set for general pure three-partite states

In this section, we will construct concurrence class for pure three-qubits states based on orthogonal complement of our POVM and then we will compare this result with concurrence monotone. For three-partite states we have two different
joint phases in our POVM, those which are sums and differences of phases of two subsystems, i.e., \( (\varphi_{Q_1:k_1,l_1} \pm \varphi_{Q_2:k_2,l_2}) \) and those which are sums and differences of phases of three subsystems, i.e., \( (\varphi_{Q_1:k_1,l_1} \pm \varphi_{Q_2:k_2,l_2} \pm \varphi_{Q_3:k_3,l_3}) \). The first one identifies the W\(^3\) class and the second one identifies the GHZ\(^3\) class. For W\(^3\) class concurrence, we have three types of entanglement; entanglement between subsystems one and two \( Q_1Q_2 \), one and three \( Q_1Q_3 \), and two and three \( Q_2Q_3 \). Let us discuss and analyze the W\(^3\) class for the three-qubit states. In this case the W\(^3\) class operators are given by

\[
\tilde{\Delta}_{Q_1,2(2,2)}^{\text{EPR}} = \Delta_{Q_1}(\varphi_{Q_1,1,2}^\ast) \otimes \Delta_{Q_2}(\varphi_{Q_2,1,2}^\ast) \otimes I_2 = \sigma_2 \otimes \sigma_2 \otimes I_2 = \begin{pmatrix}
\Delta_{1,1} & \Delta_{1,2} \\
\Delta_{2,1} & \Delta_{2,2}
\end{pmatrix},
\]

where \( \Delta_{1,1} = \Delta_{2,2} \) are 4-by-4 zero matrices and

\[
\Delta_{1,2} = \Delta_{2,1}^T = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \otimes \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

From the structure of the matrix \( \tilde{\Delta}_{Q_1,2(2,2)}^{\text{EPR}} \) we can observe that we can divide it into four reign that contribute to the degree of entanglement. For the first division we use the upper and lower diagonal elements and for second we use upper and lower anti-diagonal elements. For example, we have

\[
\langle \Psi | \tilde{\Delta}_{Q_1,2(2,2)}^{\text{EPR}} C_3 | \Psi \rangle = 2[(\alpha_{1,2,1} \alpha_{2,1,1} - \alpha_{1,1,1} \alpha_{2,2,1})
+ (\alpha_{1,2,2} \alpha_{2,1,2} - \alpha_{1,1,2} \alpha_{2,2,2})],
\]

where the factor 2 indicates that we have the same contribution one from the upper and one lower diagonal and parenthesis indicates the contribution from upper and lower anti-diagonal. Next, we would like to write the operator \( \tilde{\Delta}_{Q_1,2(2,2)}^{\text{EPR}} \) as a direct sum of the upper and lower anti-diagonal, that is,

\[
\tilde{\Delta}_{Q_1,2(2,2)}^{\text{EPR}} = U \Delta_{Q_1,2(2,2)} + L \Delta_{Q_1,2(2,2)}.
\]

For example we get

\[
\langle \Psi | U \tilde{\Delta}_{Q_1,2(2,2)}^{\text{EPR}} C_3 | \Psi \rangle = 2[(\alpha_{1,2,1} \alpha_{2,1,1} - \alpha_{1,1,1} \alpha_{2,2,1})].
\]

Thus we can define the semi W\(^3\) class concurrence as

\[
\mathcal{K}(Q_3^{W^3}(2,2,2)) = \sum_{1=r_1 < r_2 < k_1, l_1, k_2, l_2, k_3, l_3} |\langle \Psi | \tilde{\Delta}_{Q_{r_1},r_2(2,2)}^{\text{EPR}} C_3 | \Psi \rangle|^2
\]

\[
= |\alpha_{1,2,1} \alpha_{2,1,1} - \alpha_{1,1,1} \alpha_{2,2,1}|^2 + |\alpha_{1,2,2} \alpha_{2,1,2} - \alpha_{1,1,2} \alpha_{2,2,2}|^2
+ |\alpha_{1,1,2} \alpha_{2,1,1} - \alpha_{1,1,1} \alpha_{2,2,1}|^2 + |\alpha_{1,2,2} \alpha_{2,2,1} - \alpha_{1,1,2} \alpha_{2,2,2}|^2
+ |\alpha_{1,1,2} \alpha_{1,2,1} - \alpha_{1,1,1} \alpha_{1,2,2}|^2 + |\alpha_{2,1,2} \alpha_{2,1,2} - \alpha_{2,1,1} \alpha_{2,2,2}|^2.
\]

Let us now discuss and analyze the GHZ\(^3\) class for the three-qubit states. In this case the GHZ\(^3\) class operators are given by

\[
\tilde{\Delta}_{Q_1,2(2,2)}^{\text{GHZ}} = \Delta_{Q_1}(\varphi_{Q_1,1,2}^\ast) \otimes \Delta_{Q_2}(\varphi_{Q_2,1,2}^\ast) \otimes \Delta_{Q_3}(\varphi_{Q_3,1,2}^\ast) = \sigma_2 \otimes \sigma_2 \otimes I_2 = \begin{pmatrix}
\Xi_{1,1} & \Xi_{1,2} \\
\Xi_{2,1} & \Xi_{2,2}
\end{pmatrix},
\]

4
where $\Xi_{1,1} = \Xi_{2,2}$ are 4-by-4 zero matrices and

$$\Xi_{1,2} = \Xi_{2,1}^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (7)$$

Now, based on this operator, we get

$$\langle \Psi | \tilde{\Delta}_{Q_{1,2}(2,2)}^{GHZ_3} C_3 \Psi \rangle = 2 \left[ (\alpha_{1,2,2} \alpha_{2,1,1} + \alpha_{1,2,1} \alpha_{2,1,2} - \alpha_{1,1,2} \alpha_{2,2,1} - \alpha_{1,1,1} \alpha_{2,2,2}) \right],$$

where the factor 2 indicates that we have the same contribution from the upper and the lower diagonal. Next, we can write the operator $\tilde{\Delta}_{Q_{1,2}(2,2)}^{GHZ_3}$ as a direct sum of two operators, $\tilde{\Delta}_{Q_{1,2}(2,2)}^{GHZ_3} = P_3 \Delta_{Q_{1,2}(2,2)}^{GHZ_3} + N_3 \Delta_{Q_{1,2}(2,2)}^{GHZ_3}$, where the operator $P_3 \Delta_{Q_{1,2}(2,2)}^{GHZ_3}$ (resp. $N_3 \Delta_{Q_{1,2}(2,2)}^{GHZ_3}$) is constructed by elements of the POVM with sum and difference of quantum phases of subsystems such that relative phases of subsystems two and three is positive (resp. negative). For example, we get

$$\langle \Psi | P_3 \tilde{\Delta}_{Q_{1,2}(2,2)}^{GHZ_3} C_3 \Psi \rangle = 2 \left[ (\alpha_{1,2,2} \alpha_{2,1,1} - \alpha_{1,1,1} \alpha_{2,2,2}) \right].$$

Thus we can define the semi $W^3$ class concurrence as

$$\mathcal{K}(Q_3^{GHZ^3}(2,2,2)) = \sum_{1=r_1 < r_2 \forall k_1 < l_1, k_2 < l_2 \forall l_3} 3 \sum ||\langle \Psi | \tilde{\Delta}_{Q_{1,2}(2,2)}^{GHZ_3} C_3 \Psi \rangle||^2$$

and the lower diagonal. Next, we can write the operator

$$\Delta_{Q_{1,2}(2,2)}^{GHZ_3} = \begin{pmatrix} \alpha_{1,2,2} \alpha_{2,1,1} & \alpha_{1,2,1} \alpha_{2,1,2} & \alpha_{1,1,2} \alpha_{2,2,1} & \alpha_{1,1,1} \alpha_{2,2,2} \\ \alpha_{1,2,2} \alpha_{2,1,1} & -\alpha_{1,2,1} \alpha_{2,1,2} & -\alpha_{1,1,2} \alpha_{2,2,1} & -\alpha_{1,1,1} \alpha_{2,2,2} \\ -\alpha_{1,2,2} \alpha_{2,1,1} & \alpha_{1,2,1} \alpha_{2,1,2} & -\alpha_{1,1,2} \alpha_{2,2,1} & \alpha_{1,1,1} \alpha_{2,2,2} \\ -\alpha_{1,2,2} \alpha_{2,1,1} & -\alpha_{1,2,1} \alpha_{2,1,2} & \alpha_{1,1,2} \alpha_{2,2,1} & -\alpha_{1,1,1} \alpha_{2,2,2} \end{pmatrix}.$$

Next, we can define a concurrence for three-qubit states as square roots of semi concurrence classes as follows

$$\mathcal{C}(Q_3(2,2,2)) = \left( \mathcal{N} [\mathcal{K}(Q_3^{W^3}(2,2,2)) + \mathcal{K}(Q_3^{GHZ^3}(2,2,2))] \right)^{1/2}.$$
for pure quantum system $Q_3^p(N_1, N_2, N_3)$ we can define

$$K(Q_3^W(N_1, N_2, N_3)) = \sum_{1=r_1<r_2}^{3} \sum_{l_1>k_1=1}^{N_1} k_2=1 \sum_{l_2>k_2=1}^{N_2} k_3=1 \sum_{l_3>k_3=1}^{N_3} \sum_{l_1>l_2>k_1=1}^{N_1} k_3=1 \sum_{l_2>l_3>k_2=1}^{N_2} k_1=1 \sum_{l_3>l_4>k_3=1}^{N_3} \sum_{l_1>l_2>k_1=1}^{N_1} k_3=1 \sum_{l_2>l_3>k_2=1}^{N_2} k_1=1 \sum_{l_3>l_4>k_3=1}^{N_3} \left|\langle\Psi|\Delta_3^{EPR}(N_1, N_2, C_3\Psi)\rangle\right|^2$$

$$= \left[ \sum_{l_1>k_1=1}^{N_1} \sum_{l_2>k_2=1}^{N_2} \sum_{l_3>k_3=1}^{N_3} \left|\alpha_{l_1,l_2,k_3} - \alpha_{l_1,k_2,k_3}\right|^2 \right]$$

$$+ \sum_{l_1>k_1=1}^{N_1} \sum_{l_2>k_2=1}^{N_2} \sum_{l_3>k_3=1}^{N_3} \left|\alpha_{l_1,k_2,l_3} - \alpha_{l_1,k_2,k_3}\right|^2$$

$$+ \sum_{l_2>k_2=1}^{N_2} \sum_{l_3>k_3=1}^{N_3} \sum_{l_1>k_1=1}^{N_1} \left|\alpha_{l_1,k_2,l_3} - \alpha_{l_1,k_2,k_3}\right|^2$$

$$+ \sum_{l_3>k_3=1}^{N_3} \sum_{l_1>k_1=1}^{N_1} \sum_{l_2>k_2=1}^{N_2} \left|\alpha_{l_1,k_2,l_3} - \alpha_{l_1,k_2,k_3}\right|^2.$$

We can also generalize the $GHZ^3$ class concurrence as follows. The first linear operator is given by

$$\Delta_{Q_3^W(N_1, N_2, N_3)} = \Delta_1(\phi_{Q_3^W}(1, 1, l_1)) \otimes \Delta_2(\phi_{Q_3^W}(2, 2, l_2)) \otimes \Delta_3(\phi_{Q_3^W}(3, 3, l_3)).$$

$\Delta_{Q_3^W(N_1, N_2, N_3)}$ and $\Delta_{Q_3^W(N_1, N_2, N_3)}$ are defined in a similar way. Moreover, we decompose these operators as in the case of three-qubits. Then for a pure quantum system $Q_3^p(N_1, N_2, N_3)$ the $GHZ^3$ class concurrence for general pure three-partite states is given by

$$K(Q_3^{GHZ}(N_1, N_2, N_3)) = \sum_{1=r_1<r_2}^{3} \sum_{l_1>k_1=1}^{N_1} k_2=1 \sum_{l_2>k_2=1}^{N_2} k_3=1 \sum_{l_3>k_3=1}^{N_3} \sum_{l_1>l_2>k_1=1}^{N_1} k_3=1 \sum_{l_2>l_3>k_2=1}^{N_2} k_1=1 \sum_{l_3>l_4>k_3=1}^{N_3} \left|\langle\Psi|\Delta_{Q_3^W}(N_1, N_2, C_3\Psi)\rangle\right|^2$$

$$= \sum_{l_1>k_1=1}^{N_1} \sum_{l_2>k_2=1}^{N_2} \sum_{l_3>k_3=1}^{N_3} \left|\alpha_{l_1,l_2,k_3} - \alpha_{l_1,k_2,k_3}\right|^2$$

$$+ \sum_{l_1>k_1=1}^{N_1} \sum_{l_2>k_2=1}^{N_2} \sum_{l_3>k_3=1}^{N_3} \left|\alpha_{l_1,k_2,l_3} - \alpha_{l_1,k_2,k_3}\right|^2$$

$$+ \sum_{l_2>k_2=1}^{N_2} \sum_{l_3>k_3=1}^{N_3} \sum_{l_1>k_1=1}^{N_1} \left|\alpha_{l_1,k_2,l_3} - \alpha_{l_1,k_2,k_3}\right|^2$$

$$+ \sum_{l_3>k_3=1}^{N_3} \sum_{l_1>k_1=1}^{N_1} \sum_{l_2>k_2=1}^{N_2} \left|\alpha_{l_1,k_2,l_3} - \alpha_{l_1,k_2,k_3}\right|^2.$$

Next, we can define a concurrence for three-partite states as square roots of semi concurrence classes as follows

$$C(Q_3(N_1, N_2, N_3)) = \langle N|K(Q_3^W(N_1, N_2, N_3)) + K(Q_3^{GHZ}(N_1, N_2, N_3))\rangle^{1/2}. $$

Based on these construction we are going to propose a minimal entanglement generating set for three-partite states. For some other works on classification of three-partite states, see Ref. [22, 23, 24]. According to our expression for the concurrence, we have $EPR_{Q_1,Q_2}$ class, $EPR_{Q_1,Q_3}$ class, $EPR_{Q_2,Q_3}$ class, $W^3$
class, and $\text{GHZ}^3_{Q_1,Q_2,Q_3}$ class concurrences for three-partite states. Thus, the minimal entanglement generating set for three-partite states is given by

$$\mathcal{E}_{\text{MEGS}}^3 = \{ \text{EPR}_{Q_1,Q_2}, \text{EPR}_{Q_1,Q_3}, \text{EPR}_{Q_2,Q_3}, \text{GHZ}^3_{Q_1,Q_2,Q_3} \}.$$  

Note that the $W^3$ class concurrence is not generic and it is only the sum of EPR class concurrences.

4 Minimal entanglement generating set for multipartite states

In section 2, we have reviewed the concurrence classes for multi-partite states and in section 3, we have constructed MEGS for three-partite states. Now, based on these results, we are going to propose the MEGS for the general quantum system $Q_m(N_1, \ldots, N_m)$ as

$$\mathcal{E}_{\text{MEGS}}^m = \{ \text{EPR}_{Q_1,Q_2}, \ldots, \text{EPR}_{Q_1,Q_m}, \ldots, \text{EPR}_{Q_{m-2},Q_{m-1}}, \ldots, \text{EPR}_{Q_{m-1},Q_m}, \text{GHZ}^3_{Q_1,Q_2,Q_3}, \ldots, \text{GHZ}^3_{Q_{m-2},Q_{m-1},Q_m}, \ldots, \text{GHZ}^{m-1}_{Q_1,Q_2,\ldots,Q_{m-1}}, \ldots, \text{GHZ}^m_{Q_1,Q_2,\ldots,Q_m} \},$$

where for $m$-partite states we have $C(m, 2) = \frac{m(m-1)}{2}$ EPR classes and $C(m, k) \text{GHZ}^k_{Q_1,Q_2,\ldots,Q_k}$ classes, for $2 < k \leq m$. This set gives information about different generic types of entanglement in a given multipartite state. Other classes of entanglement, e.g., the $W^m$ class is only a combination of EPR classes and therefore we call the set (13) the MEGS. We would like to mention that there is another set called minimal reversible entanglement generating set (MREGS) for three partite state, which is different from MEGS.

Next, we will discuss in detail the MEGS of the four-partite states. For four-partite states we have six EPR class concurrences, namely $\text{EPR}_{Q_1,Q_2}$ class, $\text{EPR}_{Q_1,Q_3}$ class, $\text{EPR}_{Q_2,Q_3}$ class, $\text{EPR}_{Q_2,Q_4}$ class, $\text{EPR}_{Q_3,Q_4}$ class, and one $W^3$ class concurrence that is constructed by EPR class concurrences. Moreover, we have four $\text{GHZ}^3_{Q_1,Q_2,Q_3}$ class concurrences, namely $\text{GHZ}^3_{Q_1,Q_2,Q_4}$, $\text{GHZ}^3_{Q_1,Q_3,Q_4}$, $\text{GHZ}^3_{Q_2,Q_3,Q_4}$ class concurrences. Finally, we have the $\text{GHZ}^4_{Q_1,Q_2,Q_3,Q_4}$ class concurrence for four-partite states. Thus the minimal entanglement generating set for four-partite states is given by

$$\mathcal{E}_{\text{MEGS}}^4 = \{ \text{EPR}_{Q_1,Q_2}, \text{EPR}_{Q_1,Q_3}, \text{EPR}_{Q_1,Q_4}, \text{EPR}_{Q_2,Q_3}, \text{EPR}_{Q_2,Q_4}, \text{EPR}_{Q_3,Q_4}, \text{GHZ}^3_{Q_1,Q_2,Q_3}, \text{GHZ}^3_{Q_1,Q_2,Q_4}, \text{GHZ}^3_{Q_1,Q_3,Q_4}, \text{GHZ}^4_{Q_1,Q_2,Q_3,Q_4} \}.$$
Based on the definition of concurrence classes, $E_{MEGS}^{4}$ contains all information about the different classes of entanglement for four-partite states. Thus the MEGS set gives a good classification of entanglement for any multipartite states.

5 Conclusion

We have proposed the minimal entanglement generating set for multipartite states based on our concurrence classes. The MEGS set is not equivalent to minimal reversible entanglement generating set, but there is some similarity between these two sets. The members of MEGS are inequivalent under LQCC by construction as in the case of three-partite states and in general between $W^m$ and GHZ$^m$ classes. This MEGS set can give information about the nature of multipartite entangled states in a time where there is no well known and accepted classification of multipartite states available.

Acknowledgments: The author acknowledges useful comments from Jonas Söderholm. The author gratefully acknowledges the financial support of the Japan Society for the Promotion of Science (JSPS).

References


