A GENERALIZED VENEZIANO MODEL FOR THE $N$-POINT FUNCTION

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ABSTRACT

A prescription is proposed for the explicit construction of a Veneziano-type formula for the $N$-point function where the external lines may have any (integral) spins. The resultant amplitude is expected to have the following properties:

i) analyticity,
ii) crossing symmetry,
iii) Dolen-Horn-Schmid duality,
iv) resonance poles on linear rising trajectories,
v) Regge asymptotic behaviour in all Mandelstam channels.

In addition, the model yields:

vi) decay correlations for particles with spins,
vii) bootstrap consistency in treating all particles as bound states of one another.

So far, the prescription has only been verified explicitly for $N = 4, 5, 6,$ and $7$, but it is believed to be of general validity. If taken seriously, it could provide the basis for a new phenomenology for virtually all hadronic processes.

22 November 1968
Ref.TH.963
1. For simplicity we consider here a system with only bosons and without internal symmetries. Our first goal is the construction of an N-point function for spinless particles, which satisfies the following conditions:

i) analyticity,
ii) crossing symmetry,
iii) Dolen-Horn-Schmid duality\(^1\),
iv) resonance poles on linear rising trajectories,
v) Regge asymptotic behaviour in all Mandelstam channels.

The solutions for the special cases \(N=4\) and \(\bar{N}\) have been given respectively by Veneziano\(^2\) and by Bardakçi and Ruegg\(^3\). We shall first examine these solutions from a slightly different viewpoint, in order to elucidate the problem in the general case.

a. The four-point function. (Veneziano)\(^2\)

The amplitude is given as a sum of three terms, each corresponding to a particular permutation of the four external lines. In this context, all cyclic or anticyclic permutations will be considered as identical; thus, e.g., \((1,2,3,4) = (2,3,4,1) = (2,1,4,3)\). It can readily be seen that there are only three distinct cases corresponding to the three diagrams in Fig. 1. We need consider here only the first term since the others are readily obtained by a permutation of indices.

![Diagrams](image)

(a)  
(b)  
(c)

Fig. 1

Veneziano gave for Fig. 1a the form:
\[ B_4(x_1, x_2) = \frac{\Gamma(x_1 + 1) \Gamma(x_2 + 1)}{\Gamma(x_1 + x_2 + 2)} \]  \hspace{1cm} (1)

where

\[ x_1 = -1 - x_{1,2} \quad \alpha_{1,2} = \alpha_{1,2}^0 + \alpha' S_{1,2} \]

\[ x_2 = -1 - x_{2,3} \quad \alpha_{2,3} = \alpha_{2,3}^0 + \alpha' S_{2,3} \]  \hspace{1cm} (2)

The function \( B_4(x_1, x_2) \) is analytic in \( x_1 \) and \( x_2 \) apart from simple poles at negative integers, i), is symmetric under interchange of \( x_1 \) and \( x_2 \), ii), contains poles in both the "direct" (12→34) and "crossed" (14→23) channels, iii), but no double poles: while its residue at \( x_i = -\ell - 1 \) (i.e., \( \alpha_{1,2} = \ell \)) is a polynomial of order \( \ell \) in \( s_{23} \), iv). Furthermore, it has the correct Regge asymptotic behaviour, i.e., for \( s_{1,2} \to \infty \) at fixed \( s_{23} \).

\[ B_4(x_1, x_2) \to (s_{1,2})^{2\ell} \]  \hspace{1cm} (3)

Most of these properties can also be seen in the integral representation 3):

\[ B_4(x_1, x_2) = \int_0^1 du_1 \int_0^1 du_2 \left( \frac{u_1}{u_2} \right)^{x_1} \left( \frac{u_2}{u_1} \right)^{x_2} \]  \hspace{1cm} (4)

where

\[ u_1 = 1 - \omega_1 \quad u_2 = 1 - \omega_2 \]  \hspace{1cm} (5)

The integral (4) is obviously symmetric between \( x_1 \) and \( x_2 \). Singularities in \( x_1 \) come only from the region \( u_1 \sim 0 \). Expanding the integrand in a Taylor series in \( u_1 \) using (5) and integrating term by term, one obtains poles in \( x_1 \) at negative integers. The residue at
the pole $x_1 = -\ell^{-1}$ is given by:

$$\mathbb{R} \mathcal{L}_\ell \mathcal{B}_\ell(x_1, x_2) = \frac{1}{\ell!} \left[ \frac{d^\ell}{du^{\ell}_1} \cdot u_2^x \right]_{u_1 = u} \tag{6}$$

which is clearly a polynomial of degree $\ell$ in $x_2$. Furthermore, double poles are excluded, since by (5) $u_1$ and $u_2$ cannot vanish simultaneously.

b. The five-point function \hspace{1em} (Bardakči-Ruegg) \hspace{1em} 3)

The amplitude is again written as a sum of 12 terms, corresponding to the 12 "distinct" permutations (in the sense defined above) of the five external lines. A typical term is represented by Fig. 2:

![Fig. 2](image)

The ingenious construction of Bardakči and Ruegg 3) gives:

$$\mathcal{B}_\ell(x_1, x_2, x_3, x_4, x_5) = \int \left[ \frac{d^\ell}{du^{\ell}_1} \cdot u_2^x \cdot \frac{1}{u_5} \cdot u_4^x \cdot u_3^x \cdot u_5^x \right] \tag{7}$$

where

$$x_1' = \nu_1' - \alpha'_{12}, \quad x_1'' = \alpha_1' - \alpha_1' S_{12}, \quad x_5' = \nu_5' + \alpha' C. \tag{8}$$

$$\alpha_2' = \nu_2' - \nu_2', \quad \alpha_2'' = \nu_2' - \nu_2'. \quad \nu_5' = \nu_5' + \alpha C. \tag{9}$$

Although (9) represents altogether five conditions, only three are in fact independent, so that $u_2$, $u_3$, and $u_5$ can be solved in terms of
$u_1$ and $u_4$. A cyclic change of the independent variables from $u_1, u_4$ to $u_2, u_5$ gives

$$
\frac{\mathcal{D}(u_1, u_5)}{\mathcal{D}(u_2, u_4)} = \frac{U_1}{U_2}
$$

(10)

which guarantees that (7) is invariant under a cyclic permutation of $x_i$, ii). As in i., one shows that (7) is analytic, i), except for poles at negative integers for every $x_i$, iii). There is a double pole when both $x_1$ and $x_4$ are negative integers, which correspond to the Feynman diagram in Fig.3. The residue of this double pole is a polynomial in $x_2, x_3$ and $x_5$ of the correct degree, iv). However, there is no double pole (as is physically necessary) when $x_1$ and $x_2$ are both negative integers. This is guaranteed by (9) which does not allow $u_1$ and $u_2$ to vanish simultaneously. Regge asymptotic behaviour for (7) has also been verified, v).

![Diagram](image)

Fig.3

c. **The six-point function**

$N = 4$ and $5$ are for the present purpose rather special cases in that there is in each only one type of channels. Thus, for $N = 4$, we have only channels of the type : 2 particles $\rightarrow$ 2 particles, and for $N = 5$, only : 2 particles $\rightarrow$ 3 particles. In order to infer, hopefully, the rule for general $N$, we shall consider the six-point function in which two types of channels occur, namely : 2 particles $\rightarrow$ 4 particles, and 3 particles $\rightarrow$ 3 particles.

Consider the typical term in the sum for the amplitude represented by Fig.4. This is required by "duality", iii), to contain nine trajectories corresponding to the six two-body channels:
\[ \alpha_{12} = \alpha_{12}^c + \alpha_{12}^S, \quad \alpha_{23} = \alpha_{23}^c + \alpha_{23}^S, \quad \alpha_{24} = \alpha_{24}^c + \alpha_{24}^S, \quad \alpha_{34} = \alpha_{34}^c + \alpha_{34}^S. \]  

(11)

and the three three-body channels:

\[ \alpha_{234} = \alpha_{234}^c = \alpha_{234}^c + \alpha_{234}^S, \quad \alpha_{234}^c. \]  

(12)

Fig. 4

There can be triple poles in the amplitude, e.g., that corresponding to the diagram in Fig. 5. However, coincident poles between, e.g., \( \alpha_{12} \) and \( \alpha_{23} \), or \( \alpha_{12} \) and \( \alpha_{234} \) are physically disallowed, since they do not correspond to Feynman graphs.

