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FROISSART BOUNDS WITH ANY SPIN

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A B S T R A C T

For elastic scattering of particles of arbitrary spin, an invariant amplitude is constructed which has the "positivity" required for Martin's proof of Froissart bounds.

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This paper does not obtain any new result. It gives only an alternative to an argument by Martin and Mahoux ¹⁾. The subject is the elastic scattering of two stable particles

$$A + B \rightarrow A + B$$

in a hypothetical world without particles of zero mass. Martin and Mahoux extended to particles of arbitrary spin Martin's demonstration ²⁾ for spinless particles of the Froissart bounds. Their work, because crossing and analyticity are essential, deploys a considerable machinery associated with the singularities of helicity amplitudes. These singularities are inferred ³⁾ from the supposed regularity of spinor amplitudes. This suggests that a more direct use of spinor amplitudes, or of related invariant amplitudes, may be economical. It is shown here that there is indeed in the general case a certain invariant amplitude to which the original argument of Martin can be applied, and that from this the desired results follow.

Section 1 gives a sketch of the background. Section 2 describes the invariant amplitude with Martin's "positivity" properties. Section 3 shows that, as for Martin and Mahoux, the Froissart bound once obtained for a particular amplitude implies similar bounds for all.

1. INTRODUCTION

We work in the centre of mass system. Let $\vec{k}$ and $\vec{k}'$ be initial and final momenta of particle A. Let a and a' specify initial and final spin states of A, and b and b' the spin states of B; these indices give spin projections on an arbitrarily chosen z axis with the particles de-boosted to rest in the centre-of-mass system. We denote by

$$\langle \vec{k}' a' b' | T | \vec{k} a b \rangle$$

the elastic scattering amplitude, so normalized that the unpolarized differential cross-section is

$$\frac{d\sigma}{d\Omega} = \sum_{a'b'} \bar{\sum}_{ab} |\langle \vec{k}' a' b' | T | \vec{k} a b \rangle|^2$$

where $\bar{\sum}$ denotes averaging rather than summing. Consider in particular the spin averaged amplitude

$$f(\theta) = \bar{\sum}_{ab} \langle \vec{k}' a b | T | \vec{k} a b \rangle$$

With spins averaged over, orbital angular momentum is conserved, so that $f(\theta)$ has the same partial wave expansion as the scattering amplitude for spinless particles :

$$f(\theta) = k^{-1} \sum (2l+1) f_l P_l(\cos \theta) \quad (1)$$

where $k = |\vec{k}| = |\vec{k}'|$ and with the partial wave amplitudes given by

$$f_l = \bar{\sum}_{ab} \langle l, m, a, b | T | l, m, a, b \rangle \quad (2)$$

where $|l, m, a, b\rangle$ is a suitably normalized state of specified orbital angular momentum l with projection m . From the unitarity equation

$$T - T^\dagger = T^\dagger \Delta T \quad (3)$$

(where Δ is essentially an energy momentum conserving δ -function) diagonal elements of $T - T^\dagger$ can be bounded by retaining on the right-hand side as intermediate states only the states of the elastic channel. With the given normalization then

$$9_m \langle l m a b | T | l m a b \rangle \geq \sum_{\substack{l'm' \\ a'b'}} |\langle l'm'a'b' | T | l m a b \rangle|^2 \quad (4)$$

where $m' + a' + b' = m + a + b$. In particular then

$$1 \gg \text{Im} \langle l m a b | T | l m a b \rangle \gg |\langle l m a b | T | l m a b \rangle|^2 \quad (5)$$

and because the mean square exceeds the squared mean

$$1 \gg \text{Im} f_l \gg |f_l|^2 \quad (6)$$

The partial wave expansion (1) and the unitarity bound (6) for the spin averaged amplitude are just the forms familiar for the scattering amplitude of spinless particles; this observation is due to Yamamoto ⁴).

Suppose now it were known that the imaginary part of the series (1) continued to converge for unphysical values

$$1 < \cos \theta_1 < 1 + t_1 / 2k^2 \quad (7)$$

where t_1 is some energy independent value. Then because $P_l(\cos \theta)$ is in this region positive and rapidly increasing with l , $\text{Im} f_l$ must decrease rapidly with l . In fact ⁵), provided only that the continuous function $f(i\theta_1)$ is bounded at high energy by some power of k ,

$$\log \text{Im} f_l < \frac{L-l}{L} C' \log k \quad (8)$$

where

$$L = C'' k \log k \quad (9)$$

and C' and C'' are positive constants. Thus the series (1) would effectively terminate at l of order L , and we would have the Froissart bounds

$$|f(\theta)| < C k \log^2 k \quad (10)$$

$$\sigma < 4\pi C \log^2 k \quad (11)$$

where $\sigma = 4\pi \text{Im} f(0)/k$ is the total cross-section, and C is some constant.

