Junction conditions in General Relativity with spin sources

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Abstract: The junction conditions for General Relativity in the presence of domain walls with intrinsic spin are derived in three and higher dimensions. A stress tensor and a spin current can be defined just by requiring the existence of a well defined volume element instead of an induced metric, so as to allow for generic torsion sources. In general, when the torsion is localized on the domain wall, it is necessary to relax the continuity of the tangential components of the vielbein.

In fact it is found that the spin current is proportional to the jump in the vielbein and the stress-energy tensor is proportional to the jump in the spin connection. The consistency of the junction conditions implies a constraint between the direction of flow of energy and the orientation of the spin.

As an application, we derive the circularly symmetric solutions for both the rotating string with tension and the spinning dust string in three dimensions. The rotating string with tension generates a rotating truncated cone outside and a flat spacetime with inevitable frame dragging inside. In the case of a string made of spinning dust, in opposition to the previous case no frame dragging is present inside, so that in this sense, the dragging effect can be “shielded” by considering spinning instead of rotating sources. Both solutions are consistently lifted as cylinders in the four-dimensional case.

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1. Introduction

The junction conditions for General Relativity have been studied first in a systematic way in Ref. [1], where one of the assumptions was that the induced metric is well defined on the domain wall. In the first order formalism, where the vielbein and the spin connection are varied independently, this condition can be expressed, in a suitable frame, by the fact that the vielbein should have no discontinuity tangential to the domain wall.

It has been conjectured [2] that, for reasonable matter obeying certain energy conditions, singular shells should produce a continuous metric. We should emphasise that this conjecture is in the context of General Relativity without torsion.

However, it is shown here that the requirement of a continuous metric is no longer consistent in general with torsion sources, which are generated by spin currents along the domain wall. Thus, the problem of finding the junction conditions for gravity in such situations must be analyzed from scratch.
We derive the junction conditions for General Relativity in the presence of domain walls with intrinsic spin in three and higher dimensions. The new set of junction conditions reduce to Israel’s ones in the absence of torsion sources. We will show that a stress tensor and a spin current can be defined by requiring the existence of a well-defined volume element instead of an induced metric. This allows one to deal with situations where the torsion is localized on the domain wall, where it is necessary to relax the continuity of the tangential components of the vielbein. We consider distributions of spin currents and stress-energy tensors that can be obtained as the limit of a smooth distribution. Consequently, the junction conditions consist of dynamical equations relating the stress-energy tensor with the jump in the spin connection, and the spin current with the jump in the vielbein, as well as on certain constraints for the jump in the geometry.

In order to see how the junction conditions work in a simple setup, we first consider 2+1 gravity without cosmological constant, where space-time is flat outside the sources. The junction condition approach is then especially suitable, being simply a matter of piecing together two flat manifolds. Following this approach it is simple to obtain the circularly symmetric solutions for both the rotating string with tension and the spinning dust string in three dimensions.

Static closed string sources without torsion in 2+1 gravity have been studied in Ref. [3] solving the Einstein field equations with a distributional source. Here, instead of solving directly the Einstein field equations, we will extend the static solution in two ways by using the new junction conditions.

We first discuss the extension to rotating case [4], where it is found that the metric outside corresponds to a rotating truncated cone, while the inner metric describes flat space-time with inevitable frame dragging. The matching conditions impose for the rotating string a stress energy tensor of a fluid with nonzero pressure. This non-static case therefore circumvents the situation described in [3] where a closed string with tension can only generate a cylinder space-time as solution.

Next we discuss the case of a closed static string made of spinning dust, which has a homogeneous torsion distribution on it. As torsion is related to the spin density [5] we will refer to this as the “spinning string”. In the case of a string made of spinning dust, the metric outside is the same as for the rotating case, but in this case no frame dragging is present inside.

It is worth pointing out that the presence of torsion concentrated on the string forces a tangential discontinuity in the vielbein, but the induced volume element is well-defined. Here we have a concrete example of a physical situation where an extended gravitating object can be treated even though there is no a well defined induced metric.

From this concrete example, one concludes that having a spinning string as consistent solution of the field equation is generically incompatible with having a continuous metric. This is apparent from the fact that the discontinuity in the viel-
bein cannot be avoided by absorbing the singularity of the torsion in a δ-distribution in the spin connection, since in this case, the Einstein tensor would acquire a delta function squared, and thus the solution would be meaningless in the distributional sense. This example then shows the need for generalizing the standard junction conditions.

As an example of the junction conditions in higher dimensions, we show that the two solutions can be consistently lifted to the rotating and spinning cylinder in four dimensions in a similar way a point particle solution in 2 + 1 is lifted to a cosmic string [3, 7]. The lifting to higher dimensions is trivial.

The plan of the paper is as follows: In Section II we consider the junction conditions in 2+1 dimensions, and then apply them to obtain the rotating and spinning strings. Section III is devoted to the four- and higher-dimensional case, where the junction conditions are used to obtain the lifting of the solutions previously discussed. Section IV contains the conclusions.

2. Junction conditions for torsion sources in three dimensions

In the three-dimensional case the junction conditions can be derived in a straightforward way, and it is instructive to discuss it first in order to gain insights about how to proceed in higher dimensions.

2.1 2+1 General Relativity in first order formalism

The Lagrangian for General Relativity in first order formalism in 2+1 dimensions is

$$\mathcal{L} = \Omega^{ab} \wedge e^c \epsilon_{abc} + \mathcal{L}_{\text{matter}}.$$ 

where $e^a = e^a_\mu dx^\mu$ is the vielbein one-form, $\omega^{ab}$ is the spin connection and $\Omega^{ab} = d\omega^{ab} + \omega^{ac}_a \wedge \omega^{cb}$ denotes the curvature 2-form.

The Euler-Lagrange variation with respect to the vielbein $e^a_\mu$ gives the Einstein equations:

$$\Omega^{ab} \wedge dx^\mu \epsilon_{abc} = -2T^\mu_c Vol,$$  \hspace{1cm} (2.1)

$$\text{Vol} := \frac{1}{3!} e^a \wedge e^b \wedge e^c \epsilon_{abc} = \sqrt{-g} d^3 x$$

Here $T^\mu_c$ is the stress-energy tensor. The more familiar form is $T^\mu_c = e^c_\nu T^\mu_c$.

The Euler-Lagrange equation with respect to the spin connection $\omega^{ab}_\mu$ gives

$$dx^\mu \wedge T^c \epsilon_{abc} = S^\mu_{ab} \text{Vol}$$ \hspace{1cm} (2.2)

where $T^a = De^a := de^a + \omega^a_b \wedge e^b$ is the torsion, and $S^\mu_{ab}$ is the spin current. In any region of space-time where there is no matter, these field equations imply that both the curvature and torsion vanish. Note that we chose units where $8\pi G = 1$. 