Fig. 5

We wish to write down an integral representation for \( \mathcal{B}_6 \) similar to (4) and (7). For this, introduce the variables \( x_i \) (\( i = 1, \ldots, 6 \)) corresponding to the two-body trajectories \( \alpha_{12} \), etc., and the variables \( y_j \) (\( j = 1, 2, 3 \)) corresponding to the three-body trajectories \( \alpha_{234} \), etc.; the exact relations between \( x_i, y_j \) and the \( \alpha \)'s will be specified later. Further, in analogy to (4) and (7), we introduce nine more variables; \( u_i \) "conjugate" to \( x_i \).
(i = 1, ..., 6) and \( v_j \) "conjugate" to \( y_j \) \((j = 1, 2, 3)\). In order to avoid coincident poles which are physically disallowed, we wish to ensure that the corresponding "conjugate" variables do not vanish simultaneously. This is guaranteed by requiring that:

\[
\begin{align*}
\Psi_i &= 1 - \Psi_1 \Psi_2 \Psi_3, \\
\Psi_4 &= 1 - \Psi_1 \Psi_2 \Psi_3 \Psi_4, \\
\Psi_5 &= 1 - \Psi_1 \Psi_2 \Psi_3 \Psi_4 \Psi_5, \\
\Psi_6 &= 1 - \Psi_1 \Psi_2 \Psi_3 \Psi_4 \Psi_5 \Psi_6.
\end{align*}
\]

(13) represents altogether nine equations, only six of which are however independent. To show this, one can choose a set of three independent variables and solve for the remaining six. Since the three independent variables can (by definition) vanish simultaneously, they must correspond to allowed triple poles. We may choose, for example, \( u_1, u_5 \) and \( v_3 \) corresponding to the diagram in Fig. 5. One obtains:

\[
\begin{align*}
u_1 &= \frac{1 - \Psi_1}{1 - \Psi_1 \Psi_3}, & \nu_4 &= \frac{1 - \Psi_5}{1 - \Psi_5 \Psi_3}, \\
\nu_2 &= \frac{1 - \Psi_1 \Psi_3}{1 - \Psi_1 \Psi_3 \Psi_5}, & \nu_5 &= \frac{1 - \Psi_5 \Psi_3}{1 - \Psi_1 \Psi_3 \Psi_5}, \\
\nu_3 &= \frac{(1 - \Psi_1)(1 - \Psi_5 \Psi_3)}{(1 - \Psi_1 \Psi_3)(1 - \Psi_5 \Psi_3)}, & \nu_6 &= 1 - \Psi_1 \Psi_5 \Psi_3.
\end{align*}
\]

which satisfies all the conditions in (13). Under a cyclic change of independent variables from \( u_1, u_5 \) and \( v_3 \) to \( u_2, u_6 \) and \( v_1 \), we have for the Jacobian,

\[
\frac{\mathcal{J}(\Psi_1, \Psi_5, \Psi_3)}{\mathcal{J}(\Psi_1, \Psi_5, \Psi_3)} = \frac{\Psi_1^2 \Psi_3}{\nu_1 \Psi_6^2}.
\]

The integral:
\[ B_6(x_i, y_j) = \left( \frac{d\mu_1}{\mu_2} \right)^4 \left( \frac{v_3}{\mu_2} \right)^{x_1 \cdot x_2 \cdot x_3 \cdot y_1 \cdot y_2 \cdot y_3} \] (16)

is then guaranteed to be invariant under any simultaneous cyclic permutation of the variables \( x_i \) and \( y_j \), ii).

It is seen that \( B_6 \) is analytic 4), i), apart from poles at

\[ x_i = 1, 2, \ldots ; y_j = 1, 2, \ldots \] (17)

We therefore define:

\[ x_i = 1 - \chi_{1i} ; \quad \chi_{1i} \]

\[ y_j = 2 - \chi_{1j} ; \quad \chi_{1j} \] (18)

Using the techniques developed by Biedakö and Ruegg 3), it is then straightforward to prove:

a) the residue at the triple pole corresponding to Fig.5 at

\[ \alpha_{12} = j, \quad \alpha_{123} = k, \text{ and } \alpha_{56} = \ell \] is a polynomial in the other variables of degree

\[ x_2 - \lambda - k \quad x_3 - \lambda - k \quad x_4 - \lambda - k \quad x_5 - \lambda - k \quad \chi_{11} \quad \chi_{12} \] (19)

with all exponents \( \geq 0 \). This is as expected from Feynman rules for the exchange of particles with maximum spins \( j, k \) and \( \ell \). This guarantees iv).

