Asymptotic behavior and Hamiltonian analysis of anti-de Sitter gravity coupled to scalar fields

Marc Henneaux\textsuperscript{1,2}, Cristián Martínez\textsuperscript{1}, Ricardo Troncoso\textsuperscript{1}, Jorge Zanelli\textsuperscript{1}

\textsuperscript{1}Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile.
\textsuperscript{2}Physique théorique et mathématique, Université Libre de Bruxelles and International Solvay Institutes, ULB-Campus Plaine C.P.231, B-1050 Bruxelles, Belgium.

April 13, 2006

Abstract

We examine anti-de Sitter gravity minimally coupled to a self-interacting scalar field in $D \geq 4$ dimensions when the mass of the scalar field is in the range $m_+^2 \leq m^2 < m_+^2 + l^{-2}$. Here, $l$ is the AdS radius, and $m_+^2$ is the Breitenlohner-Freedman mass. We show that even though the scalar field generically has a slow fall-off at infinity which back reacts on the metric so as to modify its standard asymptotic behavior, one can still formulate asymptotic conditions (i) that are anti-de Sitter invariant; and (ii) that allows the construction of well-defined and finite Hamiltonian generators for all elements of the anti-de Sitter algebra. This requires imposing a functional relationship on the coefficients $a$, $b$ that control the two independent terms in the asymptotic expansion of the scalar field. The anti-de Sitter
charges are found to involve a scalar field contribution. Subtleties associated with the self-interactions of the scalar field as well as its gravitational back reaction, not discussed in previous treatments, are explicitly analyzed. In particular, it is shown that the fields develop extra logarithmic branches for specific values of the scalar field mass (in addition to the known logarithmic branch at the B-F bound).

1 Introduction

Anti-de Sitter gravity coupled to scalar fields with mass above the Breitenlohner-Freedman bound \(^1\) \cite{1, 2}

\[ m^2 = -\frac{(D - 1)^2}{4l^2}, \quad (1.1) \]

has generated considerable attention recently as it admits black hole solutions with interesting new properties \cite{3, 4, 5, 6, 7, 8, 9}. The theory supports solitons \cite{10} and provides a novel testing ground for investigating the validity of the cosmic censorship conjecture \cite{11, 12}. It is, of course, also relevant to the AdS/CFT correspondence \cite{13}.

Boundary conditions in anti-de Sitter space are notoriously known to be a subtle subject as information can leak out to or get in from spatial infinity in a finite time. Following \cite{1}, precise AdS asymptotic boundary conditions on the metric were given in \cite{14, 15, 16} in the absence of matter fields (or for localized matter). It turns out, however, that these boundary conditions do not accommodate generic scalar fields compatible with anti-de Sitter symmetry when the mass \(m\) of the scalar field is in the range

\[ m_s^2 \leq m^2 < m^2_s + \frac{1}{l^2}. \quad (1.2) \]

The main point can be already grasped by considering a free scalar field \(\phi\) in anti-de Sitter space,

\[ ds^2 = -\left(1 + \frac{r^2}{l^2}\right) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad (1.3) \]

\(^1\)Here \(l\) is the radius of \(D\)-dimensional anti-de Sitter spacetime.
that behaves asymptotically as $\phi \sim r^{-\Delta}$ with $\Delta$ real. If the exponent $\Delta$ is strictly greater than
\[
\Delta_R = \frac{D-3}{2},
\] (1.4)
the scalar field is normalizable in the sense that the spatial integral of the zeroth component $j^0$ of the Klein-Gordon current is finite. The condition $\Delta > \Delta_R$ is thus necessary for the scalar field configuration to be physically acceptable. If, furthermore, the exponent is strictly greater than
\[
\Delta_* = \frac{D-1}{2},
\] (1.5)
the Hamiltonian for the scalar field
\[
H = \frac{1}{2} \int d^{D-1}x \sqrt{g} \sqrt{-g_{00}} \left( \left( \frac{\pi}{\sqrt{g}} \right)^2 + g^{ij}\partial_i \phi \partial_j \phi \right)
\] (1.6)
is a well-defined generator \[^{[17]}\] as it stands and does not need to be supplemented by a surface integral at infinity. But if $\Delta \leq \Delta_*$ (while remaining greater than the normalizability bound $\Delta_R$), then the scalar field does contribute to surface integrals at infinity and, when coupled to gravity, modifies the standard analysis of asymptotically anti-de Sitter spaces.

Now, it follows from the Klein-Gordon equation that at large spatial distances, neglecting the self-interaction, a scalar field of mass $m$, minimally coupled to an AdS background, is asymptotically given by
\[
\phi \sim \frac{a}{r^{\Delta_-}} + \frac{b}{r^{\Delta_+}},
\] (1.7)
where
\[
\Delta_\pm = \frac{D-1}{2} \left( 1 \pm \sqrt{1 + \frac{4l^2m^2}{(D-1)^2}} \right),
\] (1.8)
are the roots of $\Delta(D-1-\Delta) + m^2 l^2 = 0$. The coefficients $a, b$, which depend on $t$ and the angles, are different from zero for generic solutions. In particular, they do not vanish for the black hole or soliton solutions considered in the literature. Comparing (1.8) with (1.4) shows that in the range (1.2), both the $a$-branch and the $b$-branch fulfill $\Delta_+ > \Delta_R$ and are physically acceptable. However, only the $b$-branch fulfills the stronger condition $\Delta_+ > \Delta_*$ and hence does not contribute to surface integrals at infinity (except when the B-F
bound is saturated, in which case $\Delta_- = \Delta_+^2$). The $a$-branch is always such that $\Delta_- \leq \Delta_+$ and does contribute to surface integrals at infinity.

In the coupled Einstein-scalar system, taking into account the back reaction of the scalar field on the geometry, one finds that the metric approaches anti-de Sitter space at infinity more slowly when $a \neq 0$ than in the absence of matter: the boundary conditions of [14] cannot accommodate the $a$-branch and must be modified. Because of this, the standard surface term giving the energy in an asymptotically anti-de Sitter space [18, 14] diverges. At the same time, there is a further contribution to the surface integrals coming from the scalar field with the possibility of cancelation of the divergences.

An additional effect arises when the mass reaches the value

$$m_*^2 + \frac{(D - 1)^2}{36 l^2},$$

(1.9)

(which is in the allowed range if $D < 7$). Indeed, the asymptotic behavior given in (1.7) is then changed, for generic potentials, by a term of order $r^{-2\Delta_-} \ln(r)$, which dominates over the $b$-branch and is thus also non-negligible at infinity, contributing with further divergences to the surface integrals. When the mass exceeds the value (1.9), the asymptotic behavior of the scalar field instead picks up a relevant term of order $r^{-2\Delta_-}$. Similarly, when the mass equals the value

$$m_*^2 + \frac{(D - 1)^2}{16 l^2},$$

(1.10)

(which is in the allowed range if $D \leq 4$), an additional term of the form $r^{-3\Delta_-} \ln(r)$ becomes relevant and also contributes to the surface integrals. When the mass exceeds (1.10), the scalar field acquires instead an extra term of order $r^{-3\Delta_-}$ which is also relevant. Finally, when the mass takes the value

$$m_*^2 + \frac{9(D - 1)^2}{100 l^2},$$

(1.11)

(which is also in the allowed range if $D \leq 4$), a term $r^{-4\Delta_-} \ln(r)$ becomes also relevant and must be taken into account, and for larger mass the scalar field possesses an extra term of the form $r^{-4\Delta_-}$ instead of a logarithmic branch.

The purpose of this paper is to show that it is possible to relax the boundary conditions on the scalar and gravitational fields in a way that

---

2This limiting case where $\Delta_- = \Delta_+$ needs a separate discussion as the asymptotic behavior of $\phi$ involves then also a logarithmic term. That discussion was given in [19] and is recalled in Section 3 below.
allows for a non-vanishing $a$-branch of the scalar field. These conditions are fully compatible with asymptotic anti-de Sitter symmetry in the sense that they allow for a consistent Hamiltonian formulation of the dynamics with well-defined, finite, anti-de Sitter generators\(^3\). With these boundary conditions, all divergences, including those arising from the subleading terms $r^{-2\Delta_-}$, $r^{-3\Delta_-}$ and $r^{-4\Delta_-}$, whenever relevant, consistently cancel. This also holds when the logarithmic branches are present.

A notable feature of these boundary conditions is that they force $a$ and $b$ to be functionally related since otherwise the surface terms giving the variations of the charges would not be integrable and hence the charges would not exist. Hence, the functions $a$ and $b$ are found not to be independent. The precise functional relationship between $a$ and $b$ is furthermore fixed by anti-de Sitter symmetry (but is arbitrary if one demands only existence of the surface integral defining the energy).

Our paper extends and completes previous work on the subject.

- The case of three spacetime dimensions was studied in [3] for a particular value of the mass of the scalar field. It was already found there that a relationship must be imposed between $a$ and $b$ and that the surface terms get scalar field contributions that cancel the divergences.

- The particular case when the B-F bound is saturated was treated in all dimensions in [19]. This case is peculiar on two accounts: first, there is a logarithmic term in the expansion for the scalar field because $\Delta_+ = \Delta_-;$ second, both branches are relevant to the surface integrals.

- An analysis which turns out to be valid when the scalar mass is in the range $m_*^2 \leq m^2 < m_*^2 + \frac{(D-1)^2}{36l^2}$ was provided in all dimensions in [5].