\hspace{1cm}
2.2 Junction conditions for a string

We will consider a string which is an object of 1 spatial dimension supporting a singular distribution of matter possibly with spin current. The 1 + 1-dimensional worldsheet of the string will be denoted by $\Sigma$, which divides the spacetime into two disconnected pieces $M_+$ and $M_-$. and it is assumed not to be a null surface. Consider a region, $O$, of space-time which contains $\Sigma$ and is of arbitrarily small width in the direction normal to $\Sigma$. The stress-energy distribution $T^\mu_a$ with support on the surface $\Sigma$ is defined by:

$$\lim_{O \to \Sigma} \int_{\partial O} T^\mu_a \text{Vol} = \int_\Sigma (T^\mu_a) \text{Vol}_\Sigma,$$  \hspace{1cm} (2.3)

where $\text{Vol}_\Sigma$ stands for the induced volume element on $\Sigma$. Alternatively we can say $T^\mu_a = (T^\mu_a) \delta(\Sigma)$ where $\delta(\Sigma)$ is the Dirac delta function. Similarly, we define the spin current distribution to be

$$\lim_{O \to \Sigma} \int_{\partial O} S^\mu_{ab} \text{Vol} = \int_\Sigma (S^\mu_{ab}) \text{Vol}_\Sigma.$$  \hspace{1cm} (2.4)

As matter is confined on $\Sigma$, the geometry is necessarily non-smooth. As usual, the stress-energy of the string will generate a jump in the spin connection. However, it can be seen that the concentration of torsion on the string requires also a discontinuity in the vielbein at $\Sigma$. This is further discussed in section 2.3. So, instead of requiring that the geometry be continuous, we impose a weaker condition: we shall require only that the vielbein and spin connection have a bounded discontinuity at $\Sigma$.

We integrate the l.h.s of the Einstein field equations (2.1) over the region $O$. In the limit, only the exterior derivatives of the spin connection contributes, giving the following boundary term,

$$\lim_{O \to \Sigma} \int_{\partial O} \Omega^{ab} \wedge dx^\mu \epsilon_{abc} = \lim_{O \to \Sigma} \int_O (d\omega^{ab} + \cdots) \wedge dx^\mu \epsilon_{abc} = \lim_{O \to \Sigma} \int_O d(\omega^{ab} \wedge dx^\mu \epsilon_{abc}) + (\cdots)$$

$$= \int_{\Sigma_+} \omega^{ab} \wedge dx^\mu \epsilon_{abc} - \int_{\Sigma_-} \omega^{ab} \wedge dx^\mu \epsilon_{abc}$$

$$= \int_{\Sigma} \Delta \omega^{ab} \wedge dx^\mu \epsilon_{abc},$$  \hspace{1cm} (2.5)

where $\Delta \omega^{ab} = \omega^{ab}_+ - \omega^{ab}_-$ is the discontinuity in the spin connection across $\Sigma$, and the dots $(\cdots)$ represents those terms which vanish in the limit. Proceeding in the same way with the torsion equation (2.2) one obtains the following boundary term

$$\lim_{O \to \Sigma} \int_O dx^\mu \wedge T^c_{abc} = \lim_{O \to \Sigma} \int_O dx^\mu \wedge (dc^c + \cdots) \epsilon_{abc} = \lim_{O \to \Sigma} \int_O d(dx^\mu \wedge e^c \epsilon_{abc}) + (\cdots)$$

$$= \int_{\Sigma_+} dx^\mu \wedge e^c \epsilon_{abc} - \int_{\Sigma_-} dx^\mu \wedge e^c \epsilon_{abc}$$

$$= \int_{\Sigma} dx^\mu \wedge \Delta e^c \epsilon_{abc}$$  \hspace{1cm} (2.6)
where $\Delta e^a = e^a_+ - e^a_-$ is the discontinuity in the vielbein.

So, comparing (2.3) and (2.6) with (2.3) and (2.4), we obtain the junction conditions:

\begin{align*}
  i^*(\Delta \omega^{ab} \wedge dx^\mu \epsilon_{abc}) &= -2(T_\Sigma)^\mu_c \text{Vol}_\Sigma, \quad (2.7) \\
  i^*(dx^\mu \wedge \Delta e^c \epsilon_{abc}) &= (S_\Sigma)^\mu_{ab} \text{Vol}_\Sigma, \quad (2.8)
\end{align*}

where $i^*$ denotes the pull-back of differential forms to the surface $\Sigma$.

According to Eq. (2.8), one can see that in the absence of spin currents on $\Sigma$, the tangent components of the vielbein are continuous, and thus, Eq (2.7) reduces to the standard Israel Junction conditions. Thus, we conclude that the presence of spin current on $\Sigma$ in (2.8) necessarily forces the discontinuity in the tangential components of the vielbein, i.e., the induced metric on the string worldsheet $h^{+\mu\nu}$ induced by the geometry in $M_+$ is different to $h^{-\mu\nu}$ induced by $M_-$.

Note that on the right hand side of the junction conditions, the volume element of the string appears explicitly. This comes from the definitions (2.3) and (2.4). In order for this to make sense, we clearly need the intrinsic volume element on $\Sigma$ to be single valued, i.e.,

\begin{equation}
  \text{Vol}_{\Sigma^+} = \text{Vol}_{\Sigma^-} \equiv \text{Vol}_\Sigma. \quad (2.9)
\end{equation}

In terms of the metric, equation (2.9) means that, although $h^{\mu\nu}_+ \neq h^{-\mu\nu}$, the determinants are equal: $h_+ = h_-$. In terms of the vielbein this means that $i^*e_+^a \wedge e_+^b = i^*e_-^a \wedge e_-^b$ even though $e_+^a \neq e_-^a$. Therefore instead of requiring a well defined induced metric one we can relax to the weaker condition of having a well defined volume.

Upon a more careful treatment of the distributional field equations, we will find in section 3 that extra conditions are needed. In the case of 2+1 dimensions, there is one extra condition $i^*\Delta e^a \wedge \Delta e^b = 0$.

Note that even in the absence of an induced metric on $\Sigma$, the junction conditions (2.7), (2.8) can be written in terms of purely anholonomic indices,

\begin{align*}
  i^*(\Delta \omega^{ab} \wedge e^d \epsilon_{abc}) &= -2(T_\Sigma)^d_c \text{Vol}_\Sigma, \\
  i^*(e^d \wedge \Delta e^c \epsilon_{abc}) &= (S_\Sigma)^d_{ab} \text{Vol}_\Sigma.
\end{align*}

provided the following conditions are satisfied

\begin{align*}
  i^*(\Delta e^a \wedge \Delta e^a) &= 0, \quad (2.10) \\
  i^*(\Delta e^d \wedge \Delta e^d) &= 0. \quad (2.11)
\end{align*}

This can be seen contracting Eqs. (2.7) and (2.8) with $e^d_\mu$, and requiring for the left hand side of these equations to be independent wether we contract with $e^d_+$ or $e^d_-$. Note that equation (2.11) is the same as the condition in the above paragraph.
It is important to stress that the fundamental equations are the junction conditions (2.7) and (2.8), plus the conditions (2.9) and (2.11). The other condition, (2.10) should be regarded as a weaker alternative to having an induced vielbein on \( \Sigma \) for the purposes of changing to anholonomic indices. This extra condition does not come from the field equations. We will see that this is not the case in higher dimensions, where the condition (2.10) is strictly needed.

In the presence of a cosmological constant, the junction conditions are the same since the cosmological constant term does not contain derivatives.

