Gravitational lensing due to dark matter modelled by vector field

V.V.Kiselev*† and D.I.Yudin†

* Russian State Research Center “Institute for High Energy Physics”, Pobeda 1, Protvino, Moscow Region, 142281, Russia
† Moscow Institute of Physics and Technology, Institutskii per. 9, Dolgoprudnyi Moscow Region, 141700, Russia
E-mail: kiselev@th1.ihep.su

Abstract. The specified constant 4-vector field reproducing the spherically symmetric stationary metric of cold dark matter halo in the region of flat rotation curves results in a constant angle of light deflection at small impact distances. The effective deflecting mass is factor $\pi/2$ greater than the dark matter mass. The perturbation of deflection picture due to the halo edge is evaluated.

PACS numbers: 04.25.-g, 04.40.-b, 04.20.-q, 98.62.Sb, 95.35.+d, 98.62.Gq

1. Introduction

For spherically symmetric stationary dark halos in spiral galaxies one could explore the weak field approximation of pressureless matter implying the metric

$$ds^2 = \{1 + 2\phi_{GR}(r)\}dt^2 - \frac{dr^2}{1 - \frac{2GM_{DM}(r)}{r}} - r^2[d\theta^2 + \sin^2\theta d\phi^2],$$

(1)

where $M_{DM}(r)$ denotes the dark matter mass inside the sphere of radius $r$, since to linear order in small $M_{DM}(r)/r$ and $\phi_{GR}(r)$ the Einstein equations give

$$8\pi G\rho(r) = G^t_t = R^t_t - \frac{1}{2}g^t_t R = \frac{2GM_{DM}(r)}{r^2}$$

(2)

exactly yielding

$$\int_0^r d^3r \rho(r) = M_{DM}(r),$$

(3)

while $\phi_{GR}(r)$ represents the gravitational potential of dark mass

$$\phi_{GR}(r) = \int_0^r dr \frac{GM_{DM}(r)}{r^2}$$

(4)
Gravitational lensing... by vector field

producing the force

\[ F = \nabla \phi_{GR}(r) = -\frac{GM_{DM}(r)}{r^2} \frac{r}{r} \]  

(5)
as well as a negligible pressure

\[ 8\pi G p_n(r) = -\frac{G \rho}{n} \approx 0, \quad n = \{r, \theta, \varphi\}. \]  

(6)

For flat rotational curves with the characteristic velocity \( v_0^2 \) we get

\[ \frac{v_0^2}{r} = \frac{G M_{DM}(r)}{r^2} \]  

(7)

and, hence,

\[ M_{DM}(r) = \frac{1}{G} v_0^2 r, \]  

(8)

which argues for the metric

\[ ds^2 = f(r) dt^2 - \frac{1}{h(r)} dr^2 - r^2 [d\theta^2 + \sin^2 \theta d\phi^2], \]  

(9)

with

\[ h(r) = 1 - 2v_0^2, \quad f'(r) = \frac{2v_0^2}{r}. \]  

(10)

We consider the metric is close to the Minkowskian one up to small corrections in \( v_0^2 \to 0 \), so that

\[ f(r) = 1 + 2v_0^2 \ln r. \]  

(11)

Surprisingly, metric (9) was derived for the dark matter modelled by 4-vector field \( A_\nu \) with constant components [1]

\[ A_0 = \frac{1}{\sqrt{8\pi G}}, \quad A = \frac{v_0^2}{\sqrt{4\pi G}} \frac{r}{r}. \]  

(12)

In [2] we supposed a treating the vector field in terms of ghost condensate [3–10]: \( \partial_t \phi = \dot{\phi} = A_0 \), and global spherically symmetric monopole \( \psi(r) = \nabla \chi(r) = A(r) \sim n = r/r \), combined into the 4-vector \( A_\nu \). The phenomenon of ghost condensate is an analogue of Higgs mechanism for the condensation of scalar field with negative square of mass near zero values of field. This mechanism of ghost condensation usefully works in the case of expanding homogeneous universe, that has a specific reference frame with a selected temporal component in comparison with spatial ones. Then, the scalar field with negative sign of kinetic term near zero values of kinetic energy in its action composed of invariant derivative terms \( X = \partial_\mu \phi \partial^\mu \phi \), condenses at a nonzero value of \( \dot{\phi} \), which represents a temporal component of 4-vector field. The ghost condensation removes the problem with negative kinetic energy, since near the condensate the fluctuations have correct sign of kinetic energy, of course. The studies [1] show, that field \( \dot{\phi} \) should be constant due to the current evolution of universe, with a high accuracy. Next, one could combine the ghost condensate with the global monopole, that produces a correct static spherically symmetric profile of mass distribution \( \rho \sim 1/r^2 \) in the region of flat rotation curves due to the dark
Gravitational lensing... by vector field

