Physics of Quantum Relativity through a Linear Realization

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The idea of quantum relativity as a generalized, or rather deformed, version of Einstein (special) relativity has been taking shape in recent years. Following the perspective of deformations, while staying within the framework of Lie algebra, we implement explicitly a simple linear realization of the relativity symmetry, and explore systematically the resulting physical interpretations. Some suggestions we make may sound radical, but are arguably natural within the context of our formulation. Our work may provide a new perspective on the subject matter, complementary to the previous approach(es), and may lead to a better understanding of the physics.

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I. INTRODUCTION

In recent years, a new form of (special) relativity has been introduced under the names deformed special relativity or doubly (or triply) special relativity (DSR, TSR) [1, 2, 3]. It is motivated from a desire to incorporate additional invariant (dimensional) parameters into the theory beyond the speed of light, particularly an invariant quantum scale such as the Planck scale. (This idea actually dates back to a paper by Snyder [4] which is also a precursor to the idea of non-commutative geometry.) As a result, the new relativity can really be thought of as the quantum relativity. Relativity, of course, involves the behavior of frames of reference in physics and the relativity algebra is the algebra of transformations of the reference frames which also reflects the algebra of “space-time symmetry”. Therefore, one can view the algebra of quantum relativity as the algebra of transformations of quantum reference frames or the symmetry algebra of quantum space-time. We note that some discussion about quantum frames of reference already exists in literature and we want to refer the reader particularly to Refs.[5, 6]. The first of the two references discusses such issues within the context of non-relativistic quantum mechanics while the second concerns theories under the influence of gravity. The shared conclusion of the two papers is that a quantum frame of reference has to be characterized by its mass. In the case of gravity, it is illustrated that the gravitational properties of the reference frame itself need to be taken into account in order to define local gauge invariant observable. On the other hand, there is also the very intriguing notion of a quantum space-time structure. Space-time structure beyond a certain microscopic scale certainly cannot be physically or operationally defined as the commutative geometry of (Einstein) special relativity. One hopes that understanding the (special) quantum relativity would pave the way to understanding the quantum structure of space-time, and eventually even a quantum theory of gravity may be constructed as the general theory of quantum relativity. The present article is an attempt in this direction.

Even when one knows the mathematical description of the transformations, the physical interpretation may be much harder to come by. As we know, while Lorentz had already studied the (Lorentz) transformations, it was Einstein who brought out the correct physical meaning of these transformations [7]. In trying to obtain the correct physical picture from a mathematical description, one has to have an open mind for unconventional perspectives (that may arise) on some of the most basic notions about physics, in general, and space-time
structure, in particular. Most of the discussions of the (new) relativity so far have focused on nonlinear realizations of the symmetry algebra that arises. There is still no agreement among different authors on what the ultimate algebra of quantum relativity is. Here we will focus on $SO(1,5)$ as the ultimate Lie algebra of the symmetry of quantum relativity which has essentially been advocated within the context of triply special relativity (TSR), although most of our discussion on doubly special relativity (DSR) as an immediate structure will still be valid if that corresponds to quantum relativity, namely, if the intermediate structure in our discussions coincides with the final. From the point of view of Lie algebra deformation, $SO(1,5)$ has actually been identified essentially with the natural stabilizer of the “Poincaré + Heisenberg” symmetry. Physical observations with limited precision can never truly confirm an unstable algebra as the symmetry. Correspondingly, this perspective is strongly suggestive of taking $SO(1,5)$ as the natural candidate for the symmetry algebra of quantum relativity. It is also very reassuring that this symmetry algebra arises naturally from the perspective of quantum relativity as a deformation of special relativity.

We take the structure of the Lie algebra seriously as denoting the symmetry of “space-time” and focus on a linear realization with a classical or commutative geometry as the background of quantum relativity. Such a linear realization has been discussed to some extent in the literature, but neither systematically nor in detail. Our approach here is to try to understand the true physical implications of quantum relativity directly from a study of the transformations of the quantum reference frames and try to deduce the underlying geometric structure as an extension of the conventional space-time. We consider such an analysis as being complementary to the earlier studies involving nonlinear realizations. We note that our approach is largely inspired by Ref.[10] which, in our opinion, has brought up some interesting perspectives related to linear realization without putting all of them on more solid foundation. In trying to do this, we arrive at some very interesting and unexpected results. The most interesting among them is the extension of Einstein space-time structure into a higher dimensional geometry which is not to be interpreted as an extended space-time in the usual sense. This result is unconventional, but is of central importance to our discussions. We obtain the symmetry of quantum relativity through the approach of deformations and look for direct implications. We ask for sensible interpretations of mathematical results, and make suggestions along the way. Our analysis should be thought of as an initial attempt, rather than a final understanding. We believe that there are still
a lot more questions to be understood than the ones we discuss with suggested possible answers. We have put forward ideas here which seem to fit the physical problem at hand. Some of this is unconventional, but we think that they are quite reasonable and plausible and are in the right direction.

The article is organized as follows. In the next section, we write down explicitly the two-step deformation procedure to arrive at the quantum relativity, more or less following Ref. [10]. In Sec.III, we focus only on the first deformation introducing the invariant ultra-violet scale (the Plank scale), which gives a DSR structure. Here, the linear realization necessitates the introduction of a new geometric dimension along with the 4D space-time. The corresponding new coordinate has the canonical dimension of time over mass while having a spacelike geometric signature. It also suggests a new definition of energy-momentum as a coordinate derivative in the “nonrelativistic limit”. This section has the most dramatic or radical results and all of this fits in well with the notion of quantum frames of reference. In Sec.IV, we discuss relations to noncommutative or quantum (operator) realization of 4D space-time. In Sec.V, we focus on the geometric structure of the last deformation introducing an invariant infrared scale (the cosmological constant). Subsequently we address some important issues about the quantum relativistic momentum in Sec. VI before we conclude the article in the last section.

II. QUANTUM RELATIVITY THROUGH DEFORMATIONS

Let us start by writing down the Lie algebra of $SO(m, n)$ (with signature convention for the metric starting with a $+$)

$$[J_{AB}, J_{LN}] = i (\eta_{BL} J_{AN} - \eta_{AL} J_{BN} + \eta_{AN} J_{BL} - \eta_{BN} J_{AL}), \quad (1)$$

where indices $A, B, L, N$ take values from 0 to $d-1$. For $d = 4$, we have the familiar algebra $SO(1, 3)$ describing special relativity. However, we would like to start our discussion with $SO(0, 3)$ (which coincides with $SO(3)$) as a relativity algebra. As we know, Newtonian physics is described on a three-dimensional space. The symmetry algebra for the rotational invariance represented by $SO(3)$ with the corresponding generators given by

$$M_{ij} = i (x_i \partial_j - x_j \partial_i), \quad i, j = 1, 2, 3. \quad (2)$$
In this case, there is no index 0 and the metric has the (special) signature \( \eta_{ij} = (-1, -1, -1) \). We have the coordinate representation of the 3-momentum given by \( p_i = i\hbar \partial_i = i\hbar \frac{\partial}{\partial x_i} \) (we take \( \hbar = 1 \) in the following). The rotations can be augmented by the three-dimensional translations to define the complete symmetry group of 3D space. An arbitrary symmetry transformation can be taken as a transformation between two (inertial) frames of reference. In this case there is an alternative special way of getting a translation, namely,

\[
x^i \rightarrow x^i + \Delta x^i , \quad \Delta x^i(t) = v^i t ,
\]

where \( t \) denotes a parameter outside of the three-dimensional manifold with \( v^i \) given by \( \frac{dx^i}{dt} \), the velocity. The parameter \( t \) is identified with the (absolute) time and such translations are known as Galilean boosts. To the extent that time is just an external parameter, the Galilean boosts do not distinguish themselves from translations. The relevant symmetry group describing the admissible transformations between reference frames is the group \( \text{ISO}(3) \equiv \text{SO}(3) \otimes_\text{S} \mathbb{R}^3 \) where \( \otimes_\text{S} \) represents the semi-direct product. The generators of Galilean boosts (or special translations) can be denoted by \( N_i \) and satisfy

\[
[M_{ij}, N_k] = i \left( \eta_{jk} N_i - \eta_{ik} N_j \right).
\]

Note that much like the momentum, we can have the coordinate representation \( N_i = i \partial_i \) which satisfies Eq.(4). Of course, \( N_i \)'s commute among themselves and we have Galilean relativity.

Within the framework of Galilean relativity, the speed of a particle (as well as the speed of inertial reference frames) can take any value. Einstein realized that one has to go beyond Galilean relativity in order to accommodate an invariant speed of light, \( c \). From the present perspective, one can extend the three dimensional manifold to a four dimensional one by including (the external parameter) \( t \) so that \( x^\mu = (x^0, x^i) = (c t, x^i), \mu = 0, 1, 2, 3 \). Furthermore, if one introduces the velocity four-vector on this manifold as \( u^\mu = (u^0, u^i) \) with \( u^0 = c/\sqrt{c^2 - v^2} = \gamma, u^i = v^i/\sqrt{c^2 - v^2} = \gamma \beta^i \) and \( \beta^i = v^i/c \) (\( c \) represents the speed of light) so that

\[
\eta_{\mu\nu} u^\mu u^\nu = (u^0)^2 - (u^i)^2 = 1 ,
\]

then Eq.(5) equivalently leads to

\[
-\eta_{ij} v^i v^j = v^2 = c^2 \left( 1 - \frac{1}{\gamma^2} \right) \leq c^2 ,
\]
with equality attained only in the limit $\gamma \to \infty$. With this constraint the velocity (speed) takes values on the coset space $v^i \in SO(1,3)/SO(3)$ in contrast to the case of Galilean relativity where $v^i \in R^3 \equiv ISO(3)/SO(3)$. Furthermore, extending the manifold to a four dimensional one, we obtain a linear realization of the transformation group of reference frames which is deformed to $SO(1,3)$, the Lorentz group. The deformation of the algebra is given by

$$[N_i, N_j] \rightarrow -i M_{ij} ,$$

and the $N_i$'s can now be identified with the $M_{\mu\nu}$'s in this extended “space”, the space-time manifold, so that the full set of six $M_{\mu\nu}(= J_{\mu\nu})$ satisfying Eq.(1) can be written as

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) .$$

Furthermore, adding the four-dimensional translations with $p_0 = E/c = i \partial_0 = i \frac{\partial}{\partial x^0} (\hbar = 1)$, we obtain the full symmetry for Einstein special relativity described by the Poincaré group, $ISO(1,3) \equiv SO(1,3) \otimes_s R^4$ which represents the complete transformation group of inertial reference frames.

The idea of quantum relativity is to introduce further invariant(s) into the physical system. In special relativity, for example, we note that the momentum four vector can take any value. An invariant bound for the four-momentum (energy-momentum four-vector) that serves as a bound for elementary quantum states is one such example and such a generalization is commonly referred to as “doubly special relativity” (DSR) [1]. One can follow the example of Einstein relativity and derive this generalization as follows. Let us consider a parameter $\sigma$ outside of the four-dimensional space-time manifold and consider special translations in this manifold of the form (similar to Galilean boosts, see Eq.(3))

$$x^\mu \rightarrow x^\mu + \Delta x^\mu , \quad \Delta x^\mu(\sigma) = V^\mu \sigma ,$$

where we have identified $V^\mu = \frac{dx^\mu}{d\sigma}$. The generators $O_\mu$’s of these translations have the same commutation relations as the conventional four-dimensional translations. Furthermore, let us extend the 4D space-time manifold by adding the coordinate $\sigma$ as $x^A = (x^\mu, x^4) = (x^\mu, \kappa c \sigma), A = 0, 1, 2, 3, 4$, where $\kappa$ denotes the Planck mass and $c$ the speed of light. With this particular choice of the extra coordinate, we recognize that $\sigma$ has the dimensions of time/mass. As a result $V^\mu$ in Eq.(9) has the dimension of a momentum and we identify $V^\mu = p^\mu$. (We will discuss more on this identification in the next section.) It is clear that like
the velocity in the case of Galilean relativity, here $p^\mu$ can take any value. However, following the discussion in the case of Einstein relativity, we see that on this five-dimensional manifold, if we define a momentum 5-vector $\pi^A$ as

$$\pi^A = (\pi^\mu, \pi^4) = \left(\frac{p^\mu}{\sqrt{\kappa^2 c^2 - p^\mu p^\mu}}, \frac{\kappa c}{\sqrt{\kappa^2 c^2 - p^\mu p^\mu}}\right) = (\Gamma^{\alpha^\mu, \Gamma}), \quad (10)$$

with $\alpha^\mu = p^\mu / \kappa c$, this will satisfy

$$\eta_{AB} \pi^A \pi^B = \eta_{\mu\nu} \pi^\mu \pi^\nu - (\pi^4)^2 = -1. \quad (11)$$

This, in turn, would imply that (see Eq.(6))

$$p_\mu p^\mu = \eta_{\mu\nu} p^\mu p^\nu = \kappa^2 c^2 \left(1 - \frac{1}{\Gamma^2}\right) \leq \kappa^2 c^2, \quad (12)$$

where as we have said earlier $\kappa$ stands for the Planck mass. Similar to the earlier discussion on Einstein relativity, we see that in this case the four momentum lives on the coset space $p^\mu \in SO(1, 4)/SO(1, 3)$ instead of $p^\mu \in R^4$ and the de Sitter group $SO(1, 4)$ corresponds to the symmetry of the deformed relativity here.

In this extended (five-dimensional) manifold, the extra generators needed to complete $SO(1, 4)$ and to lead to a linear realization can now be taken as

$$O_\mu \equiv J_\mu^4 = i (x_\mu \partial_4 - x_4 \partial_\mu). \quad (13)$$

Like the conventional 4D translation generators, the generators $O_\mu$’s also satisfy (we note the identification made earlier $M_{\mu\nu} = J_{\mu\nu}$)

$$[M_{\mu\nu}, O_\lambda] = i (\eta_{\mu\lambda} O_\mu - \eta_{\mu\lambda} O_\nu). \quad (14)$$

However, with the identification in Eq.(13), the algebra is deformed to $SO(1, 4)$ with

$$[O_\mu, O_\nu] \longrightarrow i M_{\mu\nu}. \quad (15)$$

We call the transformations generated by $O_\mu$’s as de Sitter momentum boosts, or simply as momentum boosts. Adding the five-dimensional translations, the full symmetry group of this manifold becomes $ISO(1, 4) \equiv SO(1, 4) \otimes_s R^5$. We want to emphasize here that although we have a natural five-dimensional Minkowski geometry to realize the new relativity, the

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1 We scale by a factor $\kappa$ relative to the common notation as first introduced by Snyder [4].
fifth dimension here should be considered neither as space nor time ($\sigma$ has the dimension of time/mass). In the following section, we will try to explore the physics meaning of this extra coordinate from two different points of view.

Before ending this section, let us consider, for reasons to be clarified below, one more deformation in relativity by imposing a third invariant. Here, there is even less physical guidance on what should be the appropriate quantity to consider, but an infrared bound seems to be quite meaningful [3, 8]. As we will argue, it can, for example, end the many possible iterative deformations that can be introduced along these lines. As in the earlier discussions, let us introduce a parameter $\rho$ external to the five-dimensional manifold that we have been considering and for which the coordinates are $(x^0 = ct, x^i, x^4 = \kappa c \sigma)$. Following the discussion of Galilean boost, let us introduce special translations of the form

$$x^A \rightarrow x^A + \Delta x^A, \quad \Delta x^A(\rho) = V_A^\rho,$$

(16)

where we have identified $V_A = \frac{dx_A}{d\rho}$. The generators, $O'_A$, of these translations will obey the same commutation relations as the conventional five-dimensional translation generators. Furthermore, let us extend the 5D manifold by including this new coordinate $\rho$ as $x^M = (x^A, \ell \rho), M = 0, 1, 2, 3, 4, 5$ where $\ell$ is an invariant length. With this choice of the fifth coordinate we see that the new coordinate $\rho$ is dimensionless. As a result, $V^A$ defined in Eq.(16) has the dimension of length and we identify this with a coordinate vector of translation $V^A = z^A$. (The meaning of this vector will be discussed in Sec.V.) As in Galilean relativity, we see that $z^A$ can take any arbitrary value. However, following the earlier discussion on special relativity, let us introduce a coordinate vector on this six dimensional manifold as

$$X^M = (X^A, X^5) = \left( \frac{z^A}{\sqrt{\ell^2 - \eta_{AB} z^A z^B}}, \frac{\ell}{\sqrt{\ell^2 - \eta_{AB} z^A z^B}} \right) = (G\gamma^A, G),$$

(17)

with $\gamma^A = z^A / \ell$. It is clear that this coordinate vector will satisfy the condition

$$\eta_{MN} X^M X^N = \eta_{AB} X^A X^B - (X^5)^2 = -1,$$

(18)

which, in turn, will imply that

$$\eta_{AB} z^A z^B = \ell^2 \left( 1 - \frac{1}{G^2} \right) \leq \ell^2.$$

(19)
The constraint introduces an invariant length scale into the problem. This construction also makes clear that the special five-dimensional translations $z^A$ live on the coset $z^A \in SO(1, 5)/SO(1, 4)$, instead of $z^A \in \mathbb{R}^5$ and the enlarged de Sitter group $SO(1, 5)$ corresponds to the symmetry of this deformed relativity. The new translations can now be thought of as a new kind of boost described by the generators

$$J_{5\alpha} \equiv O'_\alpha = i \left( x_\alpha \partial_5 - x_5 \partial_\alpha \right),$$

with

$$[J_{AB}, O'_C] = i \left( \eta_{BC} O'_A - \eta_{AC} O'_B \right).$$

The deformation relative to the $ISO(1, 4)$ algebra is now obtained to be

$$[O'_A, O'_B] \rightarrow i J_{AB},$$

for $A, B = 0, 1, 2, 3, 4$. Indeed, all the symmetry generators can be written as

$$J_{MN} = i \left( x_M \partial_N - x_N \partial_M \right),$$

for $M, N = 0, 1, 2, 3, 4, 5$, giving the linear realization of the $SO(1, 5)$ symmetry. This is what we consider to be the true (full) symmetry of quantum relativity. Discussions in the rest of the paper will attempt to justify this choice as a sensible one.

We note that because of the constraint of Eq. (18), the relevant part of the six-dimensional geometry is actually a five-dimensional hypersurface given by

$$\eta_{MN} X^M X^N = -1.$$  

(As we discuss later in Sec.V, the coordinates $x^M$ and $X^M$ give different parametrizations of a point in this six dimensional manifold.) The five-dimensional hypersurface does not admit simple translational symmetry along any of the six coordinates anymore. Hence, the above scheme of relativity deformations naturally ends here. To be more specific, it ends once we put in a deformation that imposes an invariant bound on the displacement vector generalizing the space-time coordinate itself. The transformations generated by operators in Eq. (20) are isometries of the 5D hypersurface mixing $x^5$ with the other coordinates. We call them (de Sitter) translational boosts. Taking this as the quantum relativity forces us to consider the 5D hypersurface $dS_5$, which is a de Sitter space compatible with a positive cosmological constant, as the arena for (quantum) space-time. However, we not only have
dS, having one more dimension than the dS curved space-time conventionally considered in the cosmology literature, but quantum relativity also suggests that the extra coordinates $x^4$ and $x^5$ are quite different from the conventional (spacelike) space-time coordinates.

III. SOME PHYSICS OF THE MOMENTUM BOOSTS AND THE $x^4$ COORDINATE

In this section, we focus on the relativity with only one extra invariant scale, $\kappa c$. Discussions here can be considered as relevant only to the intermediate case without the last deformation involving the invariant length $\ell$. The deformed relativity up to the level of $SO(1,4)$ is essentially the same as the DSR constructions, with a different parametrization for the energy-momentum surface defined by Eq. (11) [11]. The linear realization of the transformations presented here has been discussed implicitly, most notably in Ref. [10], although the physics involved is not clearly discussed. We view the transformations here as what they should be, namely, transformations of (quantum) reference frames, in order to extract a better understanding of a sensible interpretation of physics issues involved. We want to emphasize right away that the deformation, introducing the new momentum boosts as distinct from the Lorentz boosts in the linear realization through the 5-geometry, is characterized by a central idea contained essentially in the defining relation $p^\mu \equiv \frac{dx^\mu}{d\sigma}$. This is nothing less than introducing a new definition of the energy-momentum 4-vector, whose implications will be discussed later in this section. To emphasize, we note that $\sigma$ (or $x^4$) as a coordinate is external to the four-dimensional space-time, and hence $p^\mu$ so defined is different from the old definition of (Einstein) energy-momentum in the 4D space-time. In fact, we consider it necessary to take special caution against thinking of $x^4$ as simply an extra space-time coordinate.

The new relativity (in this intermediate case) only adds a new dimension parametrized by $x^4$ and we note that the set of Lorentz transformations continue to be a part of the isometry group of the extended 5D manifold characterizing rotations within any 4D space-time submanifold. However, there are also new symmetry transformations in this 5D manifold. These are the momentum boosts. To better appreciate the physics of the momentum boosts generated by the $O_\mu$'s, let us analyze the finite transformations under such boosts and, in particular, examine the transformation of the energy-momentum 4-vector. To keep the
discussion parallel to what we know in Einstein relativity, let us summarize some of the essential formulae from the latter. We recall that in an inertial frame characterized by the velocity $\vec{\beta} = \vec{v}/c$ (note that we use the $\vec{\cdot}$ notation in this paper to denote a generic vector defined on a manifold of any dimension; whenever ambiguities are likely to arise, we will use a notation such as $\vec{\cdot}^{n}$ to define explicitly the dimensionality of the vector), the coordinates transform as

$$
\begin{align*}
 x'^0 &= \gamma (x^0 - \vec{\beta} \cdot \vec{x}) , \\
 \vec{x}' &= \vec{x} + \vec{\beta} \left( \frac{(\gamma - 1)}{\beta^2} \vec{\beta} \cdot \vec{x} - \gamma x^0 \right),
\end{align*}
$$

(25)

where $\beta^2 = \vec{\beta} \cdot \vec{\beta}$ and $\gamma = 1/\sqrt{1 - \beta^2}$ denotes the Lorentz contraction factor. Furthermore, if we have a particle moving with a velocity $\vec{\beta}_1 = \vec{v}_1/c$, then in the boosted frame, it will have a velocity given by

$$
\vec{\beta}'_1 = \frac{\gamma^{-1}}{1 - \vec{\beta} \cdot \vec{\beta}_1} \left[ \vec{\beta}_1 + \vec{\beta} \left( \frac{(\gamma - 1)}{\beta^2} \vec{\beta} \cdot \vec{\beta}_1 - \gamma \right) \right].
$$

(26)

This gives the formula for the composition of velocities and, in particular, when $\vec{\beta} = \beta(1, 0, 0)$ and $\vec{\beta}_1 = \beta_1(1, 0, 0)$, reduces to the well-known formula

$$
\beta'_1 = \frac{\beta_1 - \beta}{1 - \beta \beta_1},
$$

(27)

which can also be written as

$$
v'_1 = \frac{v_1 - v}{1 - \frac{vv_1}{c^2}}.
$$

(28)

The momentum boosts of the five-dimensional geometry generated by $O_{\mu}$ can also be understood along the same lines. If we consider an inertial frame characterized by the momentum 4-vector $\vec{\alpha} = \vec{p}/\kappa c$, then the 5D coordinates will transform as

$$
\begin{align*}
 x'^{\mu} &= \Gamma (x^{\mu} - \vec{\alpha} \cdot \vec{x}) , \\
 x'^{\mu} &= x^{\mu} + \alpha^{\mu} \left( \frac{(\Gamma - 1)}{\alpha^2} \vec{\alpha} \cdot \vec{x} - \Gamma x^\mu \right),
\end{align*}
$$

(29)

where $\vec{\alpha} \cdot \vec{x} = \eta_{\mu\nu} \alpha^{\mu} x^{\nu}$, $\alpha^2 = \vec{\alpha} \cdot \vec{\alpha}$, and $\Gamma = 1/\sqrt{1 - \alpha^2}$ is the analogous “contraction factor”. It can be checked easily that under these transformations, the metric of the manifold remains invariant, namely,

$$
\eta'^{AB} = \eta^{AB}, \quad \eta'_{AB} = \eta_{AB},
$$

(30)
so that the new transformations correspond to isometries of the manifold.