To summarize, the two junction conditions (2.7) and (2.8), together with (2.1) and (2.2) in the interior of \( M^+ \) and \( M^- \), determine completely the solution for the string.

### 2.3 The static dust string

The results obtained from the junction conditions agree with solving the Einstein equation with delta function sources. This can be seen explicitly for a static closed dust string [8]. In this case the interior is a piece of Minkowski space and that the exterior is locally Minkowski, with the spatial section having the shape of a cone of deficit angle \( 2\pi (1 - B) \). A simple way to derive the mass of the string in terms of \( B \) is to write the metric for the whole space-time as

\[
ds^2 = -dt^2 + \left\{ 1 - (1 - B^{-2})\theta(r - r_0) \right\} dr^2 + r^2 d\phi^2.
\]

Above, \( \theta \) is the Heaviside distribution, which takes value 0 for \( r < r_0 \) and 1 for \( r > r_0 \). The location of the string will be \( r = r_0 \). Although in this form the metric appears discontinuous, this is an artefact of the choice of coordinates\(^1\). Indeed, the induced metric on the surface \( r = r_0 \) is well defined:

\[
ds^2_\Sigma = -dt^2 + r_0^2 d\phi^2,
\]

which is the metric of the cylinder with radius \( r_0 \). We can choose the vielbein to be:

\[
e^0 = dt,
\]

\[
e^1 = \left\{ 1 - (1 - B^{-1})\theta(r - r_0) \right\} dr,
\]

\[
e^2 = rd\phi.
\]

Using the zero torsion condition, we find the non-vanishing component of the spin connection:

\[
\omega^{12} = - \left\{ 1 - (1 - B^{-1})\theta(r - r_0) \right\}^{-1} d\phi = - \left\{ 1 - (1 - B)\theta(r - r_0) \right\} d\phi. \tag{2.12}
\]

\(^1\)The coordinates outside the source can be chosen so that the spatial section of exterior metric is conformally flat

\[
\left( \frac{r}{r_0} \right)^{-2(1 - B)} (dr^2 + r^2 d\phi^2),
\]

and in this way the metric and the normal are continuous across \( \Sigma \).
The only non-trivial curvature component is:

$$\Omega^{12} = d\omega^{12} = (1 - B)\delta(r - r_0)dr \wedge d\phi.$$ 

Here $\delta$ is the Dirac delta distribution. Inserting this ansatz into the field equations (2.1) we obtain

$$2(1 - B)\delta(r - r_0)dr \wedge d\phi \wedge dt = -2T_0^0 dt \wedge e^1 \wedge e^2.$$ 

The integration constant $B$ is related with the mass which can be defined as the integral of $-T_0^0 e^1 \wedge e^2$ over a spatial cross section

$$M = \int drd\theta(1 - B)\delta(r - r_0) = 2\pi(1 - B),$$

and so the mass (or $8\pi G$ times the mass, restoring Newton’s constant) is equal to the deficit angle.

As a check, we apply the junction conditions to the dust string to rederive the above results. The discontinuity in the connection is $\Delta\omega^{12} = (1 - B)d\phi$. Putting this into the junction conditions (2.7) and (2.8), we get

$$(T_\Sigma)^0_0 = -(1 - B)/r_0,$$

and the spin current vanishes. Integrating this around the string’s length, we get $M = 2\pi(1 - B)$ as expected. Note that the junction conditions work even for a discontinuous vielbein. This discontinuity contributes nothing to the torsion because it is purely normal to the string i.e. $i^*\Delta e^a = 0$.

### 2.4 The rotating string with tension

Here we extend the previous result to the rotating case with tension making use of the junction conditions (2.7), (2.8). For the exterior region $M_+$, the metric corresponds to the one of a rotating conical spacetime[8]. This can be written in the familiar form

$$ds^2_+ = -\left(dt - \frac{J}{2}d\phi\right)^2 + \tilde{r}^2 + B^2\tilde{r}^2d\phi^2,$$  

$$(2.13)$$

In this section we will, by a rescaling $\tilde{r} \rightarrow r(\tilde{r})$, write the metric in the following form

$$ds^2_+ = -\left(dt - \frac{J}{2}d\phi\right)^2 + (1 + \chi^2)\left(\frac{r}{r_0}\right)^{2(B-1)}(dr^2 + r^2d\phi^2),$$  

$$(2.14)$$

where the constant $\chi$ is defined as

$$\chi := \frac{J}{2r_0}$$
and we choose the vielbein as
\[ e_0^0 = dt - \frac{J}{2} d\phi, \quad e_1^1 = \sqrt{1 + \chi^2} \left( \frac{r}{r_0} \right)^{(B-1)} dr, \quad e_2^2 = \sqrt{1 + \chi^2} \left( \frac{r}{r_0} \right)^{(B-1)} rd\phi. \]

The string is located at \( r = r_0 \) so that the induced metric on its worldsheet is:
\[ -dt^2 + J d\phi dt + r_0^2 d\phi^2, \]
which ensures the absence of closed timelike curves in the outer region.

The interior, \( M_- \), is a region of Minkowski space,
\[ ds_-^2 = -dt^2 + dr^2 + r^2 d\phi^2. \]
Assuming that there is no spin current on the string, by virtue of (2.8) the induced metric is continuous. As a consequence, the inner and outer frames and co-ordinates must be related. Thus, the matching of the coordinates gives
\[ t' = \sqrt{1 + \chi^2} t, \quad \phi' = \phi + \frac{\chi}{r_0} t, \]
where \( \phi' \) has the same periodicity as \( \phi \), and the location of the string measured with respect to the inner coordinate \( r \) is \( r = r_0 \).

The vielbein of the interior region can taken to be:
\[ e_\perp^0 = dt', e_\perp^1 = dr, e_\perp^2 = r d\phi', \]
so that the inner and outer induced vielbeins on the string worldsheet are
\[ \theta_-^0 = \sqrt{1 + \chi^2} dt, \]
\[ \theta_-^2 = r_0 d\phi + \chi dt, \quad (2.16) \]
and
\[ \theta_+^0 = dt - r_0 \chi d\phi, \]
\[ \theta_+^2 = r_0 \sqrt{1 + \chi^2} d\phi, \quad (2.17) \]
respectively. They are related by a Lorentz transformation\(^2\):
\[ \theta_+^a = \Lambda_a^b \theta_-^b, \quad \Lambda_a^b = \begin{pmatrix} \sqrt{1 + \chi^2} & -\chi \\ -\chi & \sqrt{1 + \chi^2} \end{pmatrix}. \]

The spin connection in the interior is
\[ \omega_-^{12} = -d\phi'. \]

\(^2\)Here hatted indices correspond to the inner frame. This is useful because the junction conditions are formulated in a single frame.
and in the exterior region it is
\[ \omega_+^{12} = -B d\phi. \]

Now we notice that \( i^* \omega \) transforms like a tensor under the two dimensional Lorentz transformations on the worldsheet \(^3\).