b) in the limit \( s_{34}, s_{45}, s_{56} \to \infty \) at fixed values of \( s_{23}, s_{24}, s_{61}, \) \( K_1 = s_{45} s_{56}/s_{456} \), \( K_2 = s_{34} s_{45}/s_{345} \), namely the normal triple Regge limit 5), one obtains,

\[ B_6 \to (-x_{14})^{\alpha_{14}} (-x_{45})^{\alpha_{45}} (-x_{54})^{\chi_{54}} \ G(\alpha_{12}, \alpha_{34}, \alpha_{56}, K_1, K_2) \] (20)
where \( g_2 = (\alpha_{23}, \alpha_{24}, \alpha_{64}, k_1, k_1) \)

\[
\mathcal{G}_2 \left( x_{23}, x_{24}, x_{64}, k_1, k_1 \right) = \int \int \int \alpha \left( x_{23} y_{24} x_{64} y_{64} \right) \exp \left[ \frac{x_{23} y_{24} x_{64} y_{64}}{k_1 k_1} \right]
\]

The function \( B_6(x_i; y_j) \) thus satisfies all the requirements i) - v).

\[ (21) \]

d. The N-point function

Exactly the same procedure as in c., has been applied by Tsou and the present author \( n \) to the seven-point function; the result has been verified explicitly to satisfy again conditions i) - v). Since the examples considered seem to be sufficiently general, we venture to suggest that the prescription may in fact be valid for the N-point function. The crucial step seems to be the conditions (13) on the "conjugate" variables. In general there will be several types of channels, hence also several groups of "conjugate" variables. Once the equations (13) are solved and shown to have the right number of independent variables, the rest seems for some reason to be almost automatic.

2. We wish to show next that the model is consistent with the bootstrap philosophy in treating all particles consistently as bound states of others. Consider first spin zero bound states in the six-point function (Fig.4) as example. The residue of \( B_6 \) at the pole \( \chi_{12} = 0 \) is given by :

\[
\mathcal{R}_{\chi_{12}}\left( x_i; y_j \right) = \mathcal{Q} \left( \mu_1; x_2, x_3, x_4, x_5, x_6, y_4, y_5, y_6 \right) \bigg|_{\chi_{12} = 0}
\]

\[
(22) \]
where

$$Q = \int_L d\alpha \int_{c_i} d\alpha \left( \frac{V_3}{\mu_1} \right) \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 V_{3} V_{1} V_{2} V_{3}$$  \hspace{1cm} (23)$$

On putting $u_1 = 0$, one obtains immediately:

$$\text{Res. } B_6 (x_i; y_i) = B_5 (y_i + 1, y_i + 1, x_1, x_2, x_3, x_5)$$ \hspace{1cm} (24)$$

$$\alpha_{12} = 0$$

This corresponds to the reduction of the six-point function of Fig. 4 to the diagram of Fig. 6:

![Diagram](image)

Fig. 6

Similarly, the residue of $B_6$ at $\alpha_{123} = 0$ is given by:

$$\text{Res. } B_6 (x_i; y_i) = \mathcal{P} (V_3; x_1, x_2, x_3, x_4, x_5, x_6; y_i, y_i)$$ \hspace{1cm} (25)$$

where

$$\mathcal{P} = \int_L d\alpha \int_{c_i} d\alpha \left( \frac{1}{\mu_1} \right) \mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 V_{1} V_{2} V_{3}$$  \hspace{1cm} (26)$$

Putting $V_3 = 0$ yields

$$\text{Res. } B_6 (x_i; y_i) = B_4 (x_1, x_2) B_4 (x_4, x_5)$$ \hspace{1cm} (27)$$
which corresponds to the reduction of Fig. 4 to Fig. 7. Notice that in both reductions considered, there are no correlations between the two parts of the reduced diagrams as is necessary for interpreting the intermediate bound state as a spin zero particle.

Next consider the bound state at \( \alpha_{12} = j \) corresponding to the reduction of Fig. 6. The residue of \( B_6 \) at this pole is given by:

\[
R_{\lambda \delta} \quad \mathcal{F}_j (\chi, \nu, \gamma) = \frac{1}{j!} \left[ \frac{\mathcal{F}_j}{\mathcal{P}} \right]_{\nu, \gamma = 0}
\]

(28)

The result is in general a linear combination of \( B_5 \) functions. For consistency with the bootstrap philosophy, we are required to interpret (28) as the five-point function with four external lines spinless and one external line of spin \( j \). There will, in general, be correlations between the lines 1 and 2 and the rest of the diagram, which presumably should describe the decay correlations of the spin \( j \) particle. These correlations are at most of order \( j \).

Similarly, for the reduction of Fig. 7 with intermediate bound state having spin \( j \), one has

\[
R_{\lambda \delta} \quad \mathcal{F}_j (\chi, \nu, \gamma) = \frac{1}{j!} \left[ \frac{\mathcal{F}_j}{\mathcal{P}} \right]_{\nu, \gamma = 0}
\]

(29)

which gives a linear combination of products of \( B_4 \)'s. This is to be defined as the four-point function with three spinless lines and one with spin \( j \), the spin \( j \) particle decaying subsequently into three spinless particles. The decay correlations are also given.
For the six-point function, one can also have the reduction given in Fig. 8. This is given by:

\[
\mathcal{R} = \mathcal{B}_6(x_1, y_6) = \frac{1}{j!} \frac{1}{k!} \left[ \frac{\partial^j}{\partial u_1} \frac{\partial^k}{\partial \mu_6} \mathcal{R} \right]_{u_1 = 0} \quad \mu_6 = 0
\]

(30)

where

\[
\mathcal{R} = \int d
\]

\[
\mathcal{R} = \int d u_i \frac{\mathcal{V}_3}{\mathcal{U}_6} \mathcal{U}_1 \mathcal{U}_3 \mathcal{U}_4 \mathcal{U}_5 \mathcal{V}_1 \mathcal{V}_3 \mathcal{V}_5.
\]

(31)

This should represent then the four-point function with two lines having spins.

![Fig. 8](image)

It is clear then that with the general N-point function for spinless particles one can consistently construct the general N-point function for particles with arbitrary integral spins. The compound particles may have any decay modes, the correlations of which with the "production" process are also explicitly given, vi).

As an example we give the explicit form for the reduction of Fig. 7 when the intermediate particle has spin one:

\[
\frac{\mathcal{D}}{\mathcal{D} \mathcal{U}_1 \mathcal{V}_5 \mathcal{V}_6} = x_1 \mathcal{B}_4(x_1, x_4, x_5, x_6) + x_4 \mathcal{B}_4(x_1, x_2) - y_1 \mathcal{B}_4(x_1, x_4, x_5) - y_4 \mathcal{B}_4(x_4, x_5, x_6) + (x_4 - 2) \mathcal{B}_4(x_1, x_2, x_4, x_5, x_6) - x_5 \mathcal{B}_4(x_4, x_5, x_6)
\]

(32)
This could represent, for example, the reaction:

\[ \pi + \pi \rightarrow \pi + A_1 \rightarrow \pi + 3\pi \]  \hspace{1cm} (33)

Finally, we note that though in the preceding discussion we seem to have given a special place to spinless particles, this is only for convenience. One can as well consider spinless particles as bound states of several particles of non-zero spins, \textit{vii}). However, the formulation in terms of \( N \)-point functions with spinless external lines is particularly appropriate, because:

a) there is no correlation between parts of a diagram connected by a spinless particle,

b) the only bosons known to be stable under strong interactions are pseudoscalars.

3. Problems apparent already in the original Veneziano formula, such as the question of unitarity and uniqueness, have not been attacked. Trajectories are kept real throughout.

In the \( N \)-point function for spinless particles, we have assumed no satellite terms \textit{\textsuperscript{10}}. It is interesting to note that this is consistent with:

a) the reduction properties required by the bootstrap philosophy, though it would not be for particles with spins [see, \textit{e.g.}, \((32)\)],

b) the empirical observations of Lovelace \textit{\textsuperscript{10}} and the relation he suggested between the Veneziano model and the Adler self-consistency condition.
ACKNOWLEDGEMENTS

The author is indebted to Drs. K. Bardakçı and H. Ruegg for informing him of their result before publication, and for much useful conversation.

NOTE ADDED IN PROOF

The prescription suggested has now been proved by Tsou and the present author to be indeed valid for general N.
REFERENCES AND FOOTNOTES


4) We consider $B_6$ as an analytic function of nine complex variables although in the six-point function there are in fact only eight independent variables. The physical amplitude is then a restriction of $B_6$ on the eight-dimensional hypersurface defined by, e.g., the vanishing of the Grammian determinant. This fact has to be taken into account in taking the Regge limit, which is a limit in the physical region, (see below).


6) The residue function is in fact factorizable – K. Bardakçı (private communication).

7) Chan Hong-Mo and Tsou Sheung-Tsun, to be published.

8) The question concerning the parity of these spinless particles has not been clarified.

9) By a particle with spin $j$, we mean, as in the original Veneziano model, a parent with spin $j$ plus its usual sequence of degenerate daughters.

10) C. Lovelace, CERN preprint TH.950 (1968).