The plan of the paper is as follows: In the next section, the standard asymptotic conditions for matter-free anti-de Sitter gravity and the form of the charge generators are reviewed. In Section 3 the case of gravity and minimally coupled scalar fields with a logarithmic branch is summarized. A detailed analysis of the consequences of admitting both branches in four dimensions is presented in Section 4. In Section 5 the generalization for

\(^3\)To accommodate the $b$-branch alone ($a = 0$) presents no difficulty since the scalar field is then compatible with the standard fall-off of the metric. Furthermore, it does not contribute to surface integrals, except when the B-F bound is saturated, as recalled below.
higher dimensions is discussed. Section 6 contains the analysis of the special cases (1.9) when the fields develop logarithmic branches. Section 7 contains comments concerning possible extensions of these results when the AdS symmetry is broken. We conclude in section 8.

We adopt the following conventions: the action for gravity with a minimally coupled self-interacting scalar field in $D \geq 4$ dimensions is given by

$$I[g, \phi] = \int d^Dx \sqrt{-g} \left( \frac{R - 2\Lambda}{2\kappa} - \frac{1}{2}(\nabla \phi)^2 - \frac{m^2}{2}\phi^2 - U(\phi) \right), \quad (1.12)$$

where the self-interacting potential $U(\phi)$ is assumed to have an analytic expansion in $\phi$ and is at least cubic in $\phi$. When unwritten, the gravitational constant $\kappa = 8\pi G$ is chosen as 1 and the cosmological constant $\Lambda$ is $\Lambda = -l^{-2}(D-1)(D-2)/2$.

## 2 Standard asymptotic conditions for gravity and charge generators

We first recall the standard situation, from which we shall depart in the presence of a scalar field. In order to write down the asymptotic behavior of the fields, the metric is written as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is the deviation from the AdS metric,

$$ds^2 = -(1 + r^2/l^2)dt^2 + (1 + r^2/l^2)^{-1}dr^2 + r^2d\Omega^{D-2}. \quad (2.1)$$

For matter-free gravity, the asymptotic behavior of the metric is given in [14, 15, 16] and reads

$$h_{rr} = O(r^{-D-1}),$$

$$h_{rm} = O(r^{-D}),$$

$$h_{mn} = O(r^{-D+3}). \quad (2.2)$$

Here the indices have been split as $\mu = (r, m)$, where $m$ includes the time coordinate $t$ and the $D - 2$ angles.

The asymptotic symmetries correspond to the diffeomorphisms that map the asymptotic conditions into themselves, i.e., $\xi^\mu$ generates an asymptotic symmetry if

$$\mathcal{L}_\xi g_{\mu\nu} = O(h_{\mu\nu}).$$
Note that it is not necessary to require the existence of exact Killing vectors when dealing with conserved charges for a generic configuration in gravity, e.g. for the dynamics of several objects. This can be seen intuitively, since in a region far from the objects only the leading terms are relevant to compute the energy and thus only the existence of an asymptotic timelike Killing vector is required. Analogously, the linear and angular momenta can be obtained through the asymptotic symmetries.

It is easy to check that the asymptotic conditions (2.2) are invariant under $SO(D - 1, 2)$ for $D \geq 4$, and under the infinite-dimensional conformal group in two dimensions (two copies of the Virasoro algebra) for $D = 3$. The asymptotic behavior of a generic asymptotic Killing vector field $\xi^{\mu}$ is given by

$$
\begin{align*}
\xi^r &= O(r), \\
\xi^r_r &= O(1), \\
\xi^m &= O(1), \\
\xi^m_r &= O(r^{-3})
\end{align*}
$$

(2.3)

The charges that generate the asymptotic symmetries involve only the metric and its derivatives, and are given by

$$Q_0(\xi) = \int d^{D-2}S_i \left( \frac{\widetilde{G}^{ijkl}}{2\kappa} (\xi^l \dot{h}_{kl|j} - \xi^l \dot{h}_{kl}) + 2\xi^j \pi^i_j \right),
$$

(2.4)

where the supermetric is defined as $G^{ijkl} = \frac{1}{2}\dot{g}^{1/2}(\dot{g}^{ik}\dot{g}^{jl} + \dot{g}^{il}\dot{g}^{jk} - 2\dot{g}^{ij}\dot{g}^{kl})$, and the vertical bar denotes covariant differentiation with respect to the spatial AdS background. From (2.2) it follows that the momenta possess the following fall-off at infinity

$$
\pi^{rr} = O(r^{-1}), \quad \pi^{rm} = O(r^{-2}), \quad \pi^{mn} = O(r^{-5}),
$$

(2.5)

and hence, the surface integral (2.4) is finite. We have adjusted the constants of integration in the charges so that anti-de Sitter space has zero anti-de Sitter charges.

The Poisson bracket algebra of the charges yields the AdS group for $D > 3$ and two copies of the Virasoro algebra with a central charge given by

$$c = \frac{3l}{2G},
$$

(2.6)

in three dimensions [16].

The asymptotic conditions (2.2) hold not only in the absence of matter but also for localized matter fields which fall off sufficiently fast at infinity,
so as to give no contributions to the surface integrals defining the generators of the asymptotic symmetries. Note that, as pointed out above, a minimally coupled scalar field would not contribute to the charges if it goes as \( \phi \sim r^{-(\frac{D-1}{2}+\epsilon)} \) for large \( r \).

## 3 Gravity and scalar fields saturating the Breitenlohner-Freedman bound

When the scalar field mass saturates the Breitenlohner–Freedman bound, i.e., for \( m^2 = m^2_\ast \), the scalar field acquires a logarithmic fall-off. This induces a back-reaction on the metric which differs from the standard asymptotic behavior by the addition of logarithmic terms as well. This case was treated in [19] and we recall the results here for completeness. The leading terms for \( h_{\mu\nu} \) and \( \phi \) as \( r \to \infty \) are found to be

\[
\phi = \frac{1}{r^{(D-1)/2}} \left( a \ln \left( \frac{r}{r_0} \right) + b \right) + \cdots
\]

\[
h_{rr} = -\frac{(D-1)l^2a^2}{2(D-2)} \frac{\ln^2 \left( \frac{r}{r_0} \right)}{r^{(D+1)}} + \frac{l^2(a^2 - (D-1)ab)}{D-2} \ln \left( \frac{r}{r_0} \right) + O \left( \frac{1}{r^{(D+1)}} \right)
\]

\[
h_{mn} = O \left( \frac{1}{r^{(D-3)}} \right)
\]

\[
h_{mr} = O \left( \frac{1}{r^{(D-2)}} \right)
\]

(3.1)

where \( a = a(x^m) \), \( b = b(x^m) \), and \( r_0 \) is an arbitrary constant\(^5\). This relaxed asymptotic behavior still preserves the original asymptotic symmetry,

\(^4\)Here the roles of \( a \) and \( b \) are interchanged with respect to those in Ref.\([19]\).

\(^5\) Making use of the the relaxed asymptotic conditions, the momenta at infinity are found to be

\[
\pi^{rr} = O(r^{-1}), \quad \pi^{rm} = O(r^{-2}), \quad \pi^{mn} = O(r^{-5} \ln^2(r)), \quad \pi_\phi = O(r^{(D-7)/2} \ln(r)),
\]

(3.2)

(3.3)
which is $SO(D-1,2)$ for $D \geq 4$, and the conformal group in two spacetime dimensions for $D = 3$, as indicated in the previous section.

The variation of the corresponding conserved charges can be obtained following the Regge-Teitelboim approach \cite{17} and it is found to depend on $\delta a$ and $\delta b$. This differential is exact—and hence, integration of the variation of these charges as local functionals of the fields is possible—only is $a$ and $b$ are functionally related. AdS invariance fixes the relation to be of the form

$$b = -\frac{2}{(D-1)} a \ln(a/a_0),$$

where $a_0$ is a constant.

The conserved charges acquire an extra contribution coming from the scalar field which read

$$Q(\xi) = Q_0(\xi) + Q_\phi,$$

where $Q_0(\xi)$ is given by (2.4), and $Q_\phi$ is given by

$$Q_\phi = \frac{1}{2(D-1)l} \int d\Omega^{D-2} r^{(D-2)} \xi^\perp \left\{ l^2 (n^r \partial_r \phi)^2 + \frac{(D-1)^2}{4} \phi^2 \right\}$$

where $n^r = (\sqrt{g_{rr}})^{-1}$ is the only nonvanishing component of the unit normal to the boundary. Note that here, one can replace $g_{rr}$ by the background value $\bar{g}_{rr}$. These charges are finite even when the logarithmic branch is switched on, because the divergence in the gravitational piece is canceled by the divergence in the scalar piece.

In the case $a = 0$, the asymptotic behavior of the metric reduces to the standard one \cite{22}, and the original asymptotic symmetry is preserved. Nevertheless, the charges \cite{33} still give a non-trivial contribution coming from the scalar field since the exponent is just equal to $\Delta_*$. The algebra of the charges \cite{33} is identical to the standard one discussed in the previous section.

In the absence of the logarithmic branch, conserved charges have been constructed in \cite{20} following covariant methods, and a comparison of different methods to compute the mass of five-dimensional rotating black holes in supergravity has been performed in \cite{21}. We note that the AdS charges of metric-scalar field configurations with a logarithmic branch could also be computed through the method of holographic renormalization, as in \cite{22,23} in five dimensions.
4 Asymptotically AdS gravity with a minimally coupled scalar field - The case of four dimensions

We now turn to the case where the scalar field mass is strictly above the Breitenlohner–Freedman bound. We treat \( D = 4 \) first, as it is for this space-time dimension that the non-linearities due to the potential and to the interactions with the gravitational field are the most intricate. Matters simplify in higher dimensions. We expand the potential up to the relevant order,

\[
l^2 U = C_3 \phi^3 + C_4 \phi^4 + C_5 \phi^5 + O(\phi^6) .
\]

As explained in the introduction, there are four cases in which the fields have a power-law decay:

- \( m^2 < m^2 < m_\ast^2 + \frac{1}{l^2} \)
- \( m_\ast^2 + \frac{1}{4l^2} < m^2 < m_\ast^2 + \frac{9}{16l^2} \)
- \( m_\ast^2 + \frac{9}{16l^2} < m^2 < m_\ast^2 + \frac{81}{100l^2} \)
- \( m_\ast^2 + \frac{81}{100l^2} < m^2 < m_\ast^2 + \frac{1}{l^2} \)

For the limiting cases \( m^2 = m_\ast^2 + (2l)^{-2} \), \( m_\ast^2 + \frac{9}{16l^2} \), and \( m_\ast^2 + \frac{81}{100l^2} \), the fields acquire logarithmic branches, as discussed in Sect. 6.

We first give the asymptotic conditions. We explain next how they were arrived at. We start with the last range, which displays the full complexity of the problem.

4.1 \( m_\ast^2 + \frac{81}{100l^2} < m^2 < m_\ast^2 + \frac{1}{l^2} \)

In this range, the exponent lies in the range \( 1/2 < \Delta_- < 3/5 \), while \( \Delta_+ \) varies between 12/5 and 5/2. It follows that \( r^{-\Delta_-} \), \( r^{-2\Delta_-} \), \( r^{-3\Delta_-} \), and \( r^{-4\Delta_-} \) (but not \( r^{-5\Delta_-} \)) dominate asymptotically \( r^{-\Delta_+} \). We shall denote from now on \( \Delta_- \) simply by \( \Delta \).

The appropriate asymptotic conditions for the scalar field and the metric are given by

\[
\phi = ar^{-\Delta} + \beta_1 a^2 r^{-2\Delta} + \beta_2 a^3 r^{-3\Delta} + \beta_3 a^4 r^{-4\Delta} + br^{-\Delta_+} + \ldots .
\] (4.1)
\[ h_{rr} = \frac{\kappa l^2}{r^2} \left( \alpha_1 a^2 r^{-2\Delta} + \alpha_2 a^3 r^{-3\Delta} + \alpha_3 a^4 r^{-4\Delta} + \alpha_4 a^5 r^{-5\Delta} \right) \]
\[ + \frac{f_{rr}}{r^5} + \cdots \]
\[ h_{mn} = \frac{f_{mn}}{r} + \cdots \]
\[ h_{mr} = \frac{f_{mr}}{r^2} + \cdots \]
\[ \text{(4.2)} \]

where the dots (\( \cdots \)) indicate subleading terms that do not contribute to the charges. Here, \( b, f_{rr}, f_{mn} \) and \( f_{rm} \) are independent functions of time and the angles \((x^m)\). The function \( a \) is determined by \( b \) through

\[ a = b^{\Delta r} \]
\[ \text{(4.3)} \]

where \( a_0 \) is an arbitrary constant. The standard asymptotic conditions (\( a \)-branch switched off) are recovered for \( a_0 = 0 \), while \( a_0 = \infty \) corresponds to switching off the \( b \)-branch. The coefficients \( \beta_1, \beta_2, \beta_3 \) are constants given by the following expressions:

\[ \beta_1 = \bar{\beta}_1, \]
\[ \beta_2 = \bar{\beta}_2 + \kappa \frac{\Delta (3 - 2\Delta)}{4(4\Delta - 3)}, \]
\[ \beta_3 = \bar{\beta}_3 + \kappa \frac{C_3 (-153 + 327\Delta - 170\Delta^2)}{18(\Delta - 1)(4\Delta - 3)(5\Delta - 3)}, \]
\[ \text{(4.4)} \]

where the coefficients \( \bar{\beta}_1, \bar{\beta}_2 \) and \( \bar{\beta}_3 \) correspond to those found neglecting the back reaction, i.e., for a fixed AdS background,

\[ \bar{\beta}_1 = \frac{C_3}{\Delta (\Delta - 1)}, \]
\[ \bar{\beta}_2 = \frac{2C_4}{\Delta (4\Delta - 3)} + \frac{3C_3^2}{\Delta^2(\Delta - 1)(4\Delta - 3)}, \]
\[ \bar{\beta}_3 = \frac{5C_5}{3\Delta (5\Delta - 3)} + \frac{4C_3 C_4 (5\Delta - 4)}{\Delta^2(\Delta - 1)(4\Delta - 3)(5\Delta - 3)} + \frac{C_3^3 (10\Delta - 9)}{\Delta^3(5\Delta - 3)(4\Delta - 3)(\Delta - 1)^2}. \]
\[ \text{(4.5)} \]
Similarly, the constants $\alpha_1, ..., \alpha_4$ in the metric are given by

\[
\begin{align*}
\alpha_1 &= -\frac{\Delta}{2}, \\
\alpha_2 &= -\frac{4}{3} \Delta \beta_1, \\
\alpha_3 &= -\frac{\Delta}{4} \left(-\frac{\kappa \Delta}{2} + 6 \beta_2 + 4 \beta_1^2 \right), \\
\alpha_4 &= -\frac{\Delta}{5} \left(8 \beta_3 + 12 \beta_1 \beta_2 - \frac{10}{3} \kappa \Delta \beta_1 \right).
\end{align*}
\] (4.6)

Note that when the $a$-branch of the scalar field is switched on, the asymptotic fall-off of the metric acquire a strong back reaction in comparison with the standard asymptotic conditions in Eq. (2.2). In fact, the first two terms in $h^{rr}$ dominate asymptotically the 1 in the $g^{rr}$-component of the anti-de Sitter background metric. In turn, the effects of the gravitational back reaction as well as of the potential drastically modify the asymptotic behavior of the scalar field, as can be seen by comparing Eq. (4.1) with the behavior obtained for the linear approximation in a fixed AdS background in Eq. (1.7), where the asymptotically relevant powers $r^{-2\Delta}$, $r^{-3\Delta}$ and $r^{-4\Delta}$ are absent. Note also that the effects of the self interactions are relevant even if the gravitational field is switched-off.

It is interesting to point out that the back reaction of the gravitational field has a similar effect on the asymptotic form of the scalar field as the presence of cubic and quartic self-interaction terms, since even in the absence of a self-interacting potential, the term $\beta_2 a^3$ must be considered.

### 4.2

\[m_h^2 + \frac{9}{16 l^2} < m^2 < m_s^2 + \frac{81}{100 l^2}\]

In this case, the exponent $\Delta$ is in the range $3/5 < \Delta < 3/4$, while $\Delta_+$ varies between $9/4$ and $12/5$. It follows that $r^{-\Delta}$, $r^{-2\Delta}$ and $r^{-3\Delta}$ (but not $r^{-4\Delta}$) dominate asymptotically over $r^{-\Delta_+}$.

The appropriate asymptotic conditions for the scalar field and the metric are given by

\[
\phi = ar^{-\Delta} + \beta_1 a^2 r^{-2\Delta} + \beta_2 a^3 r^{-3\Delta} + b r^{-(\Delta+\gamma)} + \cdots.
\] (4.7)
\[
\begin{align*}
  h_{rr} &= \frac{\kappa l^2}{r^2} \left( \alpha_1 a^2 r^{-2\Delta} + \alpha_2 a^3 r^{-3\Delta} + \alpha_3 a^4 r^{-4\Delta} \right) \\
  &+ \frac{f_{rr}}{r^3} + \cdots \\
  h_{mn} &= \frac{f_{mn}}{r} + \cdots \\
  h_{mr} &= \frac{f_{mr}}{r^2} + \cdots 
\end{align*}
\]  
(4.8)

with \( a \) related to \( b \) as in (4.3) and \( \beta_1, \beta_2, \alpha_1, \alpha_2 \) and \( \alpha_3 \) given by (4.4), (4.5) and (4.6). Note that the coefficient \( C_5 \) of the potential does not enter the relevant expressions and hence, its precise value need not be specified.

4.3 \( m_*^2 + \frac{1}{4l^2} < m^2 < m_*^2 + \frac{9}{16l^2} \)

In this range, the exponent \( \Delta \) varies between 3/4 and 1, while \( \Delta_+ \) varies between 2 and 9/4. It follows that \( r^{-\Delta} \) and \( r^{-2\Delta} \) (but not \( r^{-3\Delta} \)) dominate asymptotically \( r^{-\Delta_+} \).

The appropriate asymptotic conditions for the scalar field and the metric are given by

\[
\begin{align*}
  \phi &= a r^{-\Delta} + \beta_1 a^2 r^{-2\Delta} + b r^{-\Delta_+} + \cdots \\
  h_{rr} &= \frac{\kappa l^2}{r^2} \left( \alpha_1 a^2 r^{-2\Delta} + \alpha_2 a^3 r^{-3\Delta} \right) + \frac{f_{rr}}{r^3} + \cdots \\
  h_{mn} &= \frac{f_{mn}}{r} + \cdots \\
  h_{mr} &= \frac{f_{mr}}{r^2} + \cdots 
\end{align*}
\]  
(4.9)

with \( a \) related to \( b \) as in (4.3) and \( \beta_1, \alpha_1 \) and \( \alpha_2 \) given by (4.4), (4.5) and (4.6). Note that now it is not necessary to specify the coefficients \( C_4 \) and \( C_5 \), since they do not enter the relevant expressions.

4.4 \( m_*^2 < m^2 < m_*^2 + \frac{1}{4l^2} \)

The range of the exponent \( \Delta \) is now 1 < \( \Delta < 3/2 \), while \( \Delta_+ \) varies between 3/2 and 2. It follows that \( r^{-\Delta} \) (but not \( r^{-2\Delta} \)) dominate asymptotically over \( r^{-\Delta_+} \).
The appropriate asymptotic conditions for the scalar field and the metric are given by

\[ \phi = ar^{-\Delta} + br^{-\Delta_+} + \ldots \]
\[ h_{rr} = \frac{k l^2}{r^2} \left( \alpha_1 a^2 r^{-2\Delta} \right) + \frac{f_{rr}}{r^5} + \ldots \]
\[ h_{mn} = \frac{f_{mn}}{r} + \ldots \]  \hspace{1cm} (4.10)
\[ h_{mr} = \frac{f_{mr}}{r^2} + \ldots \]

with \( a \) related to \( b \) as in (4.3) and \( \alpha_1 \) given by (4.6). Note that in this range it is no longer necessary to specify the potential.

We shall now justify the boundary conditions and check their consistency. First, we verify they anti-de Sitter invariance. Second, we shall derive the anti-de Sitter charges and show that all divergences cancel. To carry the analysis, we shall assume to begin with that the functions \( a \) and \( b \), as well as the constants \( \beta_i \) and \( \alpha_i \) are arbitrary. The necessity to restrict them as in the above formulas will then appear quite clearly.

### 4.5 Asymptotic AdS symmetry

It is easy to verify that, even in the presence of the extra terms in the scalar field and in the metric, the asymptotic conditions given by Eqs. (4.1) and (4.2) are preserved under the asymptotic AdS symmetry. Indeed, since the action of an asymptotic Killing vector \( \xi^\mu = (\xi^r, \xi^m) \) on the scalar field reads,

\[ \phi \rightarrow \phi + \mathcal{L}_{\xi} \phi = \phi + \xi^\mu \partial_\mu \phi , \]  \hspace{1cm} (4.11)

where \( \xi^r = \eta^r(t, x^m) r + O(r^{-1}) \), and \( \xi^m = O(1) \), the asymptotic behavior is of the same form as in Eq.(4.1) with

\[ a \rightarrow a - \eta^r \Delta a + \xi^m \partial_m a \]  \hspace{1cm} (4.12)
\[ b \rightarrow b - \eta^r \Delta_+ b + \xi^m \partial_m b , \]  \hspace{1cm} (4.13)

verifying that the asymptotic symmetries are preserved. Similarly, the Lie derivative of the metric under the anti-de Sitter Killing vectors has the requested fall-off. (Equations (4.12) and (4.13) are generically modified if \( \Delta_+ / \Delta \) is an integer, since then there appear logarithmic branches, as shown in Section 6.)
However, as discussed below (and noticed in \[3, 19, 5\]), the integration of the variation of the symmetry generators as local functionals of the fields requires \(a\) and \(b\) to be functionally related in the form \(a = a(b, x^m)\). Consistency of this assumption with the asymptotic AdS symmetry requires the compatibility of Eqs. (4.12) and (4.13), which means that

\[
\eta^r \left( \Delta a - \Delta_+ b \frac{\partial a}{\partial b} \right) + \xi^m \left( \frac{\partial a}{\partial b} \partial_m b - \partial_m a \right) = 0. \tag{4.14}
\]

Hence, since \(\eta^r\) and \(\xi^m\) are independent, the asymptotic AdS symmetry fixes the functional relationship between \(a\) and \(b\) to be of the form (4.3) given above.

### 4.6 Anti-de Sitter Charges

In order to write down the conserved charges, it is convenient to split the deviation \(h_{ij}\) from the AdS background as

\[
h_{ij} = \varphi_{ij} + \psi_{ij}, \tag{4.15}
\]

where

\[
\psi_{rr} = \frac{f_{rr}}{r^3} + O(r^{-6}), \quad \psi_{mn} = \frac{f_{mn}}{r} + O(r^{-2}), \quad \psi_{mr} = \frac{f_{mr}}{r^2} + O(r^{-3}), \tag{4.16}
\]

and \(\varphi_{ij} = h_{ij} - \psi_{ij}\). The \(\psi_{ij}\)-part contributes to finite surface integrals at infinity, while \(\varphi_{ij}\), which collects the terms that go more slowly to zero, yields divergences.

The variation of the conserved charges corresponding to the asymptotic symmetries can be found following the Regge-Teitelboim approach [17]. We shall carry out the computation in the more complex case \(m_s^2 + \frac{81}{100\alpha^2} < m^2 < m_s^2 + \frac{1}{\alpha}\) and comment later on for the other ranges of the mass.

The contributions coming from gravity and the scalar field to the conserved charges, \(Q_G(\xi)\) and \(Q_\phi(\xi)\) are respectively given by

\[
\delta Q_G(\xi) = \frac{1}{2\kappa} \int d^2 S_i \left[ G^{ijkl}(\xi^l \delta g_{ij;k} - \xi^l_k \delta g_{ij}) \right. \\
+ \left. \int d^2 S_i (2\xi_k \delta \pi^{kl} + (2\xi^k \pi^{jl} - \xi^l \pi^{jk}) \delta g_{jk}) \right] \tag{4.17}
\]

\[
\delta Q_\phi(\xi) = -\int d^2 S_i \left( \xi^l g^{1/2} g^{ij} \partial_j \phi \delta \phi + \xi^l \pi \delta \phi \right). \tag{4.18}
\]
Using the relaxed asymptotic conditions \((4.1), (4.2)\), the momenta at infinity are found to be

\[
\pi^r = O(r), \quad \pi^m = O(r^{-2}), \quad \pi^{mn} = O(r^{-(2+\Delta)}), \quad (4.19)
\]
\[
\pi_\phi = O(r^{-\Delta}), \quad (4.20)
\]

(here the indices \(m,n\) are purely angular) and hence Eq. \((4.17)\) acquires the form

\[
\delta Q_G(\xi) = \delta Q_G(\xi)\big|_{\text{finite}} + \int d^2\Omega \frac{\xi^t}{\ell^2} \left( 2\alpha_1 a\delta ar^{3-2\Delta} + 3\alpha_2 a^2 \delta ar^{3(1-\Delta)} \right) + a^3 \delta ar^{3-4\Delta} \left[ 4\alpha_3 - 3\kappa\alpha_1^2 \right] + a^4 \delta ar^{3-5\Delta} \left( 5\alpha_4 - \frac{15}{2}\kappa\alpha_1\alpha_2 \right),
\]

where \(\delta Q_G(\xi)\big|_{\text{finite}}\) stands for the terms of \(O(1)\) and is given explicitly by \(^6\)

\[
\delta Q_G(\xi)\big|_{\text{finite}} = \frac{1}{2\kappa} \int d^2\Omega G^{ijkl}(\xi^t \delta_\psi_{ijkl} - \xi^t_{ijkl} \delta\psi_{ij}) + 2 \int d^2\Omega \delta_{\pi kl}.
\]

In a similar way, Eq. \((4.18)\) takes the form

\[
\delta Q_\phi(\xi) = \frac{\Delta}{\ell^2} \int d^2\Omega \frac{\xi^t}{\ell^2} \left( a\delta ar^{3-2\Delta} + 4\beta_1 a^2 \delta ar^{3(1-\Delta)} + a^3 \delta ar^{3-4\Delta} \left[ 6\beta_2 + 4\beta_1^2 - \frac{\kappa\alpha_1}{2} \right] + a^4 \delta ar^{3-5\Delta} \left[ 8\beta_3 + 12\beta_1\beta_2 - 2\kappa\alpha_1\beta_1 - \frac{\kappa\alpha_2}{2} \right] + a\delta b + b\delta a \frac{\Delta+}{\Delta} \right). \quad (4.23)
\]

Therefore, requiring the total variation of the charges, \(\delta Q = \delta Q_G + \delta Q_\phi\), to be finite forces the coefficients \(\alpha_1, \cdots, \alpha_4\) appearing in the asymptotic form of the metric \((4.2)\) to be fixed in terms of the scalar field mass parameter and the \(\beta_1's\) appearing in the asymptotic form of the scalar field \((4.1)\) exactly as in \((4.6)\). This is the rationale behind these equations. Thus, the variation of the charges becomes

\[
\delta Q(\xi) = \delta Q_G(\xi)\big|_{\text{finite}} + \int d^2\Omega \frac{\xi^t}{\ell^2} (a\delta b\Delta + b\delta a\Delta_+). \quad (4.24)
\]

\(^6\)In the presence of the scalar field, the terms of order \(r^{-2}\) in \(h_{mr}\) give a nontrivial finite contribution to the charges. This is in contrast with the standard case, where these terms can be gauged away \((4.3)\).
Once the variation of the charges are guaranteed to be finite, one can ask the question of whether they are integrable. It is here that a functional relationship on $a$ and $b$ must be imposed. Indeed, since $\Delta_+/\Delta \neq 1$, the integrability of the variation of the matter piece of the charges given by (4.23) and by (4.24) as a local functional of the fields requires $\delta a$ and $\delta b$ not to be independent, i.e., $a$ and $b$ must be functionally related. As discussed above, the form (4.23) is then forced by asymptotic AdS symmetry.

Now, we will integrate $\delta Q_G(\xi)$ and $\delta Q_\phi(\xi)$ separately as functions of the canonical variables. For matter piece, we get

$$Q_\phi(\xi) = \frac{1}{6l} \int d^2 \Omega r^2 \xi^\perp [l^2 (n^r \partial_r \phi)^2 - m^2 l^2 \phi^2 + k_3 \phi^3 + k_4 \phi^4 + k_5 \phi^5] ,$$

with

\[
\begin{align*}
k_3 &= -2C_3 \\
k_4 &= -2C_4 - \kappa \frac{3}{8} \Delta^2 \\
k_5 &= -2C_5 - \kappa \frac{C_3 \Delta}{2(\Delta - 1)}
\end{align*}
\]

Note that the gravitational correction to $k_4$ does not depend on the potential, whereas the gravitational correction to $k_5$ is proportional to the coupling constant of $\phi^3$.

The normal is given as before by $n^r = (\sqrt{g_{rr}})^{-1}$ but now one cannot replace it by its background value $(\sqrt{g_{rr}})^{-1}$ when $m^2 \geq m_*^2 + \frac{9}{16l^2}$.

Similarly, the purely gravitational part of the charge can also be integrated to yield

$$Q_G(\xi) = Q_0(\xi) + \Delta Q(\xi) ,$$

where $Q_0$ is given by the standard formula in Eq. (2.4), and

$$\Delta Q(\xi) = -\frac{3}{4\kappa} \int_{\partial \Sigma} d^2 \Omega \frac{r^6}{l^5} \xi^\perp h_{rr}^2$$

is a nonlinear correction in the deviation from the background metric that arises because one cannot replace the supermetric $G^{ijkl}$ by its background value at infinity: the difference does contribute to the surface integral. This nonlinear term in the deviation $h_{rr}$ could not have been obtained through standard perturbative methods to construct conserved charges and is essential to make the charges finite (it cancels some divergences). A similar
phenomenon was observed in \cite{1} in 2+1 dimensions and in \cite{2} in the context of Goedel black holes.

The symmetry generators are then finite and given by

\[ Q(\xi) = Q_G(\xi) + Q_\phi(\xi), \]

with \( Q_G(\xi) \) and \( Q_\phi(\xi) \) given by Eqs. \eqref{eq:4.27} and \eqref{eq:4.25}, respectively.

An expression for \( Q(\xi) \) which is manifestly free of divergences is easily obtained by inserting the asymptotic expressions of the fields and using the relationship between \( a \) and \( b \), and reads

\[ Q(\xi) = Q_0(\xi)|_{\text{finite}} - \frac{2}{3} m^2 a_0^{-\frac{2}{N-3}} \int d^2 \Omega \xi^t a^{\frac{2}{N-3}}, \tag{4.29} \]

where

\[ Q_0(\xi)|_{\text{finite}} = \int d^2 S \left( \frac{\bar{G}^{ijkl}}{2\kappa} (\xi^+ \psi_{kl\bar{j}} - \xi^+ \xi^+ \psi_{kl}) + 2\xi^i \pi^i j \right). \tag{4.30} \]

The last term in Eq. \eqref{eq:4.29} can be written in terms of \( \phi \). Then, we obtain

\[ Q(\xi) = Q_0(\xi)|_{\text{finite}} - \frac{2}{3} m^2 a_0^{-\frac{2}{N-3}} \int d^2 \Omega \xi^t r^3 \phi^{\frac{2}{N-3}}. \tag{4.31} \]

In the case of spherical symmetry, the energy \((\xi = \partial_{\partial t})\) is given by

\[ Q\left( \frac{\partial}{\partial t} \right) = \frac{4\pi f_{rr}}{\kappa l^4} - \frac{8\pi}{3} m^2 a_0^{-\frac{2}{N-3}} a^{\frac{2}{N-3}}. \tag{4.32} \]

It should be stressed that only the sum of the gravitational contribution and the one of scalar field defines a meaningful AdS charge that is conserved. Each term separately may vary as one makes asymptotic AdS time translations. The algebra of the charges \cite{3.5} is identical to the standard one, i.e., the AdS algebra. This can be readily obtained following Ref. \cite{25}, where it is shown that the bracket of two charges provides a realization of the asymptotic symmetry algebra with a possible central extension.

For the other ranges of the mass, the analysis proceeds in the same way but is somewhat simpler as there are fewer divergent terms. The final expression for the charges is the same, but some of the terms can be dropped as they give zero. To be precise:
• $m_*^2 + \frac{81}{100} l^2 < m^2 < m_*^2 + \frac{1}{10} (1/2 < \Delta < 3/5)$: all the terms in (4.25) contribute to the surface integral and the non linear contribution (4.28) is essential;

• $m_*^2 + \frac{9}{16}l^2 < m^2 < m_*^2 + \frac{81}{100} l^2 (3/5 < \Delta < 3/4)$: the term proportional to $k_5$ can be dropped but the non linear contribution (4.28) remains essential;

• $m_*^2 + \frac{1}{4}l^2 < m^2 < m_*^2 + \frac{9}{16}l^2 (3/4 < \Delta < 1)$: both $k_4$ and $k_5$ can be dropped, as well as the non linear contribution (4.28) to the gravitational charge;

• $m_* < m^2 < m_*^2 + \frac{1}{16}l^2 (1 < \Delta < 3/2)$: the terms proportional to $k_3$, $k_4$, $k_5$ and the non linear contribution (4.28) are subleading.

4.7 Compatibility with equations of motion

When imposing boundary conditions at infinity, there is always the danger of eliminating by hand interesting solutions that would not have the prescribed fall-off. We show here that our boundary conditions are compatible with the equations of motion. This would not be the case had we not allowed terms that behave like $r^{-2\Delta}$, $r^{-3\Delta}$ or $r^{-4\Delta}$, which are forced by the non linearities of the field equations. Otherwise, the treatment would only be valid in the range when these non linear effects are subleading, i.e., $m_*^2 < m^2 < m_*^2 + \frac{1}{16}l^2 (1 < \Delta < 3/2)$.

We consider the static and spherically symmetric case for simplicity. The metric has the form

$$ds^2 = -\left[1 + \frac{r^2}{l^2} + O(r^{-1})\right]dt^2 + \frac{dr^2}{1 + \frac{r^2}{l^2} - \frac{\mu(r)}{r}} + r^2 d\Omega^2,$$

(4.33)

where $\mu(r)$ must grow slower than $r^3$ for $r \to \infty$ in order to preserve the value of the cosmological constant (but it can overcome the 1 in $g_{rr}$). The nonlinear Klein-Gordon equation then reads,

$$\left(\frac{2}{r} + \partial_r + \frac{1}{2} \partial_r \log[ - g_{tt} g_{rr}] \right) \left( g^{rr} \partial_r \phi \right) - m^2 \phi = \frac{dU}{d\phi} = l^{-2} \left( 3 C_3 \phi^2 + 4 C_4 \phi^3 + 5 C_5 \phi^4 + \cdots \right).$$

(4.34)
Expanding $\phi$ as a power series, the leading term of the scalar field is of the form

$$\phi(r) = \frac{a}{r^{\Delta}} + \cdots,$$

as in the linear case. Nonlinearities are felt at the next order and do indeed arise from the self-interacting potential if the scalar field mass is large enough. In this case, the cubic self-interacting term forces the next leading order to be of the form

$$\phi(r) = \frac{a}{r^{\Delta}} + \beta_1 \frac{a^2}{r^{2\Delta}} \cdots,$$

in order to match both sides of Eq. (4.34). Here $\beta_1$ is precisely given by (4.4) and (4.5) (this is in fact how it might be fixed).

Depending on the scalar field mass, the next relevant orders in the Klein-Gordon equation can also depend on the next terms of self-interacting potential as well as on the gravitational back-reaction through $\mu(r)$ in Eq. (4.33), which can be found as a series solving the constraint $H_\perp = 0$,

$$\mu' + \frac{r}{2}(\phi')^2 \mu = \kappa \frac{r^2}{l^2} \left[ \left( \frac{r^2}{l^2} + 1 \right) (\phi')^2 + m^2 \phi^2 + 2U(\phi) \right]. \quad (4.35)$$

Substituting the asymptotic form of the scalar field in (4.35) provides a series expression for the back reaction $\mu(r)$ which can be plugged back into the Klein-Gordon equation (4.34) to determine the next terms in the series of $\phi$. These equations can be solved consistently as a power series to yield both Eq. (4.1) and

$$\mu = \frac{\kappa r^3}{l^2} \left( \alpha_1 a^2 r^{-2\Delta} + \alpha_2 a^3 r^{-3\Delta} + (\alpha_3 - \alpha_1) a^4 r^{-4\Delta} + (\alpha_4 - 2\alpha_1 \alpha_2) a^5 r^{-5\Delta} \right) + \mu_0, \quad (4.36)$$

where the $\beta_i$ and the $\alpha_j$ are the constants given by (4.4), (4.5) and (4.6). This behavior corresponds precisely to the asymptotic conditions (4.1) and (4.2) – which were derived in fact in this manner. It is notable that the values of the constants $\beta_i$ and $\alpha_j$ that follow upon integration of the equations of motion also cancel all divergences in the surface integrals. In other words, the same results are found solving the hamiltonian constraints and imposing finiteness of the charges.

It is worth pointing out that for the asymptotic form of the scalar field (4.1), the gravitational back reaction merely amounts to a redefinition of the coefficients $\beta_2$ and $\beta_3$, and hence, its effect mimics the nonlinearity of the
self-interaction (which are present even in a pure anti-de Sitter background). Consequently, even in the absence of a self interacting potential, the $\beta_2$ term is switched on due to the gravitational back reaction for a large enough mass. In fact, both effects can even cancel each other out. For example, the effects of the self interaction can be completely screened by choosing a particular family of potentials. The standard asymptotic behavior of the free scalar field in Eq. (1.7) is then recovered by imposing $\beta_1 = \beta_2 = \beta_3 = 0$. This implies the following restrictions on the self interaction

$$U(\phi) = \begin{cases} 
O(\phi^3) & : 1 < \Delta < 3/2 \\
O(\phi^4) & : 3/4 < \Delta \leq 1 \\
-\kappa \Delta^2 (3 - 2\Delta) \phi^4 + O(\phi^5) & : 3/5 < \Delta \leq 3/4 \\
-\kappa \Delta^2 \phi^4 + O(\phi^6) & : 1/2 < \Delta \leq 3/5 
\end{cases}, \quad (4.37)$$

where the equalities hold even in the presence of logarithmic branches (see section 6).

## 5 Higher dimensions

The analysis becomes simpler in higher dimensions as most of the difficulties encountered in four dimensions go away. We only give the results as the derivation proceeds along identical lines. There are only three cases to be considered:

- $D = 5, 6, m_s^2 + \frac{(D-1)^2}{36 l^2} < m^2 < m_s^2 + \frac{1}{l^2}$,
- $D = 5, 6, m_s^2 < m^2 < m_s^2 + \frac{(D-1)^2}{36 l^2}$,
- $D \geq 7, m_s^2 < m^2 < m_s^2 + \frac{1}{l^2}$,

the last two cases being treated similarly. In the first case, $r^{-\Delta}$ and $r^{-2\Delta}$ (but not $r^{-3\Delta}$) dominate asymptotically over $r^{-\Delta_+}$. In the last two cases, only $r^{-\Delta}$ (but not $r^{-2\Delta}$) dominates asymptotically over $r^{-\Delta_+}$. 

21
5.1 $D = 5, 6, m_s^2 + \frac{(D-1)^2}{36l^2} < m_s^2 < m_s^2 + \frac{1}{l^2}$

In this case, the asymptotic conditions for the scalar field and the metric are given by

$$\phi = ar^{-\Delta} + \beta_1 a^2 r^{-\Delta} + br^{-\Delta} + \cdots ,$$  \hspace{1cm} (5.1)

and

$$h_{rr} = \frac{\kappa l^2}{r^2} (\alpha_1 a^2 r^{-2\Delta} + \alpha_2 a^3 r^{-3\Delta}) + \frac{f_{rr}}{r^{(D+1)}}$$

$$h_{mn} = \frac{f_{mn}}{r^{(D-3)}} + \cdots$$  \hspace{1cm} (5.2)

$$h_{mr} = \frac{f_{mr}}{r^{(D-2)}} + \cdots$$

where the indices $m, n$ are purely angular and where, as in the four dimensional case $b, f_{rr}, f_{mn}$ and $f_{rm}$ are independent functions of time and the angles $(x^m)$. The coefficients $\beta_1$, in the scalar field, and $\alpha_1$ and $\alpha_2$ in the metric, are constants given by

$$\beta_1 = \frac{C_3}{\Delta(\Delta - (D - 1)/3)},$$  \hspace{1cm} (5.3)

and

$$\alpha_1 = -\frac{\Delta}{D - 2}, \quad \alpha_2 = -\frac{8\Delta}{3(D - 2)} \beta_1.$$  \hspace{1cm} (5.4)

The conjugate momenta fulfill

$$\pi^{rr} = O(r), \quad \pi^{rn} = O(r^{-2}), \quad \pi^{mn} = O(r^{D - 2\Delta - 6}),$$

$$\pi^\phi = O(r^{D - 4 - \Delta}).$$  \hspace{1cm} (5.5)

Finally, $a$ is fixed in terms of $b$ as

$$a = a_0 b^{\frac{\Delta}{D-2}},$$  \hspace{1cm} (5.7)

where $a_0$ is an arbitrary dimensionless constant. Just as in the four-dimensional case, the existence of a relationship between $a$ and $b$ is necessary in order to get integrable charges. That relationship is then fixed to be of the form (5.7) by anti-de Sitter invariance.

Again, the asymptotic behavior of the metric acquires a strong back reaction in comparison with the standard fall-off Eq. 2.2. Unlike the four-dimensional case, the gravitational back-reaction has, however, no influence in the asymptotic form of the scalar field.
5.2 $D = 5, 6$, $m_*^2 < m^2 < m_*^2 + \frac{(D-1)^2}{36 l^2}$ and $D \geq 7$

The boundary conditions are then simpler and read

$$\phi = ar^{-\Delta} + br^{-\Delta} + \cdots ,$$

(5.8)

and

$$h_{rr} = \frac{\kappa l^2}{r^2} (\alpha_1 a^2 r^{-2\Delta}) + \frac{f_{rr}}{r^{(D+1)}} ,$$

$$h_{mn} = \frac{f_{mn}}{r^{(D-3)}} + \cdots ,$$

$$h_{mr} = \frac{f_{mr}}{r^{(D-2)}} + \cdots ,$$

(5.9)

with

$$\alpha_1 = -\frac{\Delta}{D-2} , \quad a = a_0 b^{\frac{\Delta}{D-2}} .$$

(5.10)

In this case, the self-interacting potential has no effect on the asymptotic form of the scalar field, which coincides with the one obtained for the linear approximation in a fixed AdS background as in Eq. (1.7).

5.3 Symmetries and Generators

The asymptotic conditions given above are preserved under the asymptotic AdS symmetry, and the functional relation between $a$ and $b$ required for the integrability of the symmetry generators, acquires the same form as in the four dimensional case given by Eq. (4.3).

Following Ref. [17], one can compute the charges. With our boundary conditions, the divergences cancel and the charges are found to be

$$Q(\xi) = Q_0(\xi) + Q_\phi(\xi) ,$$

(5.11)

where $Q_0$ is given by the standard formula in Eq. (2.4), and

$$Q_\phi = \frac{1}{2(D-1)l} \int d^2 \Omega r^{(D-2)} \xi^\perp \left[ l^2 (n^r \partial_r \phi)^2 - m^2 l^2 \phi^2 + k_3 \phi^3 \right] ,$$

(5.12)

with

$$k_3 = -2C_3 .$$

(5.13)
The term proportional to \( k_3 \) is needed only for \( D = 5, 6 \) and \( m_*^2 + \frac{(D-1)^2}{36l^2} \leq m^2 < m_+^2 + 1 \). (As seen in the previous section, this term is also necessary in the range \( m_*^2 + \frac{1}{4l^2} < m^2 < m_*^2 + \frac{9}{16l^2} \), and for larger values of \( m^2 \), the terms proportional to \( k_4 \) and \( k_5 \) are also necessary.) Note that Eq. (5.12) can then be extrapolated to the case when the BF bound is saturated, and it can also be seen to hold for \( \Delta_+ = 2\Delta \). The algebra of the charges (5.11) coincides with the standard one, i.e., the AdS algebra \( SO(D-1, 2) \).

It is also worth noting that in dimensions higher than four, the gravitational back reaction cannot mimic the nonlinearity of the self interaction, so that the “screening” effect observed in section 4.7 above is absent.

6 Logarithmic terms of nonlinear origin

In general, for any dimension, logarithmic branches are present when \( \frac{\Delta_+}{\Delta} = n \) is a positive integer. In this case the scalar field acquires a logarithmic branch of the form

\[
\phi = ar^{-\Delta} + \cdots + ha^n r^{-\Delta_+} \log(r) + br^{-\Delta_+} + \cdots ,
\]

where \( h \) is a fixed constant explicitly determined below.

The critical values of the spacetime dimensions and mass for which this phenomenon occurs are

- \( D = 4, 5, 6, m^2 = m_*^2 + \frac{(D-1)^2}{36l^2}, (n = 2) \),
- \( D = 4, m^2 = m_*^2 + \frac{9}{16l^2}, (n = 3) \),
- \( D = 4, m^2 = m_*^2 + \frac{81}{100l^2}, (n = 4) \).

As in the generic case, integrability of the charges requires \( a \) and \( b \) to be functionally related. The asymptotic AdS symmetry implies a functional relation given by

\[
b = a^n \left[ b_0 - \frac{h}{\Delta} \log a \right].
\]

which is different from the generic form (5.7) due to the presence of the logarithmic branch. Note that this relation also holds for \( n = 1 \), when the scalar field saturates the BF bound.
6.1 \( D = 4, 5, 6, \) \( m = m^2_* + (D - 1)^2/(36l^2) \)

In this case, \( \Delta_+ = 2\Delta = \frac{2(D-1)}{3} \), and the asymptotic behavior of the scalar field and the metric is given by

\[
\begin{align*}
\phi &= \frac{a}{r^{2\Delta}} - \frac{9C_3}{D-1} a^2 \frac{\log r}{r^{2\Delta}} + \frac{b}{r^{2\Delta}} + \ldots \\
h_{rr} &= \frac{\kappa l^2}{r^2} \left( \alpha_1 a^2 r^{-2\Delta} + \frac{8C_3}{D-2} a^3 r^{-3\Delta} \log r \right) + \frac{f_{rr}}{r^{D+1}} \\
h_{mn} &= \frac{f_{mn}}{r^{D-3}} + \ldots \\
h_{mr} &= \frac{f_{mr}}{r^{D-2}} + \ldots
\end{align*}
\]

(6.2)

where \( \alpha_1 = -\Delta/2 \), as in the generic case. Proceeding as in the generic case for these asymptotic conditions, it is found that the divergences cancel out, and the charges are still expressed in the form (5.11) and (5.12). The relationship between \( a \) and \( b \) required by asymptotic AdS symmetry is now given by

\[
b = a^2 \left[ b_0 + \frac{27C_3}{(D-1)^2} \log a \right].
\]

(6.3)

Note that as the mass approaches the critical value \( m^2_* + (D - 1)^2/(36l^2) \) from above, the coefficient \( \beta_1 \) in Eq. (5.3) develops a pole at \( \Delta = (D - 1)/3 \). Thus in the limit, the following replacement takes place:

\[
\frac{\beta_1}{r^{2\Delta}} \rightarrow (D - 1 - 3\Delta)\beta_1 \frac{\log r}{r^{2\Delta}}.
\]

(6.4)

Also to be pointed out is the fact that the logarithmic terms are absent if \( C_3 = 0 \).
6.2 \( D = 4, \ m^2 = m^2_* + \frac{9}{16l^2} \)

In this case, \( \Delta_+ = 3\Delta = 9/4 \). The asymptotic behavior of the scalar field and the metric is

\[
\phi = \frac{a}{r^{3\Delta}} + \frac{\beta_1 a^2}{r^{2\Delta}} + (3 - 4\Delta)\beta_2 a^3 \frac{\log r}{r^{3\Delta}} + \frac{b}{r^{3\Delta}} + \ldots
\]

\[
h_{rr} = \frac{\kappa l^2}{r^2} \left( \alpha_1 a^2 r^{-2\Delta} + \alpha_2 a^3 r^{-3\Delta} + (4\Delta - 3)\beta_2 a^4 \frac{9\log r}{8r^{4\Delta}} \right) + \frac{f_{rr}}{r^5}
\]

\[
h_{mn} = \frac{f_{mn}}{r} + \ldots
\]

\[
h_{mr} = \frac{f_{mr}}{r^2} + \ldots
\]

(6.5)

where \( \beta_1 \) and \( \beta_2 \) are given by (4.4), and \( \alpha_1 \) and \( \alpha_2 \), are those of (4.6), as in the generic case. The divergences of the charges cancel, and they are still expressed as \( Q = Q_G + Q_\phi \), where the term proportional to \( k_5 \) in (4.25) is subleading, but the nonlinear contribution to the gravitational part of the charge \( \Delta Q \) given by (4.28) is relevant.

The relationship between \( a \) and \( b \), required by asymptotic AdS symmetry, is now given by

\[
b = a^3 \left[ b_0 + \frac{(4\Delta - 3)\beta_2}{\Delta} \log a \right], \quad (6.6)
\]

where the factor \( (4\Delta - 3) \) cancels the pole of \( \beta_2 \) at \( \Delta = 3/4 \).

6.3 \( D = 4, \ m^2 = m^2_* + \frac{81}{100l^2} \)

Now \( \Delta_+ = 4\Delta = 12/5 \), and the asymptotic behavior of the fields is given by

\[
\phi = \frac{a}{r^{3\Delta}} + \frac{\beta_1 a^2}{r^{2\Delta}} + \frac{\beta_2 a^3}{r^{3\Delta}} - (5\Delta - 3)\beta_3 a^4 \frac{\log r}{r^{4\Delta}} + \frac{b}{r^{4\Delta}} + \ldots
\]

\[
h_{rr} = \frac{\kappa l^2}{r^2} \left[ \alpha_1 a^2 r^{-2\Delta} + \alpha_2 a^3 r^{-3\Delta} + \alpha_3 a^4 r^{-4\Delta} + \frac{24}{25\pi^2} (5\Delta - 3)\beta_3 a^5 \log r \right] + \frac{f_{rr}}{r^5}.
\]

\[
h_{mn} = \frac{f_{mn}}{r} + \ldots
\]

\[
h_{mr} = \frac{f_{mr}}{r^2} + \ldots
\]

(6.7)

where \( \beta_1, \beta_2 \) and \( \beta_3 \) are given by (4.4), and \( \alpha_1, \alpha_2 \) and \( \alpha_3 \), are those of (4.6), as in the generic case. The divergences of the charges again cancel, and they
are given by \( Q = Q_G + Q_\phi \), where the all the terms in (4.25), as well as the nonlinear contribution to the gravitational part of the charge \( \Delta Q \) given by (4.28) are relevant.

The relationship between \( a \) and \( b \), required by asymptotic AdS symmetry, is now given by

\[
b = a^4 \left[ b_0 + \frac{(5\Delta - 3)\beta_3}{\Delta} \log a \right],
\]

where the factor \((5\Delta - 3)\) cancels the pole of \( \beta_3 \) at \( \Delta = 3/5 \).

7 Remarks on non-AdS invariant boundary conditions

7.1 Breaking AdS invariance through the boundary conditions on the scalar field

The existence (integrability) of the charges (in particular, the energy) forces \( a \) and \( b \) to be functionally related in a way that is essentially unique if one insists on AdS invariance. However, one may consider different functional relationships. In that case, although the metric still has the same asymptotic AdS invariance, the scalar field breaks the symmetry to \( R \times SO(D - 1) \) because the asymptotic form of \( \phi \) is not maintained under the action of \( \xi^r \). This breaking of asymptotic AdS invariance has been considered in [10, 26], following ideas from the AdS/CFT correspondence [28]. One may still develop the formalism provided that, as above, one takes proper account of the nonlinearities in the equations when these are relevant.

It is worth pointing out that requiring the matter piece of the charges \( Q_\phi \) to be integrated as an analytic local functional of \( \phi \) and its derivatives,

\[
Q_\phi = \int_{S^{D-2}} \sqrt{h} d^{D-2} x \xi^{\perp} F(\phi, n^i \partial_i \phi, n^i n^j \partial_i \partial_j \phi,...),
\]

where \( F \) is a polynomial in its entries and \( n^i \) is a unit normal to the sphere at infinity, is strong enough to fix the relation between \( a \) and \( b \) within a one-parameter family. In the generic case \((\Delta_+ / \Delta \neq n)\), this relation is of the form

\[
a = a_0 \delta^{\frac{b_1 \Delta}{(n-1) - s_1^2}},
\]

where
where \( k_1 \) is some constant (equal to 1 in the asymptotically anti-de Sitter invariant case). The case \( \Delta_+ / \Delta = n \) will be discussed at the end of this subsection.

This can be seen as follows. Since the field satisfies a second order equation, it is expected that \( F \) should depend only on \( \phi \) and \( \partial_r \phi \) at infinity. In fact, one can see that, using the asymptotic conditions \( 5.1 \), the higher derivatives terms can always be expressed as linear combinations of \( \phi \) and \( n^r \partial_r \phi \). The leading terms are then \( \phi^2 \), \( (n^r \partial_r \phi)^2 \) and \( \phi n^r \partial_r \phi \), and by virtue of the asymptotic conditions, without loss of generality one can drop the term\(^7 \phi n^r \partial_r \phi \). In consequence, for \( D \geq 5 \) dimensions, requiring the variation of this local functional to match what one gets from the Hamiltonian constraint fixes the form of the charge to be

\[
Q^{k_1}_\phi = \frac{1}{2(D-1)t} \int d^{D-2} \Omega_r (D-2) \xi^\perp \left[ k_1 l^2 (n^r \partial_r \phi)^2 + k_2 \phi^2 + k_3 \phi^3 \right], \tag{7.3}
\]

with

\[
k_2 = \Delta(D - 1 - \Delta k_1), \tag{7.4}
\]

\[
k_3 = 2 \beta_1 \Delta \left( \frac{D - 1}{3} - k_1 \Delta \right), \tag{7.5}
\]

and requires \( a \) and \( b \) to be related as in Eq. \( 7.2 \). It is easy to see that for \( D \geq 7 \), the cubic term is subleading. Note that for \( k_1 = 1 \), the expressions for the asymptotically AdS invariant case are recovered c.f., \( 5.12 \), \( 5.13 \).

In four dimensions, higher order terms are needed, so that the matter piece of the charge reads

\[
Q^{k_1}_\phi = \frac{1}{6l} \int d^2 \Omega_r \xi^\perp \left[ k_1 l^2 (n^r \partial_r \phi)^2 + k_2 \phi^2 + k_3 \phi^3 + k_4 \phi^4 + k_5 \phi^5 \right], \tag{7.6}
\]

\(^7\)There is always a precise linear combination of these three terms that give no contribution to the charge \( Q_\phi \), except for \( m^2 = m_*^2 \).
where

\begin{align*}
    k_2 &= (3 - k_1 \Delta) \Delta \\
    k_3 &= 2\beta_1 \Delta (1 - k_1 \Delta) \quad (7.7) \\
    k_4 &= \Delta \left[ 3\beta_1^2 (k_1 \Delta - 1) - (\beta_2 + \kappa \frac{\Delta}{8}(4k_1 \Delta - 3)) \right] \quad (7.8) \\
    k_5 &= \Delta \left[ \beta_1^2 \frac{6}{5} (3 - 5k_1 \Delta) - 6\beta_1^2 (k_1 \Delta - 1) \\
    &\quad + \kappa \frac{\beta_1 \Delta}{6} (3 - 8k_1 \Delta) + \frac{12\beta_1 \beta_2}{5} (5k_1 \Delta - 4) \right]. \quad (7.9)
\end{align*}

For \( k_1 = 1 \) the expressions for the asymptotically AdS invariant case, (4.25) and (4.26), are recovered.

For the cases in which the fields develop logarithmic branches, \( \Delta_+ / \Delta = n \), the results can be summarized as follows.

- \( D = 4, 5, 6, m^2 = m_+^2 + \frac{(D-1)^2}{36l^2} \). The charge is given by (7.6), where \( k_1 = 1 \) and \( k_2 = -m^2 l^2 \), as for asymptotically AdS case, but now instead \( k_3 \) is arbitrary, labelling the relationship between \( a \) and \( b \),

\begin{equation}
    b = a^2 \left[ b_0 - \frac{27k_3}{2(D-1)^2 \log a} \right]. \quad (7.11)
\end{equation}

For \( k_3 = -2C_3 \) the expression (6.3) is recovered, so that the asymptotic AdS symmetry is restored, and all \( k \)'s are as in the generic case for AdS (5.12).

- \( D = 4, m^2 = m_+^2 + \frac{9}{14l^2} \). The charge is given by (7.6), with \( k_1 = 1 \), \( k_2 = -m^2 l^2 \), and \( k_3 = -2C_3 \) (as for AdS), but now \( k_4 \) is arbitrary, and the \( k_5 \) term is irrelevant. In this case, the relationship between \( a \) and \( b \) is given by

\begin{equation}
    b = a^3 \left[ b_0 - \frac{16k_4 + 9\beta_1^2}{9} \log a \right]. \quad (7.12)
\end{equation}

The asymptotic AdS symmetry is recovered for \( k_4 = -2C_4 - \kappa \frac{27}{128} \), in agreement with (6.6) and (4.26).

- \( D = 4, m^2 = m_+^2 + \frac{81}{100l^2} \). The charge is given by (7.6), where all the terms are relevant. In the general case (when AdS symmetry is broken by the scalar field), \( k_1, k_2, k_3, \) and \( k_4 \) are fixed as in (4.26) (as for AdS),
but now $k_5$ is arbitrary. In this case, the relationship between $a$ and $b$ is given by

$$b = a^4 \left[ b_0 + \left( -\frac{125 k_5}{54} - \kappa \frac{\beta_1}{4} + \frac{10 \beta_1^2}{3} - \frac{10 \beta_1 \beta_2}{3} \right) \log a \right]. \quad (7.13)$$

The asymptotic AdS symmetry is recovered for $k_5 = -2C_5 + \kappa \frac{3C_3}{4}$, in agreement with (4.26), and (6.8).

- $D \geq 3$, $m^2 = m^2_*$. When the BF bound is saturated, $Q^k_\phi$ has the form

$$Q^k_\phi = \frac{1}{2(D-1)l} \int d^{D-2} \Omega_r (\xi^t) \left[ k_0 l^2 \phi n^r \partial_r \phi + k_1 l^2 (n^r \partial_r \phi)^2 + k_2 \phi^2 \right], \quad (7.14)$$

with $k_0 = (k_1 - 1)(D - 1)$ and $k_2 = -m^2_* l^2 k_1$. The relation between $a$ and $b$ is

$$b = -k_1 \frac{2}{D-1} a \log a/a_0.$$ 

This means that for $k_1 = 1$ our previous results (3.4) and (3.6), which are compatible with AdS symmetry, are recovered.

For $k_1 \neq 1$, the total charges $Q = Q_G + Q^k_\phi$ are finite and generate the asymptotic symmetry group $\mathbb{R} \times SO(D - 1)$.

### 7.2 Locally asymptotically anti-de Sitter space

The surface integrals expressing the conserved charges presented here can be readily extended to configurations where the asymptotic AdS symmetry is broken through non trivial topology. For instance, the exact four-dimensional black hole solution of Ref. [6] which is dressed with a minimally coupled scalar field with a slow fall-off, has broken rotational invariance in the asymptotic region since the boundary of the spacelike surface does not correspond to a sphere in that region. Thus, in order to compute the conserved charges for the remaining asymptotic symmetries for this kind of objects, it is enough to replace the volume element $d\Omega^{D-2}$ of the sphere $S^{D-2}$,

---

8It was shown in [27] that the perturbative stability of locally AdS spacetimes with this kind of topology holds provided the mass satisfies the same BF bound.
by the volume element \( d\Sigma^{D-2} \) of the boundary of the spacelike surface. It is simple to check that the mass for the black hole in Ref. [6] can be reproduced in this way.

8 Conclusions

In this paper, we have investigated the asymptotics of anti-de Sitter gravity minimally coupled to a scalar field with a slow fall-off \((a \neq 0)\). The scalar field gives rise to a back reaction that modifies the asymptotic form of the geometry, which is consistent with asymptotically AdS symmetry for suitable boundary conditions. In turn, additional contributions to the charges, which are not present in the gravitational part and which depend explicitly on the matter fields at infinity, arise and insure finiteness of the charges.

The discussion has been carried out here for a minimally coupled self-interacting scalar field in dimensions \( D \geq 4 \) with any mass between the Breitenlohner-Freedman bound and the Breitenlohner-Freedman bound plus \( 1/l^2 \).

We have shown that one can consistently include the slow fall off of the scalar in the Hamiltonian formulation provided a functional relationship is imposed on \( a \) and \( b \) ensuring integrability of the charges. We considered only one scalar field. In the presence of many scalar fields, one must make \( a_i \delta b_i \) integrable, forcing \( a_i = \delta L/\delta b_i \) for some \( L \). This is in line with the AdS-CFT correspondence where such functional relationships (defining “Lagrangian submanifolds”) have been considered in the context of multi-trace deformations [28, 29].

We have also observed that when a non-trivial potential is considered, the asymptotic form of the scalar field obtained through the linear approximation is no longer reliable and acquires extra contributions when the mass of the scalar field is within the range \( m^2_* + (D - 1)^2 / (36 l^2) < m^2 < m^2_* + 1/l^2 \) for \( D < 7 \) dimensions. The effects of the self-interaction are absent only for a particular class of potentials. The four-dimensional case is particularly interesting since gravitational back reaction is so strong that it can even mimic the nonlinearity of the self interaction. Both effects are present but can cancel each other out for fine-tuned potentials within a particular class. Furthermore, the purely gravitational contribution to the charges acquires an extra term which is nonlinear in the deviation from the background. These effects were first observed in the three-dimensional case for an specific value...
of the scalar field mass $m$.

A somewhat unexpected outcome of our analysis is that at the critical values of the mass where new terms in the potentials become relevant, the self-interactions of the scalar field as well as its gravitational back reaction, (not discussed in previous treatments), force the fields to develop extra logarithmic branches.

One of the advantages of allowing the $a$-branch of the scalar field – and hence, considering the relaxed asymptotic behavior discussed here – is that the space of physically admissible solutions is then enlarged and includes new interesting objects. In particular, asymptotically AdS black hole solutions having scalar hair with slow fall-off have been found numerically in Refs. [4, 5, 7, 30]. Numerical black hole solutions exhibiting these features have also been found for non-abelian gauge fields with a dilatonic coupling in Ref. [31].

The possibility to consider the two scalar branches simultaneously is a feature peculiar to anti-de Sitter space, which is not available when the cosmological constant vanishes. In that case, one can only include the decaying mode $\sim \exp(-mr)/r$ at infinity.

Another interesting limit is obtained when one switches off the gravitational coupling constant. In this case, the matter piece of the charge reduces to

$$Q^{AdS}_\phi = \frac{l}{D-1} \int_{\partial\Sigma} d^{D-2} \Omega r^{(D-2)} \{ [(n^r \partial_r \phi)^2]/2 - m^2 \phi^2/2 - U(\phi) \}.$$ 

This boundary term is sufficient to regularize the generators on a fixed AdS background as

$$G^{AdS}(\xi) = \int_{\Sigma} \xi^\mu T_{\mu\nu} dS^\nu + Q^{AdS}_\phi.$$ 

In this way, there is no need to invoke an "improvement" coming from the non minimal coupling between gravity and the scalar field.

The effect of the relaxed asymptotic behavior discussed here opens new questions that deserve further study, as for instance, the positivity of the energy in this wider context, its compatibility with supersymmetry, as well as its holographic significance\footnote{While this paper was finished, we have been informed by Max Bañados, Adam Schwimmer and Stefan Theissen about a holographic interpretation of the logarithmic relations between the boundary conditions found in [19].}. Incidentally, the critical values of the mass
where a new term in the potential, say $\phi^k$, becomes relevant, corresponds precisely to the case where the $k$-th power of the dual field (which has dimension $\Delta$) becomes relevant in the sense of the dual conformal field theory. A related question is to derive the above charges through holographic methods \cite{32}.

**Acknowledgments** We thank Aaron Amsel, Riccardo Argurio, Max Bañados, Eloy Ayón-Beato, Claudio Bunster, Gastón Giribet, Stanislav Kuperstein, Don Marolf, Herman Nicolai, Rubén Portugués, Adam Schwimmer and Stefan Theisen, for useful discussions and enlightening comments. We are also grateful to Geoffrey Compère for spotting an important typo in the first version. This work was funded by an institutional grant to CECS of the Millennium Science Initiative, Chile, and Fundación Andes, and also benefits from the generous support to CECS by Empresas CMPC. The work MH is partially supported by IISN - Belgium (convention 4.4505.86), by the "Interuniversity Attraction Poles Programme – Belgian Science Policy ” and by the European Commission programme MRTN-CT-2004-005104, in which he is associated to V.U. Brussel. This research was also supported in part by FONDECYT grants N° 1020629, 1040921, 1051056, 1051064, 1061291, 7050232 as well as by the National Science Foundation under Grant No. PHY99-07949.

**References**


