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CURRENT AND DENSITY ALGEBRA AND GAUGE INVARIANCE

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1. INTRODUCTION

In current algebra ¹⁾, relations of the following kind are important ²⁾:

$$\begin{aligned} \frac{\partial}{\partial x_\mu} \langle F | T(j_\mu^l(x), j_\nu^m(y), \dots) | I \rangle \\ = \langle F | T(\partial_\mu j_\mu^l(x), j_\nu^m(y), \dots) | I \rangle \quad (1) \\ + \delta(x_0 - y_0) \langle F | T([j_0^l(x), j_\nu^m(y)], \dots) | I \rangle \\ + \dots \end{aligned}$$

where T denotes time ordering and j_μ^l and j_ν^m are some currents. Values for the equal time commutators are suggested by the quark model, in which one can define 18 currents

$$j_\mu^l = i \bar{\psi} \gamma_\mu \lambda^l \psi \quad (2)$$

The λ^l are 18 matrices having Dirac and unitary spin indices: $1, \lambda^1, \dots, \lambda^8, \gamma_5, \gamma_5 \lambda^1, \dots, \gamma_5 \lambda^8$, where $\lambda^1 \dots \lambda^8$ are the 3×3 matrices of Gell-Mann. Naive manipulation of canonical anticommutation rules leads to

$$i \delta(x_0 - y_0) [j_0^l(x), j_\nu^m(y)] = C^{lmn} j_\nu^n(y) \delta(x - y) \quad (3)$$

where

$$[\lambda^l, i \lambda^m] = C^{lmn} \lambda^n \quad (4)$$

Then (1) becomes

$$\begin{aligned} i \frac{\partial}{\partial x_\mu} \langle F | T(j_\mu^l(x), j_\nu^m(y), \dots) | I \rangle \\ = \langle F | T(\partial_\mu j_\mu^l(x), j_\nu^m(y), \dots) | I \rangle \quad (5) \\ + \delta(x - y) C^{lmn} \langle F | T(j_\nu^n(y), \dots) | I \rangle \\ + \dots \end{aligned}$$

The more specific assumption³⁾ can be added that the divergences of the currents come entirely from the mass term in the quark Lagrangian. Then they are combinations of the 18 scalar (and pseudoscalar) densities

$$u^\ell = \bar{\Psi} \lambda^\ell \Psi$$

With these also in the T products one requires in equations like (1) and (5) the commutators

$$i \delta(x_0 - y_0) [j_0^\ell(x), u^m(y)] = D^{\ell mn} u^n(y) \delta(x-y) \quad (6)$$

where

$$[\lambda^\ell, i \beta \lambda^m] = D^{\ell mn} \beta \lambda^n \quad (7)$$

It is this extended version that we aim to replace below by a gauge formalism. The more restricted case has been considered already⁴⁾.

We begin by recalling the celebrated argument, due to Goto and Imamura⁵⁾, showing that difficulties arise in the approach through time-ordered products and equal time commutators. Consider the second order electromagnetic vacuum polarization tensor, i.e., the sum of graphs of the type

where the external lines are photons and there are no photons in the black box. One could expect this to be given by, for photon momentum q ,

$$T_{\mu\nu}(q) = \langle 0 | \int d^4x \, e^{-iqx} T(j_\mu(x), j_\nu(0)) | 0 \rangle \quad (8)$$

where j_μ is electromagnetic current, a combination of j_μ^1 (we leave out leptons). Then from (1), with a partial integration, taking

account of the conservation equation $\partial_\mu j_\nu = 0$, we obtain

$$iq_\mu T_{\mu\nu} = \langle 0 | \int d^4x \delta(x_0) [j_0(x), j_\nu(0)] | 0 \rangle \quad (9)$$

According to (3) the commutator is zero, because we have the same current twice and C^{lmn} is antisymmetric in l and m . Thus

$$iq_\mu T_{\mu\nu}(q) = 0 \quad (10)$$

This is just the standard gauge invariance condition; there is no difficulty here. However, the commutator can be computed in another way. By summing over a complete set of intermediate states, and collecting together contributions from states of equal mass, one finds

$$\langle 0 | [j_\mu(x), j_\nu(y)] | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{k_0}{|k_0|} e^{ik(x-y)} \int dm^2 2\pi \delta(m^2 + k^2) \rho(m^2) (k_\mu k_\nu - k^2 \delta_{\mu\nu}) \quad (11)$$

The weight function ρ , essentially a sum of squared moduli of matrix elements, is non negative. Putting $x_0 = y_0$, the required equal time commutator is

$$\begin{aligned} & \delta(x_0 - y_0) \langle 0 | [j_0(x), j_\nu(y)] | 0 \rangle \\ &= \left(\int dm^2 \rho(m^2) \right) \frac{1}{i} \frac{\partial}{\partial x_\nu} (\delta_{\nu 4} - 1) \delta(x - y) \end{aligned} \quad (12)$$

This is not zero, in contradiction with both the gauge condition (10) and the canonical commutator (3).

Clearly the paradox must arise somehow from the singular nature of relativistic quantum field theory. It is natural to attempt a resolution by regarding the theory as a limiting case of a less singular one, and studying carefully the passage to the limit. The simplest approach is that of Pauli-Villars regularization⁶⁾. In the form due to Gupta⁶⁾, the current commutators retain the simple canonical values. On the

other hand an indefinite metric is introduced, so that ρ need no longer be positive, and the consistency condition ⁷⁾

$$\int dm^2 \rho(m^2) = 0 \quad (13)$$

is perfectly acceptable.

Of course for finite values of the regulating masses the regulated theory, because of the indefinite metric, cannot be regarded as physics. The negative values of ρ must be pushed to inaccessible high values of q^2 . This suggests that although the simple canonical commutators are relevant in gauge conditions, they may have no relevance in physical sum rules. This idea has been expounded elsewhere in terms of a simple model ⁸⁾. Here we will not be concerned with sum rules, except in so far as they follow from gauge type conditions after assumptions about subtractions in dispersion relations.

Another limiting process is that of Schwinger ⁹⁾, who uses a non relativistic cut-off. Currents previously defined as products of quark fields at a single point are given a non-local structure:

$$\bar{\psi}(x) i \gamma_\mu \lambda^l \psi(x) \rightarrow \bar{\psi}(x - \frac{1}{2}\epsilon) i \gamma_\mu \lambda^l \psi(x + \frac{1}{2}\epsilon) \quad (14)$$

in which the small vector ϵ , which has no time component, is to be averaged over all space directions. He found that with such modified currents the commutator is of the form (12) with a gradient term that does not vanish even in the limit $\epsilon \rightarrow 0$. However, it then follows at once that the time ordered product $T_{\mu\nu}$ is no longer the vacuum polarization tensor; it does not satisfy the gauge condition (10), and because of the form of (12) cannot even transform as required under Lorentz transformations. Thus, if we denote the vacuum polarization tensor by $\Pi_{\mu\nu}$ we have

$$\Pi_{\mu\nu} = T_{\mu\nu} + R_{\mu\nu} \quad (15)$$

The remainder term $R_{\mu\nu}$ has rather a complicated construction in the Schwinger theory. It serves in the end to correct the peculiar Lorentz properties of $T_{\mu\nu}$, and to restore the gauge condition

$$iq_\mu \pi_{\mu\nu}(q) = 0 \quad (16)$$

Quite apart from the Schwinger cut-off, even the simple canonical commutators have gradient terms when the currents are constructed from boson as well as fermion fields. The difference between $\pi_{\mu\nu}$ and $T_{\mu\nu}$ then arises at even the most unsophisticated level. Nevertheless the gauge condition (16) remains valid. The stability of this condition suggests that there may be some advantage in avoiding the machinery of time-ordered products and equal-time commutators, and working directly with gauge conditions. This is the motivation for what follows.

2. GAUGE INVARIANCE

We again start with the quark model, but abstract from it gauge properties rather than commutators. The infinitesimal gauge transformation ¹⁰⁾ of the quark field is

$$\psi(x) \rightarrow \psi(x) + i\epsilon^l(x)\lambda^l\psi(x) \quad (17)$$

where the λ 's are the 18 matrices already introduced and the ϵ 's are 18 infinitesimal real functions of x . We will assume the interaction invariant and concentrate on the free part of the Lagrangian. The latter,

$$L = -\bar{\psi}(\gamma_\mu \partial_\mu + M)\psi$$

where M is a mass matrix, is certainly not invariant. It can be made invariant in the traditional way by adding interactions with extra fields, which we take here to be external c-number fields. Take 18 such vector fields a^1 and 18 scalar fields b^1 , and consider

$$L = -\bar{\Psi}(\gamma_\mu \partial_\mu + i\gamma_\mu \lambda^n a_\mu^n + M + \lambda^n b^n)\Psi \quad (18)$$

This is invariant under (17) combined with

$$\begin{aligned} a_\mu^n &\rightarrow a_\mu^n + \epsilon^\ell a_\mu^m C^{\ell mn} - \partial_\mu \epsilon^n \\ b^n &\rightarrow b^n + \epsilon^\ell b^m D^{\ell mn} + \epsilon^\ell f^{\ell n} \end{aligned} \quad (19)$$

where

$$[\lambda^\ell, i\beta M] = f^{\ell n} \beta \lambda^n \quad (20)$$

and the C 's and D 's are those defined already.

Suppose now that the gauge variations ϵ vanish for large X . Then the change in fields (17) is zero asymptotically, and so the S matrix, connecting asymptotic in and out fields is unchanged. Thus, what we abstract from the model is this: S matrix elements between specified hadron states, in the presence of external fields a and b , are invariant against gauge variations (19) in those fields. Expressing this in terms of functional derivatives with respect to the external fields, we have for a given S matrix element, denoted simply by S ,

$$0 = \delta S = \int d^4x \left\{ \frac{\delta S}{\delta a_\mu^n} (\epsilon^\ell a_\mu^m C^{\ell mn} - \partial_\mu \epsilon^n) + \frac{\delta S}{\delta b^n} (\epsilon^\ell b^m D^{\ell mn} + \epsilon^\ell f^{\ell n}) \right\}$$

Integrating by parts the $\partial_\mu \epsilon$ term, and requiring the resulting expression to vanish for arbitrary ϵ ,

$$0 = \partial_\mu \frac{\delta S}{\delta a_\mu^n} + C^{\ell mn} a_\mu^m \frac{\delta S}{\delta a_\mu^n} + D^{\ell mn} b^m \frac{\delta S}{\delta b^n} + f^{\ell n} \frac{\delta S}{\delta b^n} \quad (21)$$

for arbitrary equal arguments x of all functions a and b .

In applications we are concerned with the perturbation of transition matrix elements between given hadron states I and F by a number of weak external fields, i.e., with diagrams

If S is the relevant matrix element, the required quantities are of the type

$$\left(\frac{\delta}{\delta a_{\mu}^{\ell}(x)} \frac{\delta}{\delta a_{\nu}^m(y)} \cdots \frac{\delta}{\delta b^n(z)} \cdots \right) S \Big|_{a=b=0} \quad (22)$$

The correspondence with time-ordered products is established by noting that in the naive approach to the quark model (22) is given by the appropriate matrix element of

$$T(-i J_{\mu}^{\ell}(x), -i J_{\nu}^m(y), \cdots, -i U^n(z) \cdots) \quad (23)$$

In the gauge approach we obtain relations corresponding to (5) directly for the quantities (22) by multiple functional differentiation of (21). With the correspondence rule implied by (22) and (23), these relations are exactly the same as before. In particular the δ function terms of (5), arising there from equal-time commutators of operators, come here from observing that

$$\frac{\delta}{\delta a_{\mu}^{\ell}(x)} a_{\nu}^m(y) = \delta^{\ell m} \delta_{\mu\nu} \delta(x-y) \quad (24)$$

etc.

3. KLWZ FORMALISM

To illustrate the method further, consider the theory of Kroll, Lee, Weinberg and Zumino ¹¹⁾. Their Lagrangian contains hadronic vector mesons in a particular way. They are coupled to unspecified other hadrons which we will take here to be quarks. In such a theory, gradient terms appear in the current commutators already at the level of formal canonical manipulations, and at the same level the time-ordered products are no longer the covariant transition amplitudes. Following these authors we take a hadronic Lagrangian

$$L = -\frac{1}{2} m_0^2 \rho_\mu^\ell \rho_\mu^\ell - \bar{\Psi} M \Psi + L'(\Psi, \bar{\Psi}, \rho) \quad (25)$$

where the ρ are 18 vector (and axial-vector) meson fields. The part L' is supposed invariant against infinitesimal gauge transformations

$$\begin{aligned} \Psi &\rightarrow \Psi + i \epsilon^m \lambda^m \Psi \\ \rho_\mu^n &\rightarrow \rho_\mu^n + \epsilon^l \rho_\mu^m C^{lmn} - g_0^{-1} \partial_\mu \epsilon^n \end{aligned} \quad (26)$$

where g_0 is a coupling constant. We introduce our weak external fields, as before for the b 's and by a KLZ prescription ^{11a,d)} for the a 's :

$$L = -\frac{1}{2} m_0^2 \rho_\mu^\ell \rho_\mu^\ell - \bar{\Psi} (M + \lambda^\ell b^\ell) \Psi + L'(\Psi, \bar{\Psi}, \hat{\rho}) \quad (27)$$

where

$$\hat{\rho}_\mu^\ell = \rho_\mu^\ell + g_0^{-1} a_\mu^\ell \quad (28)$$

The final Lagrangian (27) is invariant under

$$\begin{aligned}
\psi &\rightarrow \psi + i\epsilon^m \lambda^m \psi \\
p_\mu^n &\rightarrow p_\mu^n + \epsilon^\ell p_\mu^m C^{\ell mn} \\
a_\mu^n &\rightarrow a_\mu^n + \epsilon^\ell a_\mu^m C^{\ell mn} - \partial_\mu \epsilon^n \\
b^n &\rightarrow b^n + \epsilon^\ell b^m D^{\ell mn} + \epsilon^\ell f^{\ell n}
\end{aligned} \tag{29}$$

Now that the gradient term goes with a rather than p , the p mass no longer spoils the invariance. It follows as before that S matrix elements are invariant under (19), and subsequent developments are unaltered. The gauge conditions on the covariant transition amplitudes are as before, and applications depending only on these will give the same results.

4. CONCLUSION

It has to be stressed in conclusion that the mere introduction of the gauge group is by itself no more significant than the original introduction of the currents, the densities, and their commutation relations. It is the assumption that these currents enter into weak and electromagnetic interactions in a specific way, and that the current divergences and densities give rise to matrix elements dominated by certain poles, which leads to interesting results. In the gauge formalism it is the corresponding body of assumptions that makes physics.

Ideas of the kind described here are quite old. Their renewed relevance in the era of current algebra was realized by many. However, the writer wishes particularly to express indebtedness ¹²⁾ to J.D. Bjorken, M.J.G. Veltman, and M. Nauenberg.

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