The junction conditions give
\[ i^* \left( \omega_+^{ab} \wedge \theta_+^d \epsilon_{abc} - \omega_-^{\hat{a}\hat{b}} \wedge \theta_-^d \epsilon_{\hat{a}\hat{b}\hat{c}} \Lambda_d^c (\Lambda^{-1})^{\hat{c}} \right) = -2 T_c^d \text{Vol}_\Sigma. \tag{2.18} \]

Using the fact that the volume element on the string worldsheet is \( r_0 dt' \wedge d\phi' = r_0 \sqrt{1 + \chi^2} dt \wedge d\phi \), we get:
\[
T_0^0 = \frac{B}{r_0 \sqrt{1 + \chi^2}} - \frac{1 + \chi^2}{r_0}, \\
T_0^2 = -T_2^0 = \frac{\chi \sqrt{1 + \chi^2}}{r_0}, \\
T_2^2 = \frac{\chi^2}{r_0}. \tag{2.19}
\]

The stress-energy tensor (2.19) can be written as that of a perfect fluid with pressure
\[ T^a_b = (\rho + p) u^a u_b + p \delta^a_b, \]

Writing the fluid velocity vector as \( u^a = (\sqrt{1 + v^2}, v) \), we get:
\[
(1 + v^2)(\rho + p) - p = \frac{1 + \chi^2}{r_0} - \frac{B}{r_0 \sqrt{1 + \chi^2}}, \\
v \sqrt{1 + v^2} (\rho + p) = -\frac{\chi \sqrt{1 + \chi^2}}{r_0}, \\
v^2 (\rho + p) + p = \frac{\chi^2}{r_0}.
\]

So we can express the fluid velocity, the energy density and the pressure in terms of

\(^3\)Since the intrinsic spin connection on the worldsheet vanishes on both frames, for the above choice of vielbeins, \( i^* \omega_+^{ab} \) is the second fundamental form of the string worldsheet with respect to the embedding into the exterior and interior regions respectively. Thus, (2.18) is the same as the Israel junction condition. For a more general situation, we would need to calculate \( i^* (\omega_+ - \omega_0)^{ab} \) and \( i^* (\omega_- - \omega_0)^{\hat{a}\hat{b}} \), where \( \omega_0 \) is the connection associated with the intrinsic geometry of the worldsheet.
the integration constants $B$ and $\chi$ as:

$$\rho - p = \frac{1}{r_0} (1 - B') , \quad (2.20)$$

$$\rho + p = \frac{1 - B' + \chi^2}{r_0(1 + 2v^2)} , \quad (2.21)$$

$$1 + 2v^2 = \sqrt{1 + \frac{4 \chi^2(1 + \chi^2)}{(1 - B' + \chi^2) - 4 \chi^2(1 + \chi^2)}} . \quad (2.22)$$

$$B' \equiv \frac{B}{\sqrt{1 + \chi^2}} . \quad (2.23)$$

Therefore as expected, a stationary rotating string must have non-zero stress. It is worth also pointing out that, in the rotating case the presence of a tension does not imply $B = 0$, so that outside metric is given by a rotating cone in opposition to the static case [3] where it was found that the outside metric can only correspond to a cylinder.

It can be seen that the rotating string with tension generates inside a flat space-time with inevitable frame dragging. Writing the geodesic equation for the outer metric, one can show that a particular solution is $\dot{r} = \dot{\phi} = \ddot{t} = 0$. Therefore we can use $t$ as an affine parameter so that

$$\frac{\partial \phi}{\partial t} = \frac{\partial r}{\partial t} = 0 \quad (outside).$$

In the interior, a geodesic observer at constant radius has $\phi' = \text{const}.$ i.e.

$$\frac{\partial \phi}{\partial t'} = - \frac{\chi}{r_0} \quad (inside).$$

So there is a relative angular velocity between inertial observers inside and outside. For an inertial observer outside, an inertial observer inside appears accelerated. This is analogous to the case of a rotating shell in 3+1 dimensions, see e.g. [9].

### 2.5 Incompatibility of a torsion source with a continuous metric

Suppose now that we have a string with spin current, which is a source of torsion as well as curvature. In this case, the source corresponds to a continuous distribution of spinning point particles. In the case of a spinning point particle one has an exterior geometry given by the spinning cone with a $\delta$-distribution of torsion at the origin [3]. Hence, a spinning string with circular symmetry located at some fixed radius $r_0$, possesses the same exterior geometry as the spinning point particle, but now the distribution of torsion is of the form $T^0 = \gamma J \delta(r - r_0) dr \wedge d\phi$, so that it reduces to the spinning point particle in the limit of zero string length. We write the exterior metric in the simple form of equation (2.13) and we choose the vielbein as

$$e_0^0 = dt - \frac{J}{2} d\phi , \quad e_1^1 = d\tilde{r} , \quad e_2^2 = B\tilde{r} d\phi . \quad (2.24)$$
As it has been discussed above, in this case, the junction conditions imply that one must have a finite jump in the tangential components of the metric. However, one may naturally wonder if there is an alternative possibility of finding a consistent solution in this situation with a continuous intrinsic metric. Here we show that this possibility cannot be realized.

Assuming that the metric is continuous at $\Sigma$, in the presence of above torsion distribution, the spin connection necessarily acquires a delta function with support on the string worldsheet. Precisely, the torsion delta function comes only from a delta function in the contorsion part of the spin connection. Thus, the spin connection is found to be:

$$\omega^{ab} = \hat{\omega}^{ab} + \kappa^{ab}$$

where $\hat{\omega}^{ab}$ is the Levi-Civita connection of the spinning cone metric satisfying $d e^a + \hat{\omega}^{ab} e_b = 0$, and the contorsion $\kappa^{ab}$ is given by

$$\kappa^0_1 = -\gamma J \delta(r - r_0) d\phi,$$

$$\kappa^0_2 = \frac{\gamma J}{r_0} \delta(r - r_0) dr,$$

$$\kappa^1_2 = \frac{\gamma J^2}{2r_0} \delta(r - r_0) d\phi - \frac{\gamma J}{r_0} \delta(r - r_0) dt.$$  \hspace{1cm} (2.25)

The curvature 2-form can then be decomposed into a piece $\Omega^{ab} = d\hat{\omega}^{ab} + \hat{\omega}^a_c \hat{\omega}^{cb}$ which is constructed from the Levi-Civita connection, plus additional terms including the contorsion as

$$\Omega^{ab} = \hat{\Omega}^{ab} + \hat{D}\kappa^{ab} + \kappa^a_c \kappa^{cb},$$

where $\hat{D}$ stands for the covariant derivative with respect to the connection $\hat{\omega}^{ab}$. Therefore, in this way one cannot avoid the appearance of delta function squared terms in the Riemann tensor. In particular, since $(\kappa^2)^0_2 = \frac{\gamma^2 J^2}{r_0} \delta^2(r - r_0) d\phi \wedge dt$, the component $R^0_{202}$ contains a delta function squared which is ill-defined as a distribution. Since we are in 2+1 dimensions this means that the Einstein tensor possesses a delta function squared, and hence the field equations imply that the stress-energy tensor is ill-defined in the distributional sense. This means that the matter distribution is not physical because it cannot be normalized. Therefore, one concludes the assumption of having a continuous tangential metric in this case leads to an unphysical situation since the matter distribution needed to satisfy the field equations cannot exist.

Conversely, we conclude that discontinuities in the metric generically require the presence of torsion concentrated on $\Sigma$. Indeed, in the absence of torsion, the discontinuity would imply that the connection contains a delta function, leading to a delta function squared in the Riemann tensor, as is well known\[10, 11\]. We emphasise that torsion may not be required for special cases, such as travelling waves\[12\], where the Riemann tensor is linear in the distributional part of the metric.

This example then shows that, in order to have a physically consistent solution to this problem, one needs to generalize the standard junction conditions as in Eqs.
which allow for a discontinuity in the tangential metric. This is discussed in the next subsection.

2.6 The spinning closed string

The metric for the exterior region \( M_+ \) of the spinning string corresponds to the rotating cone as in Eq. (2.13) but here we use coordinates so that the spatial section is conformally flat:

\[
ds_+^2 = - \left( dt - \frac{J}{2} d\phi \right)^2 + \left( \frac{r}{r_0} \right)^{-2(1-B)} (dr^2 + r^2 d\phi^2), \tag{2.26}
\]

and we choose the vielbein as

\[
e_0^+ = dt - \frac{J}{2} d\phi, \quad e_1^+ = \left( \frac{r}{r_0} \right)^{B-1} \, dr, \quad e_2^+ = \left( \frac{r}{r_0} \right)^{B-1} \, r d\phi. \tag{2.27}
\]

For the interior region \( M_- \), the metric is flat, and the vielbein is chosen to be

\[
e_0^- = dt', \quad e_1^- = dr, \quad e_2^- = r d\phi'. \tag{2.28}
\]

Hence we have a discontinuity in \( e_0^0 \) across the worldsheet \( \Sigma \). Let us now find the stress-energy \( (T_\Sigma)^\mu_\alpha \) and spin current \( (S_\Sigma)^\mu_{ab} \) through the junction conditions (2.7) and (2.8), respectively.

The position of the string is defined to be \( r = r_0 \). We have constructed the vielbein so that the normal \( e_1 \) is continuous across \( \Sigma \). The junction condition (2.9) relating the induced volume on \( \Sigma \) measured from both regions is satisfied choosing \( t = t', \phi = \phi' \). So the vielbeins on \( \Sigma \) induced by \( M_+ \) and \( M_- \) are of the form

\[
\theta_0^0 = dt - \frac{J}{2} d\phi, \quad \theta_2^+ = r_0 d\phi, \quad \theta_0^- = dt', \quad \theta_2^- = r_0 d\phi'.
\]

respectively. Using either induced metric, the induced volume element is the same, i.e.,

\[
\text{Vol}_{\Sigma} = \text{Vol}_{\Sigma_-} = \text{Vol}_{\Sigma} = -r_0 dt \wedge d\phi.
\]

Note also that the conditions (2.10) and (2.11) are satisfied.

Since the torsion is zero outside of the string, the only nonvanishing component of the spin connection are

\[
\omega_{12}^{+} = -B d\phi, \quad \omega_{12}^{-} = - d\phi.
\]

and since the corresponding curvatures outside of the string vanish, the vacuum field equations are satisfied.
As required by the junction conditions (2.7) and (2.8), in order to find the stress-energy and spin current on $\Sigma$ we need the discontinuities of the vielbein and connection pulled back to the tangent space of $\Sigma$, i.e. $i^* \Delta e^a = \Delta \theta^a$ and $i^* \Delta \omega^{ab}$, whose only non-vanishing components are

$$i^* \Delta e^0 = -\frac{J}{2} d\phi , \quad (2.31)$$
$$i^* \Delta \omega^{12} = (1 - B) d\phi . \quad (2.32)$$

Substituting these into the junction conditions (2.7) and (2.8), we find that the only non-vanishing component of the stress-energy tensor is

$$(T_{\Sigma})^0_0 = -\frac{(1 - B)}{r_0} , \quad (2.33)$$

and analogously, for the spin current one obtains

$$(S_{\Sigma})^0_1 = -(S_{\Sigma})^0_2 = \frac{J}{2r_0} . \quad (2.34)$$

Therefore, the solution describes a string made of spinning dust.

In sum, the metric for the spinning string in the whole spacetime can be written as

$$ds^2 = -\left(dt - \frac{J}{2} \theta(r - r_0) d\phi \right)^2 + \left(1 + \left\{ \left(\frac{r}{r_0}\right)^{-2(1-B)} - 1 \right\} \theta(r - r_0) \right) (dr^2 + r^2 d\phi^2) , \quad (2.35)$$

where the only nonvanishing component of the torsion generated by the spin current turns out to be

$$T^0 = -\frac{J}{2} \delta(r - r_0) dr \wedge d\phi .$$

This can be explicitly checked writing the vielbein as

$$e^0 = dt - \frac{J}{2} \theta(r - r_0) d\phi ,$$
$$e^1 = \left(1 + \left\{ \left(\frac{r}{r_0}\right)^{-2(1-B)} - 1 \right\} \theta(r - r_0) \right) dr ,$$
$$e^2 = \left(1 + \left\{ \left(\frac{r}{r_0}\right)^{-2(1-B)} - 1 \right\} \theta(r - r_0) \right) r d\phi ,$$

and using the fact that the only non zero components of the spin connection are the same as in Eq. (2.12).

Some interesting remarks can be made comparing the spinning dust string with the rotating string with tension discussed in 2.4. Since the metric of the exterior region is the same in both cases (up to a co-ordinate rescaling of $r$ by a constant
factor), the total mass and angular momentum coincide. However, the source of the angular momentum has a different nature in both cases. For the rotating string with tension the angular momentum \( J \) is a consequence of the fact that fluid is stationary with a given angular velocity, while for the spinning string, the non-zero angular momentum of the exterior solution is caused by the torsion produced by the distribution of static spinning dust particles. Furthermore, as opposed to the rotating string with tension, where the inner flat space-time has an inevitable frame dragging, for the spinning string the inertial frames in the interior are not dragged. It is then worth pointing out that in this sense, the dragging effect can be “shielded” by considering spinning instead of rotating sources.

Let us digress for a moment to discuss the point particle [5]. In this case the metric does not distinguish between the spinning and rotating sources. In both cases we have a delta in the energy density. The source of angular momentum can either be a delta function in the torsion or a \( T_{ab} \) is the derivative of a delta function (point particle limit for a rotating object). If the source acquires a bit of length the metric distinguishes between the two situations.

Regarding the spinning string as a limiting case of some finite distribution of matter with aligned spin, makes the metric change very rapidly across the thickness of the string. In this sense, one concludes that the spin has a more dramatic effect on the geometry than does the mass: The mass causes the first derivative of the connection to blow up whereas the spin causes the first derivative of the metric to blow up.

As discussed in the next section, these effects can be seen to occur also in four and higher dimensions.

### 3. Four and higher dimensional case

The Lagrangian for General Relativity in first order formalism in arbitrary dimension \( D \) is

\[
\mathcal{L} = \frac{1}{(D-2)!} \Omega^{a_1 a_2 \cdots a_D} \epsilon_{a_1 a_2 \cdots a_D} \cdot \cdot \cdot .
\]

Hereafter wedge product between forms is understood. The variation with respect to the vielbein gives the Einstein field equations

\[
\Omega^{bc} dx^\mu \text{Vol}_{abc} = -2T^\mu_a \text{Vol} , \tag{3.1}
\]

and the variation with respect to the spin connection allows to fix the torsion in terms of the spin current as

\[
dx^\mu T^c \text{Vol}_{abc} = S^c_{ab} \text{Vol} , \tag{3.2}
\]

where

\[
\text{Vol}_{abc} := \frac{1}{(D-3)!} \epsilon^{a_1 \cdots a_D} \epsilon_{a_1 a_2 \cdots a_D} \tag{3.3}
\]
is a volume element of a $d - 3$-dimensional surface orthogonal to $e^a \wedge e^b \wedge e^c$. It is useful to define also
\[
\text{Vol}_{a_1 \cdots a_n} := \frac{1}{(D - n)!} e^{a_{n+1}} \cdots e^{a_D} \epsilon_{a_1 \cdots a_D}
\]

3.1 Junction conditions in higher dimensions

We derive the junction conditions in $D > 3$ dimensions. The geometry is assumed to be smooth on each side with a possible discontinuity on the domain wall $\Sigma$, thus we have smooth fields $e^a_+, \omega^{ab}_+$ on one side and smooth fields $e^a_-, \omega^{ab}_-$ on the other side. The geometry of the entire manifold is given then by some distributional vielbein and spin connection which coincide with the smooth functions $e^a_\pm$ and $\omega^{ab}_\pm$ outside the surface $\Sigma$.

Furthermore, we shall assume that there is a well defined normal vector on the hypersurface.

The distribution $e^a$ can be defined by a sequence of smooth vielbeins $e^a_\alpha$ which interpolate between the discontinuous values within some neighbourhood $O_\alpha$ of the hypersurface $\Sigma$. The width of region $O_\alpha$ is of order $1/\alpha$ so that the distribution $e^a_\alpha$ is obtained in the limit $\alpha \to \infty$. The distributional spin connection is defined in the same way. We assume that the fields $e^a_\pm$ and $\omega^{ab}_\pm$ can be continued smoothly across region $O_\alpha$.

It is useful to consider the classes of vielbeins and spin connections defined as
\[
[e^a]_\beta = e^a + \beta \Delta e^a, \\
[\omega^{ab}]_\gamma = \omega^{ab} + \gamma \Delta \omega^{ab},
\]
which by definition are smooth one-forms, for some arbitrary constants $\beta, \gamma$. Then, without any loss of generality, we can decompose $e^a_\alpha$ and $\omega^{ab}_\alpha$ as follows:
\[
e^a_\alpha = [e^a]_\beta + y_{\alpha, \beta} \Delta e^a + z^a_\alpha, \\
\omega^{ab}_\alpha = [\omega^{ab}]_\gamma + g_{\alpha, \gamma} \Delta \omega^{ab} + h^{ab}_\alpha,
\]
where $z^a_\alpha$ and $h^{ab}_\alpha$ are smooth one forms which vanish on $\partial O_\alpha$ and outside of $O_\alpha$, and the functions $y_{\alpha, \beta}$ and $g_{\alpha, \gamma}$ tend to the distribution $\theta(\Sigma) - \beta$, and $\theta(\Sigma) - \gamma$, respectively.

3.1.1 Constraints for well defined sources

Let us look at the right hand side of the Einstein equation (3.1). We wish to integrate over the infinitesimal region and take the limit
\[
\lim_{\alpha \to \infty} \int_{O_\alpha} (T_{\alpha})^{\mu}_\nu \text{Vol}_\alpha
\]
where $\text{Vol}_\alpha$ is the volume element constructed with $e_\alpha$.

To each $e_\alpha$ and $\omega_\alpha$ in the sequence, there is a corresponding stress-energy tensor $(T_\alpha)_\mu^\nu$. In the limit $\alpha \to \infty$ we require that $(T_\alpha)_\mu^\nu$ becomes a delta function distribution.

Expanding $\text{Vol}_\alpha$ according to equation (3.5) we obtain

$$
\frac{1}{(D-1)!} \int N_z dz \int (T_\alpha)_\mu^\nu \left( [e]_\beta + y_{\alpha,\beta} \Delta e + z_{\alpha} \right)^{a_2} \cdots \left( [e]_\beta + y_{\alpha,\beta} \Delta e + z_{\alpha} \right)^{a_D} \epsilon_{ba_2\cdots a_D} N^b
$$

Above we have split the vielbein in terms of a normal one-form $e^1 = N_z dz$ where $z$ is a co-ordinate normal to the hypersurface, and $e^2, \ldots, e^D$, which are tangential. Note that $N^b = \delta^b_1$.

We require the integral to be independent of the limiting process and converge to

$$
\int_{\Sigma} (T_\Sigma)_\mu^\nu \text{Vol}_\Sigma, \quad (3.7)
$$

with

$$
\text{Vol}_\Sigma := \frac{1}{(D-1)!} e^{a_2} \cdots e^{a_D} \epsilon_{ba_2\cdots a_D} N^b. \quad (3.8)
$$

Since the integral of the product $(T_\alpha)_\mu^\nu$ times $y_{\alpha,\beta}$ to some power depends on the limiting process, we need to require that such terms vanish identically. We assume that $z_\alpha$ tends to zero smoothly enough so that it gives vanishing contribution to the integral in the limit. Consequently, $([e] + y \Delta e)^{D-1}$ must be independent of the function $y$. This can be stated in three equivalent ways:

i)

$$
\frac{d}{d\gamma} \text{Vol}_{\gamma,a} N^a = 0 \quad (3.9)
$$

where $\text{Vol}_{\gamma,a}$ is $\text{Vol}_a$ constructed with $[e]^\gamma_{a}$;

ii)

$$
\Delta e^{a_1} \cdots \Delta e^{a_p} \Delta \text{Vol}_{ba_1\cdots a_p} N^b = 0, \quad \forall \ 0 \leq p \leq D - 2; \quad (3.10)
$$

iii)

$$
e^{a_2} \cdots e^{a_D} \epsilon_{ba_2\cdots a_D} N^b = e^{a_2} \cdots e^{a_{D-1}} e^{a_D} \epsilon_{ba_1\cdots a_D} N^b = \cdots = e^{a_2} \cdots e^{a_D} \epsilon_{ba_1\cdots a_D} N^b. \quad (3.11)
$$

This last expression means that the intrinsic volume element $\text{Vol}_\Sigma$ must be invariant under the operation of swapping $e_+$ with $e_-$ anywhere in the product. In particular, this means that $\text{Vol}_{\Sigma+} = \text{Vol}_{\Sigma-}$.

Notice that in the special case of 2+1 dimensions, we have two conditions: $\text{Vol}_{\Sigma+} = \text{Vol}_{\Sigma-}$ and $i^* \Delta e^{[a} \Delta e^{b]} = 0$, as can be seen from equation (3.10).

Looking at the equation of the spin current, we obtain the same conditions.
3.1.2 Junction conditions from the field equations

Now let us look at the left hand side of the torsion equation given by (3.2). Integrating this over the region $O_{\alpha}$ one obtains

$$
\int_{O_{\alpha}} dx^{\mu} T^{c}_{\alpha} \text{Vol}_{\alpha,abc} = \int_{\partial O_{\alpha}} dx^{\mu} \text{Vol}_{\alpha,ab} + (\cdots)
$$

In the limit that the region $O_{\alpha}$ shrinks to zero thickness, only the terms involving $de_{\alpha}^{a}$ contributes to this integral, as we suppose the discontinuities to be finite. Thus, the dots $(\cdots)$ represents those terms which vanish in the limit. Here $\text{Vol}_{\alpha,abc}$ stands for the volume in Eq. (3.3) constructed with $e_{\alpha}^{a}$.

Therefore, recalling Eq. (2.4), which is valid for any dimension, the junction condition for the spin current is

$$
i^{\ast} (dx^{\mu} \Delta \text{Vol}_{ab}) = (S_{\Sigma})^{\mu}_{ab} \text{Vol}_{\Sigma}, \quad (3.12)
$$

where $\Delta \text{Vol}_{ab} := \text{Vol}_{+,ab} - \text{Vol}_{-,ab}$ . In the three-dimensional case this reduces to Eq. (2.8).

Now, let us look at the Einstein equation (3.1). We must integrate

$$
I = \int_{O_{\alpha}} \Omega^{bc}_{\alpha} \text{Vol}_{\alpha,abc} f^{a} \quad (3.13)
$$

where we have introduced $f^{a} = f^{a}_{\mu} dx^{\mu}$ as some arbitrary smooth test one-form. In the limit that the region $O_{\alpha}$ shrinks to zero thickness, only terms involving $d\omega_{\alpha}$ or $de_{\alpha}$ will contribute to (3.13). Thus the integral (3.13) reads

$$
I = \int_{O_{\alpha}} d\omega^{bc}_{\alpha} \text{Vol}_{\alpha,abc} f^{a} + (\cdots)
= \int_{\partial O_{\alpha}} \omega^{bc}_{\alpha} \text{Vol}_{\alpha,abc} f^{a} + \int_{O_{\alpha}} \omega^{bc}_{\alpha} d\text{Vol}_{\alpha,abc} f^{a} + (\cdots), \quad (3.14)
$$

The boundary term $B_{1}$ in Eq. (3.14) in the limit turns out to be

$$
B_{1} = \int_{\Sigma} (\omega^{+}_{\alpha} \text{Vol}_{+,abc} - \omega^{-}_{\alpha} \text{Vol}_{-,abc}) f^{a}, \quad (3.15)
$$

which is expected to contribute to the junction conditions. The remaining volume integral $V$ must be handled with care, since it will give a non-zero contribution to the junction conditions.

Now let us expand $\omega_{\alpha}$ and $e_{\alpha}$ according to equations (3.5) and (3.6). Thus, the remaining volume integral reads

$$
V = \int_{O_{\alpha}} ([\omega^{bc}]_{\gamma} + h^{bc}_{\alpha}) d\text{Vol}_{\alpha,abc} f^{a} - (g_{\alpha,\gamma} dy_{\alpha,\beta}) \Delta \omega^{bc} \Delta \varepsilon^{d} \text{Vol}_{\alpha,abcd} f^{a} + (\cdots),
$$
and since $h^{bc}_\alpha$ vanishes at $\partial O_\alpha$, integrating by parts the first term we obtain

$$V = B_2 - (g_{\alpha,\gamma}dy_{\alpha,\beta}) \Delta \omega^{bc} \Delta e^d \text{Vol}_{\alpha,abcd} f^a + (\cdots) ,$$

where the boundary term $B_2$ in the limit is given by

$$B_2 := -\int_\Sigma [\omega^{bc}]_\gamma \Delta \text{Vol}_{abc} f^a . \quad (3.16)$$

The only remaining part of the volume integral $V$ which does not vanish in the limit is:

$$-\int_{O_\alpha} \{g_{\alpha,\gamma}dy_{\alpha,\beta}\} \Delta \omega^{[bc} \Delta e^{d]} \text{Vol}_{\alpha,abcd} f^a ,$$

where the object in the curly brackets depends on the limiting process. Therefore, in order to have a result which is independent of the limiting process, one needs to impose the following condition:

$$i^*(\Delta \omega^{[ab} \wedge \Delta e^{c]} ) = 0 . \quad (3.17)$$

Note that only the pull-back $i^*$ onto the surface $\Sigma$ appears because in the limit $\alpha \to \infty$ only the normal derivatives of $y$ blows up.

Since $\Delta \text{Vol}_{abc}$ can be expanded as\footnote{The explicit form of $X_{abcd}$ is}

$$\Delta \text{Vol}_{abc} = \Delta e^d X_{abcd} ,$$

the boundary term $B_2$ reads

$$B_2 := -\int_\Sigma (\omega^{bc}_- \Delta \text{Vol}_{abc} + \gamma \Delta \omega^{[bc} \Delta e^{d]} X_{abcd}) f^a ,$$

so that the second term vanishes by virtue of the condition $(3.17)$, and therefore the original integral $(3.13)$ reads

$$I = B_1 + B_2$$

$$= \int_\Sigma \Delta \omega^{bc}_- \text{Vol}_{+,abc} f^a ,$$

Note that because of the condition $(3.17)$ one can see that

$$\Delta \omega^{[ab} e^{c]}_+ = \Delta \omega^{[ab} e^{c]}_\gamma ,$$

and hence, the value of $I$ is the same if instead of using $\text{Vol}_{+,abc}$ computed with $e^a_+$, one uses $\text{Vol}_{\gamma,abc}$ which is computed with any representative of the class $[e^a]_{\gamma}$ defined as in Eq. $(3.4)$. Therefore the last junction condition is

$$i^* (\Delta \omega^{bc}_- \text{d}_+^\mu \text{Vol}_{\gamma,abc}) = -2(\mathcal{T}_\Sigma)^\mu_\gamma \text{Vol}_\Sigma . \quad (3.18)$$

The explicit form of $X_{abcd}$ is

$$X_{abcd} = \frac{1}{(D-3)!} e_{abcd\cdots a_D} \sum_{p=0}^{D-4} e^a_+ e^a_- \cdots e^{a_{5+p}}_+ e^{a_{5+p}}_- \cdots e^{a_D} .$$
3.1.3 Summary

Note that the junction condition can be written in purely anholonomic indices even without an induced metric. Because of the constraint (3.17), we can contract equation (3.18) with \(e^a_\mu\) which gives

\[
i^* \left( \Delta \omega^{cd} e^b_a \text{Vol}_{acd} \right) = -2(T_\Sigma)^b_a \text{Vol}_\Sigma .
\]

where the same result is obtained by contracting with any \(e^a_\mu\) in the class (3.4).