Furthermore, if we have a particle moving with a momentum \( \vec{\alpha}_1 = \vec{p}/\kappa c \), then in the momentum boosted frame it will have a momentum (see Eq.(26))

\[
\vec{\alpha}_1' = \frac{\Gamma^{-1}}{1 - \vec{\alpha}_1 \cdot \vec{\alpha}_1} \left[ \vec{\alpha}_1 + \vec{\alpha} \left( \frac{\Gamma - 1}{\alpha^2} \vec{\alpha} \cdot \vec{\alpha}_1 - \Gamma \right) \right].
\]

This gives the formula for the composition of momentum under momentum boosts and can also be written equivalently as

\[
\vec{p}_1' = \frac{\Gamma^{-1}}{1 - \vec{\alpha}_1 \cdot \vec{\alpha}_1} \left[ \vec{p}_1 + \vec{p} \left( \frac{\Gamma - 1}{\alpha^2} \vec{p} \cdot \vec{\alpha}_1 - \Gamma \right) \right].
\]

(31)

(32)

In particular, if we consider a momentum boost along the \( x^0 \) direction generated by \( O_0 \) characterized by \( \vec{p} = p(1, 0, 0, 0) \), then the composition of the momentum given by Eq.(32) leads to

\[
\begin{align*}
\vec{p}_1' &= \frac{p_0^L - p}{1 - \frac{p_0^L p}{\kappa^2 c^2}}, \\
\vec{p}_i^L &= \sqrt{1 - \frac{p_0^L p}{\kappa^2 c^2}} \vec{p}_i^L,
\end{align*}
\]

(33)

which can be compared with the formula Eq.(28) for velocity composition in Einstein relativity. Furthermore, if we assume that the particle characterized by a rest mass \( m_1 \) is in its rest frame so that \( \vec{p}_1 = m_1 c (1, 0, 0, 0) \) and the momentum boost of the form \( \vec{p} = mc(1, 0, 0, 0) \), then Eq.(33) leads to the composition law

\[
m_1' = \frac{m_1 - m}{1 - \frac{m m_1}{\kappa^2}}.
\]

(34)

An Einsteinian particle with the rest mass \( m_1 \) has momentum that can be parametrized as \( \vec{p}_1 = (\gamma m_1 c, \gamma m_1 c \beta_1^i) \) which satisfies the on shell condition (a terminology of relativistic quantum field theory)

\[
\eta_{\mu \nu} p_1^\mu p_1^\nu = m_1^2 c^2,
\]

in any Lorentzian frame. In particular, there is the particle rest frame in which we have \( \vec{p}_1 = (m_1, c, 0, 0, 0) \). We normally think this as the reference frame defined by the particle itself; or the frame where particle is the observer. As a result, the particle does not see its own motion, but does see its own mass or energy. The introduction of the momentum boosts relating different reference frames generalizes that perspective. Just as \( \beta_1^2 \) is not invariant
under Lorentz boosts, $p_i^2$ is also not invariant under the momentum boosts. Furthermore, just as there is the preferred rest frame with $\beta_i^2 = 0$, or equivalently with the 4-velocity characterized by $\vec{u}_i = (1, 0, 0, 0)$, similarly the linearly realized DSR introduces a preferred “particle” frame with $p_i^2 = 0$, which is equivalently characterized by the 5-momentum $\vec{p}_i = (0, 0, 0, 0, 1)$ as obtained from Eq. (34) by setting $p_\mu^i = p_\mu$. This is the “true particle frame” in which the “particle” does not see itself, neither its motion nor its mass/energy. The rest frame of Einstein relativity is only the frame that has no relative motion with respect to the particle.

As we have mentioned earlier, we would like to view the momentum boosted frames as quantum reference frames. We will see here that the interpretation is in a way necessary, as we look into how other momentum 4-vectors look like in such a reference frame. Looking at Eq. (33) we see that

$$\eta_{\mu \nu} p'_\mu p'_\nu = \frac{1}{\left(1 - \frac{P_0^2}{\kappa^2 c^2}\right)^2 \left[ (p_0^i - p)^2 - \left(1 - \frac{p^2}{\kappa^2 c^2}\right) (\vec{p}^3)^2 \right]} \neq \eta_{\mu \nu} p_\mu p_\nu. \quad (35)$$

where $(\vec{p}^3)^2$ is the magnitude of the momentum 3-vector. Because of the complicated dependence on $c$ in here, we cannot even write $\eta_{\mu \nu} p_\mu p_\nu = m_i^2 c^2$. The quantum field theoretical concept of off-shellness is what we consider applicable here. Quantum states are either on shell or off shell, as observed from a classical frame. When boosted to a quantum frame characterized by even an on shell quantum state, the state does not observe itself, and observes the other originally on shell states as generally off shell. The concept of an off shell state is related to the uncertainty principle. Unlike a classical particle, a quantum state, even on shell, has associated uncertainties. If such a state is to be taken as the reference frame, or observer (measuring apparatus), it is very reasonable to expect that the apparatus imposes its own uncertainties onto whatever it observes/measures. We have to be cautious here though about whether a quantum measuring apparatus has any practical possibility of being realized. After all, the only true practical observers, namely, human beings, are basically classical.

We see that the conceptually small step that we take here is indeed a bold one. Our explicit formulation of the natural linear realization of the momentum boosts of DSR looks highly unconventional. It begs the question if a consistent and viable phenomenological interpretation exists — a question for which we are only going to provide here some partial answer in the affirmative. In Einstein relativity, we have $\eta_{\mu \nu} p_\mu p_\nu = m^2 c^2$ and $p_\mu = mc u^\mu$. 
In quantum physics, we are familiar with the concept of off shell states which violate the first equation. Our explicit analysis of the momentum boost illustrates that the on shell condition is not preserved under a momentum boost. The momentum boost analyzed above is one of the simplest (namely, a boost along the $x^0$ axis), but our conclusions obviously hold for any other more complicated momentum boost. In fact, it is easily derived from Eq. (32) that an arbitrary momentum boost leads to

$$\eta_{\mu\nu} P_1^\mu P_1^\nu = \frac{1}{\left(1 - \frac{\vec{p} \cdot \vec{p}}{\kappa^2 c^2}\right)^2} \left[ (\vec{p}_1 - \vec{p})^2 + \frac{1}{\kappa^2 c^2} \left( (\vec{p} \cdot \vec{p})^2 - p^2 p_1^2 \right) \right] \neq \eta_{\mu\nu} P_1^\mu P_1^\nu. \quad (36)$$

Rather the condition $p^\mu = mc u^\mu$ is actually given up right at the beginning of the formulation. The linear realization of momentum boosts as distinct from velocity (Lorentz) boosts introduces the defining relation $p^\mu = dx^\mu/d\sigma = \kappa c dx^\mu/d\tau$ in contrast to the Einstein relativity limit of $mc u^\mu = m dx^\mu/d\tau$ as $mc dx^\mu/d\sigma$.

So far, we have not clarified the nature of the extra coordinate $\sigma$. This leads to $x^4 = \kappa c \sigma$ which is necessitated by the desire to have a bound on the energy-momentum 4-vector. This new coordinate has a spacelike signature, but has the dimensions of time/mass. This suggests that from the physics point of view it has the character of time as opposed to space (which its signature will suggest). In fact, $\sigma$ should be considered neither as space nor as time in this 5D space. In this sense, the frames in this 5D manifold that we are analyzing should be considered different from the ones that arise in a naive 5D extension of the usual 4D space-time by adding an extra spatial dimension. To appreciate this a bit more and also to bring out the nature of the coordinate $\sigma$, let us consider the following. Let us recall that in deforming Galilean relativity to Einstein relativity, one introduces Lorentz boosts which mix up space and time and are characterized by a velocity. The (instantaneous) velocity of a particle is defined as $v^i = dx^i/dt = c dx^i/d\tau$. This three-dimensional velocity, of course, does not transform covariantly under a Lorentz boost. However, the extra coordinate $x^0$ is in a way already there as the time parameter $t$ and the velocity is also there as the time derivative in the 3D Galilean theory. In a parallel manner, in deforming Einstein relativity to DSR, we introduce momentum boosts which mix up the extra coordinate with the 4D coordinates and are characterized by a momentum. In this case, the momentum of a particle is then to be defined as $p^\mu = dx^\mu/d\sigma = \kappa c dx^\mu/d\tau$. This momentum does not transform covariantly under momentum boosts (see Eq. (32)). However, unlike the case of deforming the Galilean velocity boosts to Lorentzian ones, here the extra coordinate $x^4$ or $\sigma$ as a parameter is
new, and the definition of the momentum as a $\sigma$ derivative does not coincide with the conventional definition of the momentum in Newtonian physics or Einstein relativity. In the latter case, one defines $p^\mu_{(ER)} = m \frac{dx^\mu}{d\tau}$ where $\tau$ represents the proper time. We know that the definition of $p^\mu_{(ER)}$ and its Newtonian limit are valid concepts. So, we do need to reconcile the two definitions of momentum in the regime of Einstein relativity. That can be achieved if we identify $\sigma = \frac{\tau}{m}$ which has the dimension of time/mass as we have pointed out earlier. With this identification we see that the $\sigma$ coordinate for a (classical) particle observed from a classical frame is essentially the Einstein proper time. (In fact, to the extent that the definition of $p^\mu_{(ER)}$ is valid in quantum mechanics, we expect the relation to be valid to a certain extent even for quantum states observed from a classical frame.) The mass factor, however, gives particles in the state of motion different $\sigma$ locations. Note that such an identification holds only for $m \neq 0$ just as the identification $p^\mu_{(ER)} = m \frac{dx^\mu}{d\tau}$ holds only when $m \neq 0$. For the massless photon, for example, $p^\mu_{(ER)}$ instead depends on the photon frequency and wavelength – an idea with origin in quantum physics. That may actually be taken as a hint that the classical notion of $p^\mu_{(ER)} = m \frac{dx^\mu}{d\tau}$ is not fully valid in true quantum physics.

Finally, we comment briefly on the $O_i$ momentum boosts here. Such a boost is one characterized, for example, by a relative energy-momentum $\vec{p} = (0, p, 0, 0)$ which cannot correspond to an on shell state observed from the original classical reference frame. However, if momentum boosts can take us to a quantum reference frame and change the observed on shell nature of states, there is no reason why one cannot accept the reference frame itself to be characterized by an off shell energy-momentum vector either. Physics described from such a frame certainly looks peculiar, though it may not concern a classical human observer.

IV. ON THE FULL QUANTUM RELATIVITY AND PHASE-SPACE SYMMETRY

There has already been a lot of work and discussion on the subject of DSR and TSR (doubly and triply special relativity). In this section, we summarize some of the results on the symmetry aspects that exist in the literature while adapting them into our point of view. In particular, we would be interested in the formulations of TSR in Ref.[3, 9] which is proposed as the full quantum relativity. The results of Ref.[3] start with the “phase-space” algebra of DSR with noncommutative space-time coordinate, which can be written according
to our conventions as

\[ [M_{\mu\nu}, M_{\lambda\rho}] = i(\eta_{\lambda\rho}M_{\mu\nu} - \eta_{\mu\lambda}M_{\nu\rho} + \eta_{\mu\nu}M_{\rho\lambda} - \eta_{\rho\lambda}M_{\nu\mu}) , \]

\[ [M_{\mu\nu}, \hat{P}_\lambda] = i(\eta_{\lambda\rho}\hat{P}_\mu - \eta_{\mu\lambda}\hat{P}_\nu) , \]

\[ [M_{\mu\nu}, \hat{X}_\lambda] = i(\eta_{\lambda\rho}\hat{X}_\mu - \eta_{\mu\lambda}\hat{X}_\nu) , \]

\[ [\hat{X}_\mu, \hat{X}_\nu] = \frac{i}{\kappa^2 c^2} M_{\mu\nu} , \]

\[ [\hat{P}_\mu, \hat{P}_\nu] = 0 , \]

\[ [\hat{X}_\mu, \hat{P}_\nu] = -i \eta_{\mu\nu} . \]  

(37)

This is identified as the algebra of DSR phase-space symmetry and we note that \( \hat{X} \) and \( \hat{P} \) correspond to generators (operators) of the algebra in contrast to the \( x \) and \( p \) coordinates described in the earlier sections. A second deformation of Eq.(37) is considered with a view to implement the third invariant as a length \( \ell \) related to the cosmological constant \( (\Lambda = \ell^{-2}) \).

In this case the commutator of the momentum operators in Eq.(37) is deformed to

\[ [\hat{P}_\mu, \hat{P}_\nu] = \frac{i}{\ell^2} M_{\mu\nu} . \]  

(38)

However, this deformation leads to a violation of the Jacobi identity which induces a further modification of the Heisenberg commutator (between \( \hat{X} \) and \( \hat{P} \)) involving a complicated (quadratic) expression in the generators. This algebra is then identified as the quantum algebra (of TSR). We also note here that it was pointed out in Ref.[3] that this algebra can be represented in terms of coordinates and derivatives of a six-dimensional manifold. It is important to recognize that in both cases (DSR and TSR), in addition to the deformations, the usual 4D coordinates are promoted to generators of the algebra, also to be interpreted as representing a noncommutative geometry of quantum space-time.

At this point, it is interesting to compare the original DSR to TSR deformation of Ref.[3] with our formulation of the quantum relativity algebra. After the deformation from Einstein relativity to \( SO(1,4) \), we have a linear realization of the algebra at the level of DSR. Our relativity algebra is set on a 5D commutative manifold. We do not have coordinate operators as generators of the algebra. However, it is worth noting that the four generators, \( O_\mu \)'s, generating momentum boosts satisfy the same commutation relations as the \( \hat{X}_\mu \) operators in Eq.(37) (see, for example, Eqs.(13) and (15)). Therefore, formally we can identify \( O_\mu \) as \(-\kappa c \hat{X}_\mu \) with the explicit form following from Eq.(13)

\[ \hat{X}_\mu = \frac{1}{\kappa c} i (x_\mu \partial_4 - x_4 \partial_\mu) , \]  

(39)
which actually seems like a very reasonable “quantum” generalization of the classical, or rather Einstein, space-time position. In the limit $\kappa c \to \infty$ (with $\sigma \to 0$ as $1/\kappa c$) and $i \partial_4 \equiv p_4 = -\kappa c (p_4 = \eta_{\mu A} p^A)$, the operators reduce to $x_\mu$. Such an identification as in Eq. (39), provides a bridge between a noncommutative geometric description of 4D quantum space-time (details of which await further investigation) and our perspective of quantum relativity as linearly realized on a 5D, or eventually 6D, commutative manifold — a geometric description beyond the space-time perspective. We believe the two pictures to be complementary.

The symmetry can now be enlarged to $\text{ISO}(1,4)$ by incorporating translations of the 5D manifold. The five of them are $p_A \equiv i \partial_A$. Dropping $p_4$ for the moment, they almost satisfy the full “phase-space” algebra of DSR. The only problem comes from the Heisenberg commutator which is no longer canonical. While one can reasonably argue that our linear realization of the DSR simply suggests that the Heisenberg commutator should be modified, it is not what we want to focus on here. We take the full quantum relativity at the next (TSR) level, namely, with the third invariant $\ell$. Again, forgetting the $\hat{X}_\mu - \hat{P}_\mu$ commutator for the moment, the algebra which is essentially $\text{ISO}(1,4)$ is deformed to $\text{SO}(1,5)$. In this case, it is possible to formally identify $\ell \hat{P}_\mu = O'_\mu (= J_{\mu 5})$ (see Eqs. (20) and (22)) which gives the explicit linear realization

$$\hat{P}_\mu = \frac{1}{\ell} i (x_\mu \partial_5 - x_5 \partial_\mu) .$$

(40)

Once again this gives a very reasonable “quantum” generalization of the “classical” $p_\mu$. This is seen by noting that in the limit $\ell \to \infty$ (with $\rho \to 0$ as $1/\ell$), $\hat{P}_\mu$ reduces to $i \partial_\mu \equiv p_\mu$. All of these fit in very nicely, except for the $\hat{X}_\mu - \hat{P}_\mu$ commutator and the issue of the extra $\hat{P}_4$, or rather $O'_4$ which has not been addressed so far.