The heart of the problem is then to demonstrate the assumed convergence of the imaginary part of the series (1) for some fixed positive t value. Martin's demonstration for the zero spin case depended on attributing the following properties to the invariant amplitude $F(s,t)$ [of which $f(\theta)$ is a θ independent multiple] :

- a) fixed t dispersion relations with finitely many subtractions. [Actually Martin and Mahoux state that the Bros-Epstein-Glaser ⁶⁾ analyticity domain would suffice.]
- b) analyticity in t in the neighbourhood of $t=0$ for s real and just below threshold, $(m_A+m_B)^2$.
- c) "positivity", by which is meant

$$\left. \frac{\partial^n}{\partial t^n} g_m F(s,t) \right|_{t=0} \gg \left| \frac{\partial^n}{\partial t^n} g_m F(s,t) \right|_{t<0} \quad (12)$$

and the same properties for the amplitude $\bar{F}(s,t) = F(u,t)$ which describes the crossed reaction.

Martin showed that a Taylor series in t for $F(s,t)$ could then be constructed by differentiation with respect to t of the fixed t dispersion relations, and that it had the required convergence properties.

We will construct in the general case an invariant amplitude $F'(s,t)$ with "positivity". If assumptions a) and b) are again made for this amplitude, it again follows that it can be continued analytically to some fixed positive value t_1 independent of s . We will see that $F'(s,t)$, although no longer a simple multiple of $f(\theta)$, is again so related to the latter that the Froissart bounds are implied.

2. INVARIANT AMPLITUDE WITH POSITIVITY

We begin with the simplest case. Let A have spin one-half, and B spin zero. Let p and p' be initial and final four-momenta of A, and q and q' of B. Let the 4x4 matrix W be the Dirac spinor scattering amplitude. The centre-of-mass amplitudes already introduced are related to W by

$$\langle \vec{k}' a' | T | \vec{k} a \rangle = N \bar{u}'(a') W u(a) \quad (13)$$

where u(a) and u'(a') are Dirac spinors for momenta p and p', and spin projections a and a' respectively. Of course, p and q are specialized to their centre-of-mass values

$$(\vec{k}, i\sqrt{m_A^2 + k^2}), (-\vec{k}, i\sqrt{m_B^2 + k^2})$$

N is a real positive normalization factor which depends only on s.

The part of W which has physical relevance is obtained by projecting onto physical spin states :

$$\begin{aligned} \sum_{a'} u'(a') \bar{u}'(a') W \sum_a u(a) \bar{u}(a) \\ = (m_A - i\gamma p') W (m_A - i\gamma p) \end{aligned} \quad (14)$$

The trace of this would give an invariant amplitude, but it is found not to have the desired "positivity". Essentially the only other invariant with simple crossing is

$$F(s, t) = - \text{Trace} (i\gamma q + i\gamma q') (m_A - i\gamma p') W (m_A - i\gamma p) \quad (15)$$

Using (8) and (9) :

$$N F(s, t) = - \sum_{a' a} \langle \vec{k}' a' | T | \vec{k} a \rangle \bar{u}(a) (i\gamma q + i\gamma q') u'(a') \quad (16)$$

Since F depends only on s and t we can also write

$$N F(s, t) = - \sum_{a'a} \langle \vec{k} a | T | \vec{k}' a' \rangle \bar{u}(a') (i\gamma q' + i\gamma q) u(a)$$

of which the Hermitian conjugate is

$$N F^*(s, t) = - \sum_{a'a} \langle \vec{k}' a' | T^\dagger | \vec{k} a \rangle \bar{u}(a) (i\gamma q + i\gamma q') u'(a')$$

Combining this with (15) and using the unitarity condition (2)

$$2N g_m F = - \sum_{a'a} \langle \vec{k}' a' | T^\dagger \Delta T | \vec{k} a \rangle \bar{u}(a) (i\gamma q + i\gamma q') u'(a') \quad (17)$$

Suppose now that $\vec{k} \times \vec{k}'$ is along the z axis - the axis of spin quantization. Then by explicit calculation

$$- \bar{u}(a) (i\gamma q + i\gamma q') u'(a') = \delta_{aa'} (X + