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LEPTONIC DECAYS OF THE KAON ACCORDING TO
CURRENT ALGEBRA AND PCAC

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In this lecture I want to discuss the recent calculations of the leptonic decays of the K mesons based on the current commutation relations of Gell-Mann ¹⁾ and the assumption of a partially conserved axial vector current ²⁾ (PCAC). For the most part this will be a review of the work of Callan and Treiman ³⁾ and Weinberg ⁴⁾, although some new results, obtained in collaboration with McNamee ⁵⁾, will also be discussed. The presentation will largely follow an approach due to Bell ⁶⁾ and will be rather elementary since this lecture is primarily intended for those who are not already familiar with the techniques.

The possible leptonic decays of the kaon are the following ⁷⁾

$$K_{l2} : K \rightarrow \bar{l} + \nu_l$$

$$K_{l3} : K \rightarrow \pi + \bar{l} + \nu_l$$

$$K_{l4} : K \rightarrow \pi + \pi + \bar{l} + \nu_l$$

$$K_{l5} : K \rightarrow \pi + \pi + \pi + \bar{l} + \nu_l$$

We shall discuss them all even though the latter process has never been observed to date. The K_{e5} mode is expected to be extremely rare but it is of some theoretical interest since, in calculating the K_{l5} amplitude, one encounters features that are not present in the other amplitudes.

Let us begin with the assumptions that underlie the calculations. Firstly, from the algebra proposed by Gell-Mann ¹⁾, we shall use the commutation relations

$$[Q_5^{(\alpha)}(t), V_\mu^{(\beta)}(x,t)] = -\sqrt{3} \begin{pmatrix} 8 & 8 & 8' \\ \alpha & \beta & \gamma \end{pmatrix} A_\mu^{(\gamma)}(x,t) \quad (1)$$

and

$$[Q_5^{(\alpha)}(t), A_\mu^{(\beta)}(\underline{x}, t)] = -\sqrt{3} \begin{pmatrix} 8 & 8 & 8' \\ \alpha & \beta & \gamma \end{pmatrix} V_\mu^{(\gamma)}(\underline{x}, t) \quad (2)$$

where $V_\mu^{(\alpha)}$ and $A_\mu^{(\alpha)}$ are the vector and axial vector currents, respectively, and $Q^{(\alpha)}$ and $Q_5^{(\alpha)}$ are the associated charges. That is,

$$Q^{(\alpha)}(t) = \int d^3x V_0^{(\alpha)}(\underline{x}, t) \quad (3)$$

and

$$Q_5^{(\alpha)}(t) = \int d^3x A_0^{(\alpha)}(\underline{x}, t) \quad (4)$$

We use $a, b, \dots, n, m$ to indicate the $SU(3)$ transformation properties and to label particles (antiparticles will be labelled $\bar{a}, \bar{b}, \dots$). Note that the normalization is such that the usual $\Delta S = \Delta Q = -1$, $\Delta I = \frac{1}{2}$ current is $(J_4 - iJ_5)_\mu = \sqrt{2}(V_\mu^{(K-)} - A_\mu^{(K-)})$. The $SU(3)$ Clebsch-Gordan coefficients $\begin{pmatrix} 8 & 8 & 8' \\ \alpha & \beta & \gamma \end{pmatrix}$ are the standard ones to be found in the literature ⁸⁾.

The second major ingredient is the hypothesis of a partially conserved axial vector current ²⁾ (PCAC). This can be stated in the form most convenient for our purposes by the relation

$$\sqrt{2} \partial^\mu A_\mu^{(\alpha)}(x) = c_\alpha \phi^{(\alpha)\dagger}(x) \quad (5)$$

where $\phi^{(\alpha)}(x)$ is the meson field and c_α is a real constant we shall determine below. The Hermitian conjugate field $\phi^{(\alpha)\dagger}$ enters in Eq. (5) since we follow the usual convention that, for example, the pion field $\phi(\pi^+)$ annihilates a π^+ while the axial vector current $A_\mu^{(\pi^+)}$ increases the isospin component. We have also adopted the usual phase convention for the fields and currents so that

$$\phi^{(\alpha)+} = \eta(\alpha) \phi(\bar{\alpha}) \quad (6)$$

where

$$\eta(\alpha) = (-)^{(\mathbb{I}_3 + \frac{1}{2}\gamma)_\alpha} \quad (7)$$

The normalization is chosen such that

$$\langle 0 | \phi^{(\beta)}(0) | \pi^{(\alpha)}(p) \rangle = \delta_{\alpha\beta} (2p_0)^{-1/2} \quad (8)$$

The PCAC constant C_π can be related to the pion lifetime. The pion decays entirely via the axial vector current and the relevant matrix element can be written in the form

$$\langle 0 | \sqrt{2} A_\mu^{(\beta)}(0) | \pi^{(\alpha)}(p) \rangle = i \eta(\alpha) \delta_{\alpha\beta} (2p_0)^{-1/2} m_\pi f_\pi p_\mu \quad (9)$$

where f_π is a dimensionless form factor.

Then from Eqs. (5)-(9) one finds immediately that

$$C_\pi = m_\pi^3 f_\pi \quad (10)$$

From the commutation relations and the PCAC hypothesis one can derive the following very useful low energy theorem which relates a matrix element of the vector (axial vector) current to the axial vector (vector) current matrix for the process involving the emission of an additional zero frequency (unphysical) pion.

$$\lim_{p \rightarrow 0} \sqrt{2p_0} \langle F | \pi^{(\alpha)}(p) | \frac{V_\mu^{(\beta)}}{A_\mu} | I \rangle = i \sqrt{6} (m_\pi f_\pi)^{-1} \eta(\alpha) \begin{pmatrix} 8 & 8 & 8' \\ \bar{\alpha} & \beta & \gamma \end{pmatrix} \langle F | \frac{A_\mu^{(\gamma)}}{V_\mu} | I \rangle \quad (11)$$

To obtain Eq. (11) one first reduces in the pion and uses PCAC [Eq. (5)] to relate the pion field to the divergence of the axial vector current. Then, on partial integration and taking the limit $p \rightarrow 0$, one finds the only surviving term can be evaluated using one of the commutators given above [Eqs. (1)-(2)]. Before Eq. (11) can be applied to actual processes one must assume some relation between the physical matrix element where the pion is on its mass shell ($p^2 = m_\pi^2$) and the matrix element in the soft pion limit ($p=0$). Following Bell⁶⁾ we shall make the rather optimistic assumption that in the extrapolation to vanishing pion momentum the matrix element is "as smooth as possible". That is, we shall take account explicitly of any contributions to a given matrix element that necessarily vary rapidly as the pion momentum vanishes, e.g., pole diagrams, and assume that the remainder varies so smoothly that any additional momentum dependence can be ignored. Of course, this principle of "Maximal Smoothness", which we adopt primarily because it is the simplest way to extrapolate, only assures us that we will avoid some manifest contradictions; nevertheless, a posteriori one can observe that it also leads to results consistent with the available experimental data.

As the first application we discuss the relation between the $K_{\ell 3}$ and $K_{\ell 2}$ form factors obtained by Callan and Treiman³⁾. The $K_{\ell 2}$ and $K_{\ell 3}$ form factors are defined by

$$\langle 0 | \sqrt{2} A_\mu^{(m)}(0) | K^{(m)}(k) \rangle = i (2k_0)^{-1/2} m_K F_K k_\mu \quad (12)$$

and

$$\langle \pi^{(n)}(p) | \sqrt{2} V_\mu^{(m)}(0) | K^{(m)}(k) \rangle = (4k_0 p_0)^{-1/2} [F_+(k+p)_\mu + F_-(k-p)_\mu] \quad (13)$$

From isospin considerations we must have

$$F_K = n(m) \delta_{m\bar{m}} f_K \quad (14)$$

and

$$F_{\pm} = \sqrt{2} \begin{pmatrix} g & g & g' \\ m & m & a \end{pmatrix} f_{\pm} \quad (15)$$

where f_K and $f_{\pm}$ depend only on the momenta. Applying the low energy theorem Eq. (11) to relate the $K_{\chi 3}$ amplitude, Eq. (13) to the $K_{\chi 2}$ amplitude, Eq. (12) gives

$$\sqrt{2} (f_+ + f_-) = \frac{m_K f_K}{m_{\pi} f_{\pi}} \quad (16)$$

which is the Callan-Treiman³⁾ result. In obtaining Eq. (16) we have neglected any momentum dependence of the form factors in accordance with the "Maximal Smoothness" principle discussed above, i.e., we can think of no contributions to the form factors that necessarily vary rapidly in the soft pion limit.

It is amusing to observe that had one applied the above considerations to the kaon rather than the pion one would have found the relation

$$\sqrt{2} (f_+ - f_-) = \frac{m_{\pi} f_{\pi}}{m_K f_K} \quad (17)$$

which, together with Eq. (16), implies that

$$(m_K f_K)^2 / (m_{\pi} f_{\pi})^2 = (1 + \xi) / (1 - \xi) \quad (18)$$

where $\xi = f_- / f_+$.

Hence, in the $SU(3)$ symmetric limit $f_+ = 1/\sqrt{2}$ and $f_- = 0$, i.e., $\xi = 0$. If one includes the $SU(3)$ breaking to first order by writing $(m_{\pi} f_{\pi}) / (m_K f_K) = 1 + x$ where x represents the deviation from symmetry, one finds $f_+ = 1/\sqrt{2} + \mathcal{O}(x^2)$ and $f_- = \mathcal{O}(x)$. This illustrates the Ademollo-Gatto⁹⁾ result, that $SU(3)$ violating effects in f_+ are absent in lowest order¹⁰⁾.

Proceeding to the $K_{\ell 4}$ axial vector form factors we define ¹¹⁾

$$\langle \pi^{(a)}(p_1) \pi^{(b)}(p_2) | \sqrt{2} A_{\mu}^{(m)}(0) | K^{(m)}(k) \rangle = i m_K^{-1} (8 k_0 p_{10} p_{20})^{-1/2} \cdot \quad (19)$$

$$\cdot [F_1(p_1 + p_2)_{\mu} + F_2(p_1 - p_2)_{\mu} + F_3(k - p_1 - p_2)_{\mu}]$$

where F_1 - F_3 are invariant dimensionless functions of the momenta and isospins. Before applying the low energy theorem to relate the $K_{\ell 4}$ form factors to the $K_{\ell 3}$ form factors one must recognize that there is now a contribution that does indeed vary rapidly in the soft pion limit, namely, the K pole diagram shown in Fig. 1. This contribution can be expressed in terms of the $K \rightarrow K\pi\pi$ amplitude, the kaon propagator, and the $K_{\ell 2}$ amplitude. Using Weinberg's ¹²⁾ result for the $K \rightarrow K\pi\pi$ amplitude, one finds in the soft pion limit that the contribution is proportional to

$$\frac{k(p_1 - p_2)}{k(p_1 + p_2)} (k - p_1 - p_2)_{\mu}$$

which approaches quite different limits depending on which pion momentum vanishes. Consequently, it is impossible to neglect the momentum dependence of the K pole diagram, which contributes to F_3 , when applying the low energy theorem to Eq. (19). We shall take the K pole contribution into account explicitly by defining

$$F_3 = f_3 + f \frac{k(p_1 - p_2)}{k(p_1 + p_2)} \quad (20)$$

Now, according to "Maximal Smoothness", we can neglect the momentum dependence of F_1 , F_2 , f_3 , and f , there being no other rapidly varying contributions. Of course, these form factors still depend on the isospin, F_1 and f_3 being symmetric and F_2 and f being antisymmetric as required by Bose statistics. Now, applying the low energy theorem to the $K_{\ell 4}$ amplitude yields four relations among F_1 , F_2 , f_3 and f which are readily solved. The results, first given by Weinberg ⁴⁾, are the following ¹³⁾:

$$F_1 = -\sqrt{2} \, \eta(a) \eta(m) \delta_{ab} \delta_{m\bar{a}} (m_K f_+) / (m_\pi f_\pi) \quad (21)$$

$$F_2 = 6\sqrt{2} \begin{pmatrix} 8 & 8 & 8' \\ a & b & \ell \end{pmatrix} \begin{pmatrix} 8 & 8 & 8' \\ m & m & \ell \end{pmatrix} (m_K f_+) / (m_\pi f_\pi) \quad (22)$$

$$F_3 = \frac{1}{2} (1 + \xi) \left[F_1 + F_2 \frac{k(P_1 - P_2)}{k(P_1 + P_2)} \right] \quad (23)$$

Here $\xi = f_-/f_+$.

Finally I would like to mention a calculation of the $K \ell 5$ decay amplitude done in collaboration with McNamee⁵⁾ at Stanford. If we define the $K \ell 5$ vector form factors $G_1 \dots G_4$ according to¹⁴⁾

$$\begin{aligned} \langle \pi^{(a)}(P_1) \pi^{(b)}(P_2) \pi^{(c)}(P_3) | \sqrt{2} V_\mu^{(m)}(0) | K^{(m)}(k) \rangle = m_K^{-2} (16 k_0 P_{10} P_{20} P_{30})^{-1/2} \\ \cdot [G_1 P_{1\mu} + G_2 P_{2\mu} + G_3 P_{3\mu} + G_4 (k - P_1 - P_2 - P_3)_\mu] \end{aligned} \quad (24)$$