The missing link in the above discussion actually can be obtained from the analysis in Ref. [9] which restores the Lie-algebraic description of the TSR algebra by identifying the right-hand side of the Heisenberg commutator as a central charge generator $\hat{F}$ of the original algebra with relevant commutators also deformed, yielding

$$[\hat{X}_\mu, \hat{P}_\nu] = -i \eta_{\mu \nu} \hat{F} , \quad [\hat{X}_\mu, \hat{F}] = \frac{i}{\kappa^2 c^2 \ell^2} \hat{P}_\mu , \quad [\hat{P}_\mu, \hat{F}] = -\frac{i}{\ell^2} \hat{X}_\mu .$$

(41)

This identifies the resulting algebra exactly with $\text{SO}(1,5)$, and, therefore, $\hat{F}$ loses its character as a central charge generator. It is also interesting that this algebra has been identified
as the mathematical stabilizer of the “Poincaré+Heisenberg” symmetry \[9\]. The generator \( \hat{F} \) is essentially \( O_4' \) (or equivalently \( \hat{P}_4 \)). Explicitly, \( O_4' = J_{\sigma} = -\kappa c \hat{F} \).

Nonlinear realizations of the different versions of quantum relativity focus on the description of the (four) space-time geometry as a noncommutative geometry. Here, the nonvanishing commutator among \( \hat{X}_\mu \)'s may be considered to result from the curvature of the energy-momentum space [cf. Eq. (11)] while the nonvanishing commutators among \( \hat{P}_\mu \)'s is considered a result of the “space-time” curvature [2, 12] as characterized by the nonvanishing cosmological constant. Our formulation through the linear realization yields an explicit identification of the quantum operators as generalizations of the familiar phase-space coordinates variables. This is based on a 6D geometry which is commutative. Both the extended \( x \)-space and the \( p \)-space are copies of \( \text{dS}_5 \) and the isometry of each contains the full phase-space symmetry algebra of the quantum theory of 4D noncommutative space-time. The \( x^\mu \)-variables and the \( p^\mu \)-variables are on the same symmetric footing. In our opinion, this is a very attractive and desirable feature. It also clearly suggests that the \( x^4 \) and \( x^5 \) coordinates should not be interpreted as space-time coordinates in the usual sense. The generators of the momentum and translational de Sitter boosts can be viewed as the set of quantum position and momentum operators.

It is interesting to note that the \( \text{SO}(1, 4) \) algebra, with the \( J_{\mu} \) generators taken essentially as the “position” variables is identified as the universal basis of a noncommutative space-time description of various DSR theories [13]. The full phase-space symmetry algebras, however, are taken to be \( \kappa \)-Poincaré quantum algebras in different bases [13]. The latter contains \( \hat{P}_\mu \) generators extending the \( \text{SO}(1, 4) \) algebra with different deformed commutators from the trivial case of \( \text{ISO}(1, 4) \) corresponding to taking different 4-coordinates for the coset space \( \text{SO}(1, 4)/\text{SO}(1, 3) \) for the energy-momentum [11]. By sticking to the 5-momentum, \( \pi^a \), together with a 5D geometric extension of the space-time, our linear realization allows us to put in the next deformation of \( \text{ISO}(1, 4) \) to the full quantum relativity of \( \text{SO}(1, 5) \) naturally. The latter group was proposed as the TSR algebra [3] only after the Lie-algebraic interpretation of Ref. [9], by pulling out the algebraic structure basically from a deformation of the phase-space symmetry algebra directly, as discussed above. With our formulation of the full quantum relativity of \( \text{SO}(1, 5) \), it is suggested that not only is the 4D space-time noncommutativity universal, so is that for the 4D energy-momentum space. But now both the 6-vectors \( x^m \) and \( \pi^m \) should be living on a \( \text{dS}_5 \). It would be interesting to see
how different choices of 5-coordinates on the two dS\(_5\) or canonical coordinates of the ten-dimensional “phase-space” match onto the various DSR theories or other similar structures for the different nonlinearly realized space-time structures as well as the role of the related quantum algebras.

V. THE TRANSLATIONAL BOOSTS AND DE SITTER GEOMETRY

Similar to the Lorentz and the de Sitter momentum boosts discussed earlier, the introduction of the de Sitter translational boosts is characterized by the 5-vector

\[ \vec{z} = \frac{d\vec{x}}{d\rho} = \ell \frac{d\vec{x}}{dx^5}. \]

We note that since the new coordinate \(\rho\) is dimensionless, the parameter of boost in this case carries the same dimension as the coordinates themselves. This should be contrasted with velocity which represents the parameter of transformation in the case of Lorentz boosts, and the momentum for the momentum boosts. A generic \(O'\) boost characterized by a vector \(\vec{z}_t\)

\[ \vec{x}' = \vec{x} + \vec{\gamma}_t \left( \frac{G_t - 1}{\gamma_t^2} \vec{\gamma} \cdot \vec{x} - G_t x^5 \right), \]

where we have identified (as defined earlier)

\[ \vec{\gamma}_t = \frac{\vec{z}_t}{\ell}, \quad G_t = \frac{\ell}{\sqrt{\ell^2 - \vec{z}_t^2}}, \quad \vec{\gamma}_t \cdot \vec{x} = \eta_{AB} \gamma_A^t x^B, \quad \vec{z}_t^2 = \eta_{AB} \gamma_A^t \gamma_B^t. \]

The transformations are analogous to Eqs.\((25)\) and \((29)\) representing Lorentz boosts and de Sitter momentum boosts and the 6D metric tensors \(\eta^{MN}, \eta_{MN}\) are preserved under these translational boosts.

Any 6-vector and, in particular, \(X^M\) defined in Eq.(17) will transform covariantly under a translation boost as in Eq.\((12)\). However, the 5D vector \(\vec{z}\) will transform like the velocity in Eq.\((26)\) or like the momentum in Eq.\((31)\) as

\[ \vec{\gamma}' = \frac{G_t^{-1}}{1 - \vec{\gamma} \cdot \vec{\gamma}_t} \left[ \vec{\gamma} + \vec{\gamma}_t \left( \frac{(G_t - 1)}{\gamma_t^2} \vec{\gamma} \cdot \vec{\gamma}_t - G_t \right) \right]. \]

Further support for identifying the parameter of translational boost \(z^A\) as a coordinate vector can be obtained as follows. Let us note that a point \(x^M\) on the six dimensional manifold satisfying

\[ \eta_{MN} x^M x^N = \eta_{AB} x^A x^B - (x^5)^2 = -\ell^2, \]

(45)
can be parametrized alternatively as (for $x^5 > 0$)

\[ x^4 = \ell \omega^A \sinh \zeta, \]
\[ x^5 = \ell \cosh \zeta, \]  
(46)

where $\omega^A$ denotes a unit vector on the 5D manifold satisfying $\eta_{AB} \omega^A \omega^B = 1$. It is clear now that we can identify the components of $X^M$ in Eq.(17) as

\[ G = \cosh \zeta, \quad z^A = \ell \omega^A \tanh \zeta, \quad \gamma^A = \frac{z^A}{\ell} = \omega^A \tanh \zeta, \quad G^2(1 - \gamma^2) = 1. \]  
(47)

This brings out the character of the (alternative) coordinate vector $z^A$ as a parameter of boost. (This can be contrasted with the angular representation of a Lorentz boost which has the form $\gamma = \cosh \theta, |\vec{\beta}| = \tanh \theta, \gamma^2(1 - |\vec{\beta}|^2) = 1$. Note the symbol $\gamma$ is used in both cases to represent different things.) Furthermore, with this identification, we recognize that the 6D vectors $x^M$ and $X^M$ can be related simply as

\[ x^M = \ell X^M. \]  
(48)

In fact, with the identifications in Eqs.\,(46) and (47), we note that if we identify

\[ \omega^A = \frac{x^A}{\sqrt{x^2}}, \]  
(49)

where $x^2 = \eta_{AB} x^A x^B$, we can write

\[ z^A = \ell \frac{x^A}{\sqrt{x^2}} \tanh \zeta = \ell \frac{x^A}{\ell \sinh \zeta} \tanh \zeta = \frac{x^A}{\cosh \zeta} = \ell \frac{x^A}{x^5}, \]  
(50)

which is, of course, the definition of Beltrami coordinates for the de Sitter manifold dS\(_5\) (on the Beltrami patch $x^5 > 0$). As we have mentioned earlier in connection with Eq.\,(24), the basic manifold of our theory is a 5D hypersurface dS\(_5\) of the 6D manifold and can be parametrized by five independent coordinates. The Beltrami coordinates (also known as gnomonic coordinates) provide a useful coordinate system which preserve geodesics as straight lines and, therefore, we can use $z^A, A = 0, 1, 2, 3, 4$ to parametrize our manifold.