matter. Therefore, one should suppose the presence of global monopole in the center of spiral galaxy, i.e. inside a supermassive black hole. At large distances, the monopole corresponds to the spatial component of vector field. Then, one considers the background field \( \phi = \mu^2 t - \mu^2 r \) at large distances (perturbations become important at small distances, corresponding to inner edge of halo), where \( \mu \) is the scale about the Planck mass, while the ratio of \( \mu/\mu^2 \) determines the rotation velocity \( v_0 \) (see details in [2]). So, the ghost condensate in the presence of global monopole with account of higher derivative terms in the action can be represented by the constant 4-vector field \( A_\mu \) relevant to the situation of both universe expansion and dark halo.

In the present paper we calculate the light bending by the metric modelling the dark halo. In Section 2 we in detail calculate the deflection angle in the limit of ‘infinite’ halo implying a negligible value of impact distance with respect to the halo size and compare the result with the deflection by a global monopole [11]. Then we determine the effective deflecting mass and find its difference by a factor of \( \pi/2 \) from the dark matter mass responsible for the flat rotation curves in contrast to naive expectations of their coincidence. Further, we describe the geometry of deflection revealing properties consistent with astronomical observations. In Section 3 we calculate the deflection in the case of finite halo, that demonstrates a weak dependence on the impact distance slowly perturbing the geometry discussed in Section 2. The results are summarized in Conclusion.

2. Limit of infinite halo

By the Hamilton–Jacobi equation

\[
g^{\mu\nu} \partial_\mu S \partial_\nu S = 0 \quad (13)
\]

for an action \( S \) of massless particle written in the form

\[
S = -E t + \mathcal{M} \theta + S(r), \quad (14)
\]

incorporating two integrals of motion in the spherically symmetric static gravitational field at fixed polar angle \( \phi = \text{const.} \), so that \( E \) is the conserved energy of particle and \( \mathcal{M} \) is its rotational momentum, which we put positive for definiteness, we deduce

\[
\left( \frac{\partial S}{\partial r} \right)^2 = \frac{1}{\mathcal{f} \hbar} E^2 - \frac{1}{\hbar} \frac{\mathcal{M}^2}{r^2}, \quad (15)
\]

which results in

\[
S(r) = \pm \int_{r_0}^{r(t)} \frac{dr}{\sqrt{E^2 - V^2(r)}}, \quad (16)
\]

where \( V^2 \) is an analogue of centripetal potential,

\[
V^2(r) = f \frac{\mathcal{M}^2}{r^2}.
\]
Thus, the quantity $S(r)$ represents the action for the radial motion of particle in the gravitational field with account of energy due to the angular rotation. The trajectory is implicitly determined by the equations:

\[
\frac{\partial S}{\partial E} = \text{const} = -t + \int_{r_0}^{r(t)} \frac{1}{\sqrt{\frac{f}{h} \sqrt{E^2 - V^2(r)}}} \, dr.
\]  

(17)

\[
\frac{\partial S}{\partial M} = \text{const} = \theta - \int_{r_0}^{r(t)} \frac{f}{\sqrt{\frac{f}{h} r^2 \sqrt{E^2 - V^2(r)}}} \, dr.
\]  

(18)

Taking the derivative of (17) and (18) with respect to the time, we get

\[
1 = \dot{r} \frac{\mathcal{E}}{\sqrt{\frac{f}{h} \sqrt{E^2 - V^2(r)}}},
\]  

(19)

\[
\dot{\theta} = \frac{\dot{r}}{r^2} \frac{f}{\sqrt{\frac{f}{h} \sqrt{E^2 - V^2(r)}}} \frac{M}{\mathcal{E}},
\]  

(20)

and, hence,

\[
\mathcal{E} = \frac{f}{r^2 \dot{\theta}} M,
\]  

(21)

relating the energy and rotational momentum. Let us denote the ratio of two integrals by

\[
\frac{M}{\mathcal{E}} = a
\]

with the dimension of length, which we call as ‘an impact distance’, that becomes clear from Fig. 1 shown for the flat Minkowskian space-time.