then, one can relate these to the $K \ell 4$ axial vector form factors $F_1 \dots F_3$ by means of the low energy theorem. However, a new complication arises in the extrapolation to zero pion momentum in that there are now rapidly varying contributions coming from the π pole diagram shown in Fig. 2(a) as well as from the K pole diagram shown in Fig. 2(b). We proceed as before by treating these contributions explicitly and neglecting the momentum dependence of the remainder. We define

$$\begin{aligned} G_1 P_{1\mu} + G_2 P_{2\mu} + G_3 P_{3\mu} + G_4 (k - P_1 - P_2 - P_3)_\mu = \\ = g_1 P_{1\mu} + g_2 P_{2\mu} + g_3 P_{3\mu} + g_4 (k - P_1 - P_2 - P_3)_\mu + g_\pi \Pi_\mu + g_K K_\mu \end{aligned} \quad (25)$$

where $\overline{\pi}_\mu$ and K_μ represent the π pole and K pole contributions, respectively. These can be written out explicitly in terms of the $K \rightarrow K\pi\pi$, $\pi \rightarrow \pi\pi\pi$, and $K_{\lambda 3}$ amplitudes in the soft pion limit, but are too lengthy to give here. Invoking "Maximal Smoothness" again we shall assume the form factors $g_1 \dots g_k$ vary smoothly in the soft pion limit and treat them as constants depending only on the isospin involved. Then one applies the low energy theorem to relate $K_{\lambda 5}$ to $K_{\lambda 4}$, exactly as in the previous calculations, thereby obtaining rather lengthy general expressions for $G_1 \dots G_4$ after some straightforward but tedious algebra. We give here only the result for the decay $K^+(k) \rightarrow \pi^+(p_1) + \pi^0(p_2) + \pi^-(p_3) + e^+ + \nu_e$ which is a typical case. [Complete results are contained in Ref. 5).] One finds the following:

$$G_1 = -2f_+ \left(\frac{m_K}{m_\pi f_\pi} \right)^2 \left[1 - 2 \frac{(p_1 + p_3)^2 - m_\pi^2}{(p_1 + p_2 + p_3)^2 - m_\pi^2} \right] \quad (26)$$

$$G_2 = f_+ \left(\frac{m_K}{m_\pi f_\pi} \right)^2 \left[1 + 4 \frac{(p_1 + p_3)^2 - m_\pi^2}{(p_1 + p_2 + p_3)^2 - m_\pi^2} + \frac{k(p_1 - p_3)}{k(p_1 + p_3)} \right] \quad (27)$$

$$G_3 = -2f_+ \left(\frac{m_K}{m_\pi f_\pi} \right)^2 \left[1 - 2 \frac{(p_1 + p_3)^2 - m_\pi^2}{(p_1 + p_2 + p_3)^2 - m_\pi^2} + \frac{k(p_1 - p_2)}{k(p_1 + p_2)} \right] \quad (28)$$

$$G_4 = -\frac{1}{2}(f_+ + f_-) \left(\frac{m_K}{m_\pi f_\pi} \right)^2 \left[1 - 4 \frac{(p_1 + p_3)^2 - m_\pi^2}{(p_1 + p_2 + p_3)^2 - m_\pi^2} + 2 \frac{k(p_1 - p_2)}{k(p_1 + p_2)} - \frac{k(p_1 - p_3)}{k(p_1 + p_3)} \right] \quad (29)$$

Numerically the K_{e5} decay rates turn out to be $\sim 10^{-4} - 10^{-3} \text{ sec}^{-1}$ which are a little larger than previously thought ¹⁵⁾, but still too small to be observed.

It has not been our purpose here to present a critical evaluation of these kaon decay calculations. However, in conclusion, we note that the above results agree with the data ⁷⁾, where it exists, about as well as the experiments agree with each other. This is encouraging.

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- 10) I am indebted to J.S. Bell for this observation.
- 11) There is also a vector term $\sim F_+ \epsilon_{\mu\nu\rho\sigma} k^\nu p_1^\rho p_2^\sigma$ that we shall not discuss. It cannot contribute in the soft pion limit and presumably is negligible in the physical region owing to the centrifugal barrier.
- 12) S. Weinberg, Phys.Rev. Letters 17, 616 (1966).

- 13) In Eqs. (21)-(23) we have corrected a sign error in Eq. (24) of Ref. 4).
- 14) The axial vector form factors are presumably unimportant. See Ref. 11).
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- Figure 1 - K pole diagram in $K_{\ell 4}$

- Figure 2a - π pole diagram in $K_{\ell 5}$

- Figure 2b - K pole diagram in $K_{\ell 5}$