We note here that there have been some studies on de Sitter special relativity [14] (which is Einstein special relativity formulated on a de Sitter, rather than a Minkowski space-time) where Beltrami coordinates are used. Some of the results obtained there may be used to shed more light on the physics of quantum relativity. However, we want to emphasize here that the (special) quantum relativity that we are discussing here is not just a version of de
Sitter special relativity. In particular, as we have discussed, the momentum boost transformations are expected to relate quantum frames of reference, and one should be cautious in borrowing physics results from Ref. [14]. In fact, the symmetric role of the relations of $x^4$ and $x^5$ coordinates to the 4D quantum noncommutative position and momentum operators, respectively, gives a new perspective on classical de Sitter physics.

Note that the translational boosts are reference frame transformations that correspond to taking the coordinate origin to a different location on the dS. The coordinate origin is always an important part of the frame of reference. The origin is where an observer measures locations of physical events from. Up to Einstein relativity, translation of coordinate origin is quite trivial. It does not change most of the physical quantities like velocity and energy-momentum measured. However, the situation is a bit different in the de Sitter geometry. Here, the coordinate origin can be unambiguously represented by $z^A = 0$, or location where the observer measures quantities including location $z^4$ of events from. The reference frame does not see itself, and must conclude that its own location is right at the origin of the coordinate system it measures event locations with. In terms of the 6-coordinate $X^M$, the origin has a single nonvanishing coordinate $X^5 = 1$. Consider an event seen as at location given by a nonvanishing $z^A \neq 0$, say with zero velocity and momentum for simplification, transforming to the new reference frame characterized by the event means translating the coordinate origin to that location, i.e. translating by $z^A$. One can check explicitly that the new coordinates of the event, $X'^M$ or $z'^A$, as seen from itself to be obtained from Eqs. (42) and (44) are indeed that of an origin. It also follows that composing translations gives nontrivial relativistic results beyond simple addition. In fact, comparing with the velocity and momentum composition formula of the Lorentz and de Sitter momentum boosts, respectively (cf. Eqs. (26,31)), we have the location composition formula

$$
\vec{z}'_1 = \frac{G^{-1}}{1 - \frac{\vec{z}_1 \cdot \vec{z}_2}{c^2}} \left[ \vec{z}_1 + \vec{z}' \left( \frac{G - 1}{z^2} \vec{z} \cdot \vec{z}_1 - G \right) \right].
$$

The formula gives the new location 5-vector $\vec{z}'_1$ of an original $\vec{z}_1$ boosted to a new frame characterized by $\vec{z}' \left( G^{-2} = 1 - \frac{{\eta}_{AB} z^A z^B}{c^2} \right)$. In particular, $\vec{z}'_1$ vanishes for $\vec{z} = \vec{z}_1$. On the other hand, simple addition of the 6-vector $X^M$'s does not preserve the de Sitter constraint of Eq. (45) characterizing the dS hypersurface. Likewise, conventional 6D translations are not admissible symmetries.

The dS hypersurface is obtained from a 6D manifold with a flat metric, $\eta_{MN}$, when
described in terms of the six coordinates $x^M$'s. When described in terms of the Beltrami coordinates $z^A$'s, however, the 5D metric is nontrivial and has the form

$$g_{AB} = G^2 \eta_{AB} + \frac{G^4}{\ell^2} \eta_{AC} \eta_{BD} z^C z^D .$$

(52)

The generators of the isometry group $SO(1, 5)$ can be expressed in terms of these variables as follows. We first note that $z_A = g_{AB} z^B = G^4 \eta_{AB} z^B$, (where we have used the definition $G^{-2} = 1 - \frac{\eta_{AB} z^A z^B}{\ell^2}$) leading to

$$x_A = \frac{1}{G^3} z_A , \quad \text{and} \quad x_5 = -G \ell .$$

(53)

Denoting $i \frac{\partial}{\partial z^A}$ by $q_A$, we have

$$p_A = i \frac{\partial}{\partial x^A} = \frac{i}{G} \frac{\partial}{\partial z^A} = \frac{1}{G} q_A ,$$

$$p_5 = i \frac{\partial}{\partial x^5} = \frac{\partial}{\partial z^5} i \frac{\partial}{\partial z^A} = \frac{\partial z^A}{\partial x^5} q_A = -\frac{1}{G \ell} q_A z^A .$$

(54)

Introducing a Lorentzian 5-coordinate $Z_A^{(c)} = G^{-4} z_A = \eta_{AB} z^B$, we have

$$[Z_A^{(c)}, q_B] = -i \eta_{AB} .$$

(55)

A form of Lorentzian ‘5-momentum’ as generators is given by

$$P_A^{(c)} = \frac{1}{\ell} J_A^{(c)} .$$

(56)

This is in agreement with the noncommutative momentum operator discussed in Sec.IV above. We have here

$$P_A^{(c)} = \frac{1}{\ell} \left( x_A p_5 - x_5 p_A \right) = q_A - Z_A^{(c)} \frac{1}{\ell^2} \left( \eta^{BC} Z_B^{(c)} q_C \right) .$$

(57)

The other ten generators are then given as

$$J_{AB} = Z_A^{(c)} q_B - Z_B^{(c)} q_A = Z_A^{(c)} P_B^{(c)} - Z_B^{(c)} P_A^{(c)} .$$

(58)

In fact, one can also write $J_{A5}$ formally as $Z_A^{(c)} P_5^{(c)} - Z_5^{(c)} P_A^{(c)}$ by taking $P_5^{(c)}$ as zero (since $Z_5^{(c)} = -\ell$). If one further writes an analogous relation $P_5^{(c)} = q_5 - \frac{1}{\ell} Z_5^{(c)} \left( \eta^{BC} Z_B^{(c)} q_C \right)$, it would require that $q_5 = -\frac{1}{\ell} \left( \eta^{BC} Z_B^{(c)} q_C \right)$. The latter is, of course, equivalent to $p_5 = \frac{1}{\ell^2} q_5$. It is interesting to note that $\left( \eta^{BC} Z_B^{(c)} q_C \right)$ resembles the conformal symmetry (scale transformation) generator for the 5-geometry with a Minkowski metric. A further interesting
point is that the fifth component of the ‘5-momentum’ now gives the operator responsible for the new quantum Heisenberg uncertainty relation as

\[ [\hat{X}_\mu, \hat{P}_\nu] = i \eta_{\mu\nu} \frac{1}{\kappa c \ell} J_{45} = -i \eta_{\mu\nu} \left( \frac{1}{\kappa c} P_4^{(c)} \right), \]

where \( \hat{P}_\nu = P_\nu^{(c)} \). This follows from Eq.\( (56) \) and the result of Sec.IV, namely, from Eq.\( (41) \) with \( O'_4 = J_{45} = -\kappa c \hat{F} \). The standard Heisenberg expression would be obtained for \( P_4^{(c)} \) taking the value \(-\kappa c\). The latter is likely to be admissible as an eigenvalue condition for the operator; and it invites comparison with the energy-momentum constraint \( p_4 = -\kappa c \) before the introduction of the deformation with the translational boosts. In case \( P_4^{(c)} \) takes other eigenvalues, it could be a generalization of the original energy-momentum constraint. Hence, the spectrum of \( P_4^{(c)} \) as an operator is of central importance.

VI. MORE ON THE MOMENTUM IN QUANTUM RELATIVITY

The interesting thing with the Beltrami coordinate formulation is that the first five coordinates of the 6-geometry \( x^A \) and hence the coordinates \( z^A = x^A \frac{\ell}{\kappa c} \) transform as components of coordinate vectors on a 5D space with Minkowski geometry. Similarly, \( Z^A_{(c)} \), \( q_A \), and \( P_A^{(c)} \), can all be considered Lorentzian 5-vectors. With the 5-coordinate description of the \( \text{dS}_5 \) geometry, the Lorentzian 5-coordinate \( Z^A_{(c)} \) and the 5-momentum \( P_A^{(c)} \) provide a very nice representation of the relativity symmetry algebra. Recall that we get to the relativity formulation through deformations in two steps. The first deformation is introduced by imposing the Planck scale as a constraint onto the Einstein 4-momentum (Eq.\( (12) \)). This promotes the 4-momentum into a 5-vector \( \pi^A = \Gamma_{\mu A} \frac{p^\mu}{\kappa c} \) with \( p^4 = \kappa c \). In terms of the original 6-coordinate \( x^M \) and the corresponding \( p_M = i \partial_M \), we also have the nice representation of the algebra with identification of the noncommutative quantum space-time position operators within the generators of the algebra given by Eqs.\( (39) \) and \( (40) \). There, we have Lorentzian 4-coordinate and 4-momentum \( \hat{X}^\mu \) and \( \hat{P}_\mu \) with a quite symmetric role, and an extra \( J_{45} \) characterizing the quantum uncertainty relation. The rest of the generators in the algebra are just the 4D Lorentz symmetry generators. The structure seems to be depicting a 4D quantum space-time. It is also easy to see that \( \hat{P}_4 = P_4^{(c)} \) together with the four \( \hat{P}_\mu = P_\mu^{(c)} \)'s transform as the 5-momentum of a 5D Lorentzian symmetry, with now \( \hat{X}^\mu \) forming part of the 5D angular momentum. The translational boosts mix the new, sixth momentum
component \( p^i \) with the other five making a consistent interpretation of the 5-momentum constraint of Eq.(10) questionable. It is then very reasonable to expect the momentum constraint to be taken as one on \( P^{(c)}_A \), which even has a natural extension to have a vanishing sixth component. Hence, we would like to think about \( P^{(c)}_A = \eta^{AB} P^{(c)}_B \). However, under the usual framework of dynamics, taking the operator form of the momentum in the (coordinate representation) to the momentum variable as (particle) coordinate derivative requires the equation of motion. Let us take a look at this issue from a different perspective.

We discuss below momentum as coordinate derivative and conserved vector corresponds to the relativity symmetry. We start from writing the “classical angular momenta” in terms of components of the relevant Beltrami 5-vectors. All the generators of \( SO(1,5) \) are expected to correspond to conserved quantities in the classical sense. We again start with \( L^{MN} = x^M p^N - x^N p^M \) where the “classical” momentum \( p^M \) is written as \( p^M = m_A \frac{dx^M}{ds} \). Here, \( m_A^2 \) is a mass-squarelike parameter not to be taken directly as the Einstein rest mass-square. It is likely to be some generalization of the latter. In fact, an apparent natural choice of the parameter \( m^2 \) is available from the eigenvalue of the first Casimir operator for the \( SO(1,5) \) symmetry \cite{14,15}. And the momentum as a coordinate derivative has, of course, to be defined with respect to the invariant line element \( ds \). In terms of the Beltrami 5-coordinates, we have

\[
L^A = -m_A G^2 \ell \frac{dz^A}{ds}. \tag{59}
\]

The expression should correspond to the conserved quantity for the five generators \( J^A = \ell \hat{P}_A \) [or \( \ell P^{(c)}_A \)]. Hence, we expect the conserved momentum to be given be

\[
P^A = m_A G^2 \frac{dz^A}{ds}, \tag{60}
\]

i.e., \( L^A = -\ell P^A \). It is interesting to note that \( \frac{dP^A}{ds} = 0 \) actually is equivalent to the geodesic equation within dS5 \cite{14}. The apparent “natural” momentum candidate \( q^A = m_A \frac{dz^A}{ds} \) is related to \( P^A \) in a way similar to the relation between the operator forms of \( P^{(c)}_A \) and \( q_A \). The coordinate transformation gives \( q^A = \frac{1}{G} p^A \), and hence

\[
P^A = G^2 q^A - m_A G \frac{dG}{ds}. \tag{61}
\]

Also we have

\[
L^{AB} = z^A G^2 q^B - z^B G^2 q^A = z^A P^{(c)}_B - z^B P^{(c)}_A, \tag{62}
\]
with the same form applicable to $L^A$ by taking $P^A_{(c)} = 0$ or equivalently $q^5 = m\frac{dG}{ds} = \frac{1}{G}p^5$.

The latter result may be written in a form similar to the operator $q_5$ given in the previous section, namely $q^5 = G^2\eta_{AB}z^Aq^B$. Note that $g_{AB}P^A_{(c)}P^B_{(c)} = m_0^2G^4$.

The above definition of “classical” momentum $p^M$ of course goes in line with the Newtonian/Einstein starting point of mass times velocity. However, as discussed in some details in Sec.III, we have to introduce the new nonquantum-relativistic Einstein momentum defined as $\frac{dx^A}{ds} = \kappa c\frac{dx^A}{d\tau}$ instead of the conventional Einstein proper time derivative $m\frac{dx^A}{d\tau}$. Taking $\kappa c$ as the natural momentum unit, we have from Eq.(10) the momentum $\pi^A$ as essentially nothing more than the derivative $\frac{dx^A}{ds}$. Taking this to the 6-geometry, we introduce the natural definition $\pi^M = i\frac{dx^M}{ds}$. The reason for introducing the $i$ is the following. The coordinate $x^4$ has the opposite metric signature relative to $\tau$ or $x^0$. Only with the $i$ we can have the result $\eta_{MN}\pi^M\pi^N = -1$. Now, we can go along the argument as we have done for $p^M$. Write the conserved quantities as

$$\frac{1}{\kappa c}J^{MN} = x^M\pi^N - x^N\pi^M = i\left(z^M\Pi^N_{(c)} - z^N\Pi^M_{(c)}\right),$$

where

$$\Pi^A_{(c)} = -\frac{1}{\kappa c\ell}J^A = iG^2\frac{dz^A}{ds}, \quad \text{and} \quad \Pi^5_{(c)} = 0. \quad (64)$$

Note that we have put in the factor $\kappa c$ to get the unit right. Obviously, the geodesic equation can also be considered as $\frac{d\Pi^A_{(c)}}{ds} = 0$, as $\Pi^A_{(c)}$ differs from $P^A_{(c)}$ only by a constant factor. The appearance of the $i$ in the above may be taken as an illustration of the intrinsic quantum nature of the formulation. In the place of the energy-momentum constraint of Eq.(11), we have instead

$$g_{AB}\Pi^A_{(c)}\Pi^B_{(c)} = -G^4,$$

which reduces to unity at the coordinate origin $z^A = 0$.

VII. SUMMARY

In this paper, we have proposed a simple linear realization of the $SO(1,5)$ symmetry as the Lie-algebraic description of quantum relativity. The relativity follows from successive deformations of Einstein special relativity through the introduction of invariant bounds on energy-momentum and on extended geometric (“space-time”) interval. The invariants
are related respectively to the Planck scale and the cosmological constant. We have discussed the logic of our formulation, and plausible physical interpretations that we consider to be naturally suggested by the latter. The linear realization has a six-dimensional geometric description, with the true physics restricted to a $dS_5$ hypersurface embedding the standard four-dimensional Minkowski space-time. The relativity algebra may be taken as the phase-space symmetry of the quantum (noncommutative) four-dimensional space-time with a natural Minkowski limit. We focus mostly on the five or six-dimensional geometric description with quite unconventional coordinate(s) (as we have argued) beyond the conventional space-time ones. There remain several open questions such as a new definition of energy-momentum in the nonquantum-relativistic limit. Our analysis aims at taking a first step in an exploration that may complement the previous approaches on the subject matter. It certainly raises some interesting questions that we hope to return to in future publications.

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[13] J. Kowalski-Glikman and S. Nowak, Int. J. Mod. Phys.D **12**, 299 (2003); note that the matching of the algebra to our notation here is given by $\tilde{X}_i \rightarrow -\hat{X}_i$ and $\tilde{X}_0 \rightarrow \hat{X}_0$ and $\ell \rightarrow \frac{1}{\kappa c}$ ($\ell$ in the paper is not the same as $\ell$ used here which characterizes the next deformation).