Analogously, the junction condition for the spin connection can be written with purely anholonomic indices because of condition \(i^* \Delta e^a \Delta \text{Vol}_{ab} = 0\) from equation (3.10). It is:

\[
i^* e^d \Delta \text{Vol}_{ab} = (S_\Sigma)^d_{ab}
\]

In sum we have a hypersurface \(\Sigma\) on which the tangential vielbeins and the spin connection have a bounded discontinuity and the normal is assumed to be continuous across \(\Sigma\). The set of conditions for General Relativity in the presence of spinning sources is given by (3.19) and (3.20) with the purely geometrical condition

\[
\Delta e^{a_1} \cdots \Delta e^{a_p} \Delta \text{Vol}_{ba_1 \cdots a_p} N^b = 0, \quad \forall \, 0 \leq p \leq D - 2,
\]

ensuring that the volume element on \(\Sigma\) does not depend on the induced metric, together with the constraint

\[
\Delta \omega_{[ab]} \Delta e^c = 0.
\]

Remarkably, this imposes a constraint between the direction of flow of energy and the orientation of the spin. Very roughly speaking, this is a kind of polarisation condition.

\[
\Delta \omega_{\mu} (T_\Sigma)^\mu_a = 0,
\]

\[
\Delta \omega_{\mu} (S_\Sigma)^\mu_{ab} = 0.
\]

But we note that \(\Delta \omega_{\mu}^{ab}\) is related to \((T_\Sigma)^\mu_a\) by:

\[
(T_\Sigma)^i_j = \Delta \omega^{1i} - \Delta \omega^{1k} \delta^i_j,
\]

\[
(T_\Sigma)^i_1 = \Delta \omega^{1k} \delta^i_k.
\]

where we use the frames adapted to the hypersurface so that \(e^1\) is the normal vector and \(e^i, \, i = 2, \ldots, D\) are tangential. Putting the above equations in (3.24), we obtain the following invariant equation

\[
N^a \left( (T_\Sigma)^b_c - \frac{1}{D - 2} \delta^b_c T_\Sigma \right) (S_\Sigma)^c_{ab} = 0.
\]

Remarkably, this imposes a constraint between the direction of flow of energy and the orientation of the spin. Very roughly speaking, this is a kind of polarisation condition.
In order to evaluate the junction conditions, we need to use co-ordinate patches which are smooth across \( \Sigma \).

The junction conditions for Einstein-Cartan and Lovelock-Cartan gravity and the action principle will be analysed in detail in a future publication [13].

### 3.2 Rotating vs. spinning cylinders in four dimensions

As an application, we consider a 3 + 1-dimensional example. Let us consider the case of a straight spinning cylinder, which can be obtained as the lifting of the spinning string of Section 2.6 by introducing an extra \( z \) direction, with \( \partial_z \) a killing vector. The exterior metric is

\[
\begin{align*}
\text{ds}_+^2 &= - \left( dt - \frac{J}{2} d\phi \right)^2 + \left( \frac{r}{r_0} \right)^{-2(1-B)} \left( dr^2 + r^2 d\phi^2 \right) + dz^2
\end{align*}
\]

and we choose the vielbein as

\[
\begin{align*}
e^0_+ &= dt - \frac{J}{2} d\phi ,
\quad e^1_+ = \left( \frac{r}{r_0} \right)^{B-1} dr ,
\quad e^2_+ = \left( \frac{r}{r_0} \right)^{B-1} r d\phi ,
\quad e^3_+ = dz.
\end{align*}
\]

For the interior region \( M_- \), the metric is flat, and the vielbein is chosen to be

\[
\begin{align*}
e^0_- &= dt ,
\quad e^1_- = dr ,
\quad e^2_- = r d\phi ,
\quad e^3_- = dz.
\end{align*}
\]

There is a single discontinuous component of the spin connection and one discontinuous component of the \( i^*\text{Vol}_{ab} \),

\[
\begin{align*}
i^* \Delta \text{Vol}_{12} &= -i^* \Delta \text{Vol}_{21} = -\frac{J}{2} d\phi \wedge dz ,
\quad i^* \Delta \omega^{12} = (1 - B) d\phi .
\end{align*}
\]

and the other components vanish. It is easy to verify that the conditions (3.21) and (3.22) are satisfied. Therefore this solution is compatible with the junction conditions. The computation of the stress tensor and the spin current is then straightforward. Using the junction conditions (3.19) and (3.20) and noting that \( \text{Vol}_\Sigma = -r_0 dt \wedge d\phi \wedge dz \), we get for the stress tensor:

\[
\begin{align*}
\left( T^0_\Sigma \right)_0 &= \left( T^3_\Sigma \right)_3 = - \frac{1 - B}{r_0} ,
\end{align*}
\]

This is the same result as in 2 + 1 but there is a pressure along the length of the cylinder as naturally expected. The spin current is given by:

\[
\begin{align*}
\left( S^0_\Sigma \right)_{12} &= -\left( S^0_\Sigma \right)_{21} = \frac{J}{2r_0} .
\end{align*}
\]
To summarise, the metric for the spinning cylinder in the whole spacetime can be written as

$$ds^2 = ds_{(3)}^2 + dz^2,$$  \hspace{1cm} (3.28)

where $ds_{(3)}^2$ is given by equation (2.35). The only nonvanishing component of the torsion generated by the spin current turns out to be

$$T^0 = -\frac{J}{2} \delta(r - r_0)dr \wedge d\phi.$$  

Analogously, the rotating cylinder can be solved as a straightforward extension of the rotating string in 2+1 dimensions found in Section 2.4. The stress tensor is the same as in equation (2.19) but with a pressure $T^3_3$ along the length of the cylinder satisfying $(T_3^3) = (T_3^0)$. The result is in agreement with that found in reference [14]. Since these solutions are essentially the same as in the 2+1-dimensional case, the same effect regarding the shielding of frame dragging occurs. The generalisation of these results to higher dimensional rotating and spinning domain walls with a worldsheet geometry given by $S^1 \times \mathbb{R}^{D-2}$ in $D$-dimensions is straightforward.

4. Summary and conclusions

The junction conditions for General Relativity in the presence of domain walls with intrinsic spin were derived in three and higher dimensions.

We considered the domain wall as the thin shell limit of some finite distribution of matter with aligned spin. We required the independence of the junction conditions on the limiting process i.e. that a sufficiently thin shell can be approximated by a shell of strictly zero thickness.

We have seen that the metric must change very rapidly across the thickness of the domain wall. In this sense, the spin has a more dramatic effect on the geometry than does the mass: The mass causes the first derivative of the connection to blow up whereas the spin causes the first derivative of the metric to blow up.

In general then, when the torsion is localized on the domain wall, in the zero thickness limit it is necessary to relax the continuity of the tangential components of the vielbein.

It was shown that a stress tensor and a spin current can be defined just by requiring the existence of a well defined volume element which is independent of an induced metric, so as to allow for generic torsion sources. In fact it was found that the spin current is proportional to the jump in the vielbein (see equation 3.20) and the stress-energy tensor is proportional to the jump in the spin connection (equation 3.19).

The consistency of the junction conditions implies a non-trivial constraint involving the product of the spin current and stress tensor, equation (3.27). This is
a constraint between the direction of flow of energy and the orientation of the spin. Very roughly speaking, this is a kind of polarisation condition.

As an application, we derive the circularly symmetric solutions for both the rotating string with tension and the spinning dust string in three dimensions. The rotating string with tension generates a rotating truncated cone outside and a flat space-time with inevitable frame dragging inside. In the case of a string made of spinning dust, in opposition to the previous case no frame dragging is present inside, so that in this sense, the dragging effect can be “shielded” by considering spinning instead of rotating sources. Applying the junction conditions for General Relativity in four dimensions, we found that the previously described string solutions can be lifted to the rotating and spinning cylinder with pressure along its length. The generalisation to higher dimensions is straightforward.

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