\[\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Geometrical relations between the impact distance $a$ and circular velocity $r\dot{\theta}$.}
\end{figure}\]

At the trajectory given by (19) and (20) we get

\[
\frac{d\theta}{dr} = \pm \frac{a}{r^2} \frac{\sqrt{\frac{f}{h} \sqrt{E^2 - V^2(r)}}}{\sqrt{1 - \frac{a^2 f}{r^2}}},
\]  

(22)

\[\because \text{Here we take the positive sign of square root for brevity of record.}\]

\[\because \text{As usual } \partial_t f(t) = \dot{f}.\]
where the negative sign corresponds to the branch of trajectory for the particle arriving from infinity, while the positive sign does to the branch for the particle departing to infinity. Denote the solution of

$$r^2 = a^2 f(r)$$

by \( r_{\text{min}} \). At this value the radial velocity \( \dot{r} \) becomes equal to zero (the denominator in Eq. (22) vanishes), that corresponds to returning point. So, the light ray approaches the gravitating object from the infinity, while it reaches the distance \( r_{\text{min}} \), after that the ray moves off the object. Therefore, the quantity \( r_{\text{min}} \) denotes the minimal distance between the light trajectory and the gravitational lens. Then, the overall change of \( \theta \) for the particle is given by

$$\Delta \theta_G = 2 \int_{r_{\text{min}}}^{\infty} dr \frac{a}{r^2} \sqrt{\frac{f}{h}} \frac{1}{\sqrt{1 - \frac{a^2 f}{r^2}}},$$

(23)

if we put the size of halo \( r_* \) much greater than the impact distance: \( a^2 f/r_*^2 \rightarrow 0 \), that corresponds to the limit of infinitely large halo. In contrast to [12], we expect that this limit is relevant to actual astronomical observations.

Let us integrate out (23) by expanding in small parameter of \( v_0^2 \rightarrow 0 \). So, with the accuracy up to the first order we get

$$f = f(r_{\text{min}}) + 2v_0^2 \ln \frac{r}{r_{\text{min}}} + \mathcal{O}(v_0^4),$$

$$h = 1 - 2v_0^2,$$

that allows us to write down

$$\Delta \theta_G = \Delta \theta_0 + v_0^2 (\Delta \theta_1 + \Delta \theta_2 + \Delta \theta_3) + \mathcal{O}(v_0^4),$$

(24)

where the leading term

$$\Delta \theta_0 = 2 \int_{r_{\text{min}}}^{\infty} dr \frac{a}{r^2} \sqrt{f(r_{\text{min}})} \frac{1}{\sqrt{1 - \frac{a^2 f(r_{\text{min}})}{r^2}}}$$

(25)

after the substitution

$$z = \frac{a}{r} \sqrt{f(r_{\text{min}})}$$

(26)

is reduced to

$$\Delta \theta_0 = 2 \int_0^1 \frac{dz}{\sqrt{1 - z^2}} = 2 \arcsin z \bigg|_0^1 = \pi,$$

(27)

which corresponds to the straightforward motion with no deflection. The contribution \( \Delta \theta_1 \) originates from the factor of \( 1/\sqrt{h} \approx 1 + v_0^2 \) in the integrand of (23), so that

$$\Delta \theta_1 = \Delta \theta_0 = \pi.$$  

(28)
The expansion of
\[ \sqrt{f} = \sqrt{f(r_{\text{min}})} - v_0^2 \ln z + \mathcal{O}(v_0^4) \]
results in
\[ \Delta \theta_2 = -2 \int_0^1 \frac{\ln z}{\sqrt{1 - z^2}} \] (29)
where we have neglected the deviation of \( f(r_{\text{min}}) \) from unit
\[ f(r_{\text{min}}) = 1 + \mathcal{O}(v_0^2). \]
Analogously, the expansion of
\[ \sqrt{1 - \frac{a^2 f}{r^2}} = \sqrt{1 - z^2} - v_0^2 \frac{z^2 \ln z}{\sqrt{(1 - z^2)^3}} + \mathcal{O}(v_0^4) \]
produces
\[ \Delta \theta_3 = -2 \int_0^1 \frac{z^2 \ln z}{\sqrt{(1 - z^2)^3}}. \] (30)
Summing up (29) and (30) gives
\[ \Delta \theta_2 + \Delta \theta_3 = -2 \int_0^1 \frac{\ln z}{\sqrt{(1 - z^2)^3}}. \] (31)
The integral is easily calculated by parts
\[ -2 \int \frac{\ln z}{\sqrt{(1 - z^2)^3}} = - \frac{2z}{\sqrt{1 - z^2}} \ln z + 2 \int \frac{dz}{\sqrt{1 - z^2}}, \]
so that
\[ \Delta \theta_2 + \Delta \theta_3 = \pi. \] (32)
Thus, the deflection of light is determined by the angle
\[ \Delta \theta = \Delta \theta_G - \pi = v_0^2 (\Delta \theta_1 + \Delta \theta_2 + \Delta \theta_3) = 2\pi v_0^2. \] (33)
Note, this limit of deflection angle is independent of impact distance. This result is similar to the calculation of light deflection by the global monopole metric in [11], where the radial component \( h \) can be taken in the same form, while the temporal component of metric is constant, \( f \rightarrow \text{const.} \), which corresponds to the term of \( \Delta \theta_1 \) in calculations above. Therefore, we emphasize that the deflection angle in the metric of global monopole is twice less than the calculated value of \( \Delta \theta = 2\pi v_0^2 \).

The bending angle derived is in agreement with results of [13], where the deflection due to the dark matter with a significant pressure was calculated numerically at various equation of state parameters \( w_n = p_n/\rho \), and the limit of pressureless matter was mentioned, too. The same result could be extracted from [12] by calculating the limit of \( v_0^4 \rightarrow 0 \) at infinitely small impact distances from analytical expressions in [12] also discussed in Section 3.
2.1. Deflecting dark matter

The main feature of deflection angle under study is its independence of impact distance. Such the deflection can be ascribed to an effective dark matter mass. Indeed, let us compare the effect with the deflection by a nonrelativistic spherical mass distribution described by the Schwarzschild metric:

$$\Delta \theta_{BH} = \frac{4MG}{r_{\text{min}}}.$$  \hfill (34)

Putting

$$\Delta \theta = \frac{4\tilde{M}_{\text{DM}}(r_{\text{min}})G}{r_{\text{min}}}$$  \hfill (35)

we derive the expression for the effective dark matter mass responsible for the deflection

$$\tilde{M}_{\text{DM}}(r) = \frac{\pi}{2G} v_0^2 r,$$  \hfill (36)

which \textit{linearly depends on the distance}. Compare it with the dark matter mass responsible for the centripetal acceleration in the region of flat rotational curves,

$$\frac{v_0^2}{r} = \frac{GM_{\text{DM}}(r)}{r^2} \Rightarrow M_{\text{DM}}(r) = \frac{1}{G} v_0^2 r.$$  \hfill (37)

Therefore, in the metric under consideration the effective deflecting mass of dark matter is factor $\pi/2$ greater than the dark matter mass producing flat rotational curves in Newtonian limit.

The same conclusion is drawn from virial estimates of dark matter mass in galaxy clusters, where we get the relation

$$\langle v^2 \rangle = \frac{G\langle M_{\text{DM}} \rangle}{2\langle r \rangle},$$

since the average kinetic energy of $N$ galaxies with average masses $\langle M_{\text{galaxy}} \rangle = \langle M_{\text{DM}} \rangle/N$ at $N \gg 1$

$$\langle T_{\text{kin}} \rangle = \frac{1}{2} N \langle M_{\text{galaxy}} \rangle \langle v^2 \rangle = \frac{1}{2} \langle M_{\text{DM}} \rangle \langle v^2 \rangle$$

is related with the average potential energy of galaxies posed at average distance $\langle r \rangle$

$$\langle V \rangle = -\frac{N^2}{2} \frac{G \langle M_{\text{galaxy}} \rangle^2}{\langle r \rangle} = -\frac{1}{2} \frac{G \langle M_{\text{DM}} \rangle^2}{\langle r \rangle}$$

by the virial theorem

$$2\langle T_{\text{kin}} \rangle + \langle V \rangle = 0.$$

Both estimates produce the similar linear dependence of dark matter mass versus the distance, as we retrieve from the gravitational lensing.

Remarkably, astronomical observations of galaxy clusters give

$$\langle M_{\text{DM}} \rangle \sim 200\langle M_B \rangle,$$  \hfill (38)
where $\langle M_B \rangle$ is the mass of luminous baryonic matter in the clusters. Note, that the mass of luminous matter is related with the velocity of rotation in spiral galaxies at distances of flat rotational curves by the empirical Tully–Fisher scaling law

$$\langle M_B \rangle \sim v_0^4,$$

which is natural, if one suggests the existence of universal critical Milgrom acceleration [14]

$$a_0 = \frac{v_0^2}{r_0} = \frac{G \langle M_B \rangle}{r_0^2},$$

where $r_0$ is a characteristic distance for a galaxy, where the dominance of baryonic matter in the gravitation comes to competition with the dark matter contribution becoming dominant at larger distances. The value of $r_0$ is close to the visible size of galaxy, i.e. it is the characteristic size of baryon distribution in the galaxy. So, let us compare the masses of luminous and dark matters:

$$\langle M_B \rangle \approx \frac{1}{G} v_0^2 r_0, \quad \langle M_{DM} \rangle \approx \frac{1}{G} v_0^2 r_\star,$$

where $r_\star$ is the size of dark halo. Therefore, we conclude that

$$r_\star \approx 200 r_0,$$

i.e. the dark halo spreads far away from the visible size of luminous object. This point supports the relevance of considering the limit of infinite halo in the problem of gravitational lensing at realistic impact distances about the visible size of lens.

Thus, the vector field model of dark matter qualitatively reproduces both dark matter effects for astronomical objects: the flat rotational curves and light bending. The numerical effect on about 50%-difference between the effective deflecting mass and dark matter gravitational mass in rotational curves could probably be used in future for falsifying the model, though such a combination of astronomical data seems to be quite exclusive.

Another aspect of comparing the rotation curve picture in a galaxy with the gravitational lensing by the same galaxy has been recently considered in [15], where the potentials responsible for the rotation and lensing have been introduced. The difference between the potentials would signalize on a non-zeroth pressure of dark matter. In the case under study these potentials coincide, so the lensing of dark mass distribution could be evaluated by standard techniques for the cold pressureless matter. In contrast to the potentials, we operate by the effective deflecting mass allowing us to compare it with the mass one could expect from the rotation velocity. This could be important for a treatment of some misalignment between the lensing mass and dark matter mass in astronomical measurements of X-band lenses [16].

### 2.2. Geometry of deflection

Let us consider the geometric aspects of gravitational lensing. In Fig. 2 point S denotes a star, which light reaches an observer posed in point O by path SBO curved by dark matter object somewhere placed at interval BD. Since all of angles are extremely small, while distances between the observer, dark matter object and star are astronomically large,
Gravitational lensing... by vector field

we can perform calculations in the first order versus angles. Then, the distance between the deflection point and position of dark matter object projected to the line between the observer and star \( r = BD \) (see Fig. 2) equals
\[
r = l \alpha; \quad r = d \beta. \tag{43}
\]
Points \( D \) and \( D' \) coincide because \(|DD'| \approx l \alpha^2 = \mathcal{O}(\alpha^2)\) is negligible. The angle of deflection \( \Delta \theta \) is independent of \( r \) and
\[
\Delta \theta = \alpha + \beta. \tag{44}
\]
Therefore,
\[
\beta = \Delta \theta \frac{l}{l + d}. \tag{45}
\]
The same deflection takes place by the path in the upper half of plane shown in Fig. 2.

\[\text{Figure 2. The plane geometry of light bending.}\]

\[\text{Figure 3. The gravitational lensing of linear image by the dark matter object (left) in comparison with the initial pattern without the light bending (right).}\]

Thus, the angle between two images of star is given by
\[
\Delta \beta = 2 \beta = 2 \Delta \theta \frac{l}{l + d}. \tag{46}
\]
Gravitational lensing... by vector field

This geometry can be easily applied to draw samples of gravitational lensing in the model with constant $\Delta \theta$.

So, Fig. 3 represents the gravitational magnifying of linear image at $d = 1$, $l = 10$ and small angles. The picture illustrates the characteristic distortion caused by such lenses. For simplicity of picturing, in Figs. 3-6 we put the parameter $v_0^2 = 1/(40\pi) \approx 0.008 \approx (0.09)^2$ in order to enlarge the visual effect. Nevertheless, in practice, an angular magnification of real astronomical facilities is sufficiently large for the observation of gravitational lensing by the dark matter (see numerical estimates in the end of section).

One can easily recognize basic features of deflection and distortion:

(i) The dark matter object is posed in the center of inversion symmetry.

(ii) Hence, the angle distances between the points inside two images coincide. (If one displaces the star image by angle $\gamma$, then the second image is displayed to the same angle increment $\gamma$.)

(iii) Therefore, small radial widths of images are coincide, while the radial sizes are proportional. The coefficient of proportionality $\kappa$ is equal to the ratio of angle fractions given by division of $\Delta \beta$ by the center of symmetry, i.e. the dark matter object. This point is shown in Fig. 4.

(iv) Then, the ratio of surface areas is equal to $\kappa$ as illustrated in Fig. 5 for the distortion of circle object.

---

**Figure 4.** The gravitational lensing of centered circular image by the dark matter object (*left*) in comparison with the initial pattern without the light bending (*right*).

**Figure 5.** The gravitational lensing of circular image by the dark matter object (*left*) in comparison with the initial pattern without the light bending (*right*).
Thus, for the model of gravitational lensing, we arrive to the following conclusion:

The relative values of surface brightness for two images of lensed small object is given the inverse ratio of angle fractions, \( \kappa \), or visible impact parameters.

So, the ordinary picture of gravitational lensing due to the dark matter modelled by the vector field is shown in Fig. 6. We have taken two galaxies as elliptic figures at different distances \((d = 1, l_1 = 9, l_2 = 3)\). Their images are narrow arcs surrounding the dark matter object, that is generically similar to the real astronomical picture inserted as illustration.

![Figure 6](image.png)

**Figure 6.** The gravitational lensing of sample galaxies at different distances by the dark mater object (*top-left*) in comparison with the initial pattern without the light bending (*top-right*) and with the real picture of Abell 2218 cluster lensing (*bottom*) [from NASA, 1998].

### 2.3. Numerical values

The characteristic angle between two images for an object due to the gravitational lensing is about

\[
\Delta \beta \simeq 2 \Delta \theta = 4\pi v_0^2 \approx 2.6 \cdot 10^6 v_0^2 \text{ arcsec.}
\]
Gravitational lensing... by vector field

A typical velocity of rotation is in the range of $100 - 400$ km/sec, i.e. $v_0^2 \sim 10^{-6}$. Then,

$$\Delta \beta \sim 2 \text{arcsec.}$$  \hspace{1cm} (48)

Such the value is consistent with astronomical observations of gravitational lensing, where the characteristic angle sizes between the dark matter object and image of lensed luminous object is about 1 arcsec.

3. Edge of halo

If the impact distance is comparable with the size of dark halo, then one has to take into account the halo edge at $r = r_*$ (see Fig. 7). So, the formulae exploited above give the angle increment in the halo,

$$\Delta \theta_G(r_*) = \pi - 2 \arcsin z_* + v_0^2 \left(2\pi - 4 \arcsin z_* + 2 \frac{z_* \ln z_*}{\sqrt{1 - z_*^2}}\right),$$  \hspace{1cm} (49)

with $z_* = a/r_*$.  

![Figure 7. The propagation of light through the dark halo of finite size $r_*$.

Let us suppose that beyond the halo we get the Schwarzschild metric with the gravitational radius $r_g$: $f, h \mapsto 1 - \frac{r_g}{r}$. Then, the increment of angle beyond the halo is determined by the integral

$$\Delta \theta_{BH}(r_*) = 2 \int_{r_*}^{\infty} dr \frac{a}{r^2} \frac{1}{\sqrt{1 - \frac{a^2}{r^2} \left(1 - \frac{r_g}{r}\right)}},$$  \hspace{1cm} (50)

which can be easily evaluated after the substitution

$$r = \rho - \frac{1}{2} r_g,$$
Gravitational lensing... by vector field

and expansion to the first order in \( r_g/a \). So, we get

\[
\Delta \theta_{\text{BH}}(r_*) \approx 2 \int_{r_*}^{\infty} \frac{d\rho}{\rho^2} \frac{a \left( 1 + \frac{r_g}{\rho} \right)}{\sqrt{1 - \frac{a^2}{\rho^2}}} = 2 \arcsin \frac{z_*}{a} + 2 \frac{r_g}{a} \left( 1 - \sqrt{1 - z_*^2} \right). \tag{51}
\]

The value of gravitational radius could be fixed by the dark matter mass at \( r_* \),

\[
r_g = 2GM_{\text{DM}}(r_*) = 2v_0^2r_* \tag{52}
\]

Then, the overall deflection angle

\[
\Delta \theta_* = \Delta \theta_{\text{C}}(r_*) + \Delta \theta_{\text{BH}}(r_*) - \pi
\]

is given by

\[
\Delta \theta_* = v_0^2 \left\{ 2\pi - 4 \arcsin z_* + 2 \frac{z_* \ln z_*}{\sqrt{1 - z_*^2}} + \frac{4}{z_*} \left( 1 - \sqrt{1 - z_*^2} \right) \right\} \tag{53}
\]

valid at \( 0 \leq z_* \leq 1 \), while at \( z_* > 1 \) we get the ordinary angle of bending \( \Delta \theta_{\text{BH}} = 2r_g/a \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig8.png}
\caption{The deflection angle light by the dark matter with the edge at \( r = r_* \) versus \( z_* = a/r_* \) under matching with the Schwarzschild metric at \( r_g = 2v_0^2r_* \) (left) and \( r_g = 3v_0^2r_*/2 \) (right) in comparison with the deflection by the massive point of \( M = r_g/(2G) \) (dotted line).}
\end{figure}

Note, that the derivative of deflection angle has a singularity at \( z_* \rightarrow 1 \),

\[
\frac{d\Delta \theta_*}{dz_*} \rightarrow \frac{v_0^2}{\sqrt{1 - z_*^2}} \left( \frac{2r_g}{v_0^2r_*} - 3 \right), \quad z_* \rightarrow 1. \tag{54}
\]

The singularity reveals as a fracture point of curve in Fig. The fracture disappears at \( r_g = \frac{3}{2}v_0^2r_* \). However, such the redefinition of parameter \( r_g \), removing the fracture of bending curve (the derivative of \( \Delta \theta_* \) with respect to \( z_* \) becomes a continuous curve), results in a discontinuity of metric coefficient \( h \) at the edge of halo, that is absolutely unacceptable.
The origin of singularity in Eq. (54) is the specific dependence of halo metric-coefficient \( f \) on the distance \( r \). So, it produces a slow deflection with negative sign in vicinity of halo edge \((z_\star = 1)\), i.e. there is a small repulsive component of force acting to the light ray in comparison with the case of point-like mass. It is clearly seen in derivation of Eqs. (30) and (31). Then, the attraction by the halo near the edge is small in comparison with the case of point-like mass, hence, the substitution of trajectory passing the halo instead of the same path in the field of point-like mass results in an essentially smaller deflection, and we even get the fracture of curve for the bending angle, while the deflection increases when the path in the halo becomes longer. Thus, the fracture is the direct consequence of metric, specified for the dark halo. Moreover, one could speak about a gravitational refraction of light on the border of two media with different metrics (in the case under study, we get the transition from the vacuum to the halo medium).

The analysis shows that in quite a wide region of impact factors, the deflection by the dark halo with the edge is slowly perturbed due to the edge: the dependence on the impact distance is weak in comparison with the rapid change in the case of point-like mass. So, the picture of deflection considered in Section 2 qualitatively remains the same.

**Comparison with the paper by Nucamendi, Salgado, Sudarsky [12].** In [12] authors in part considered the same problem of light bending by the dark halo with the edge. The difference is twofold. First, the metric coefficient \( f \) in [12] has been taken in the form

\[
f = A \left( \frac{r}{r_0} \right)^{2v_0^2}.
\]

Second, the deflection by the Schwarzschild metric calculated by expanding in powers of inverse distance has been written in a form, which is literally different, but in fact it should coincides with the formula (54) with the accuracy up to \( O(r_0^2/a^2) \). These facts allow us to make a direct comparison in the limit of \( v_0 \to 0 \). So, we find that our result is consistent with the corresponding limit of expression in [12] with the accuracy mentioned above. In part, the limit of infinite halo \( z_\star \to 0 \) in [12], reproduces the value of \( \Delta \theta = 2\pi v_0^2 \), which is actually in the evident contradiction with the main result of [12] shown graphically as the function of \( z_\star \) (see Fig. 1 in [12]). Indeed, instead of \( 2\pi \) one could observe a number, which is factor \( 4/3 \) greater. The same factor remains, when we compare the complete curve in [12]: the deflection angle of dark halo with the edge is multiplied by \( 4/3 \). Literally, the curve of [12] could be reproduced from our right curve in Fig. 8 multiplied by \( 4/3 \), that gives the correct value for the bending by the Schwarzschild metric, but it misleads on the deflection by the halo. The factor \( 4/3 \) removes the fracture point from the curve, of course, as it is clearly seen in Fig. 1 in [12].

Thus, our analysis on the deflection angle by the dark halo with edge is consistent with the formal limit of expressions in [12], but we disagree with the numerical in-figure-shown result of [12].
An inner edge. We have considered the external edge of halo at large distances. However, the luminous matter essentially contributes to the deflection at short impact distances, where the gravitational field is dominantly determined by distributions of stars and gas. So, there is the inner edge of halo. This problem has been comprehensively investigated in terms of Modified Newtonian Dynamics (MOND) [14]. Recently, Bekenstein formulated a consistent relativistic theory of gravitation including gravitational scalar and vector fields in addition to the ordinary tensor of metric [17], so that it incorporates general MOND effects originated from the introduction of critical acceleration \( a_0 \). The authors of [18] have found the same value of deflection for the case of infinite halo at impact parameters greater than the inner edge, that corresponds to the isothermal halo of circular velocity \( v_0 \). The comparison with observational data has been performed, so that the conclusion drawn is the following: MOND is not in conflict with the data on the gravitational lensing. Then, the analysis of [18] supports the model considered in this paper, which includes general features of MOND, too.

4. Conclusion

Briefly summarizing the consideration, we have described the deflection of light by the dark mater halo modelled by the constant 4-vector field, in detail. The analytical results are in agreement with the limit of small velocity of rotation \( v_0^2 \to 0 \) in previous studies.

The deflection angle for the infinite halo is twice of the value for the global monopole. The effective lensing mass is factor \( \pi/2 \) greater than the dark matter mass at the same distance. For the typical rotation velocity in spiral galaxies, the characteristic value of deflection angle is about several arcseconds, that is consistent with values observed astronomically. The relative surface brightness of two images of lensed object is determined by the inverse ratio of impact angles with respect to the dark lens.

The effect of halo edge is reduced to slow dependence of deflection angle on the impact distance, that weakly perturbs the gravitational lensing.

In conclusion, we have to make several important comments:

First, it would be wrong to identify the model of halo accepted in the paper with known model of isothermal halo. Such the opinion could not be correct because of the following reasons:

i) The physics of vector field model in the paper and the physics of general isothermal model are certainly different.

ii) The vector field model of dark matter includes inner and outer edges, so that between the edges the metric is described by the same metric as for the isothermal halo. Hence, the deflection at impact distances greater than the radius of inner edge and much less that the radius of outer edge coincides with the result of isothermal halo. That is a new statement for the limit of infinite halo in the vector field model with different physics. The final equation for the deflection is known [19], but the physical motivation is different. For instance, the model does not contain the singularity of isothermal halo.
(that is quite evident). So, the identification with the singular isothermal halo looks misleading.

iii) The effects related with the inner edge absent in the case of isothermal halo were investigated in the literature [18].

iv) The effects related with the outer edge absent in the case of isothermal halo as was represented in the literature [12], contain some errors, i.e. the presence of fracture point in the bending curve at the edge was missed. The representation of correction in the paper has required further application of method, which gives the correct result for the infinite halo, that has been usefully made in the initial part of Section 2.

Thus, the paper describes the physics different from the singular isothermal halo. The results concerning the limit of infinite halo in the vector field model and the effects of outer edge of halo are new.

Second, we have to emphasize that the reason for the difference between the effective mass deflecting the light at the impact distance $r$ and the dark matter mass inside the same radius $r$ by a factor found in the paper is quite clear: the deflection takes place due to the mass inside the radius $r$ and effectively due to the dark matter mass outside the same radius. This reason is evident and transparent.

This work is partially supported by the grant of the president of Russian Federation for scientific schools NSc-1303.2003.2, and the Russian Foundation for Basic Research, grant 04-02-17530.

References

(Kiselev V V 2005 Preprint arXiv:gr-qc/0404042)


(Krause A and Ng S P 2004 preprint arXiv:hep-th/0409241)

[7] Piazza F and Tsujikawa S 2004 JCAP 0407 004


[9] Dubovsky S L 2004 JCAP 0407 009


(Nucamendi U, Salgado M and Sudarsky D 2000 Preprint gr-qc/0011049)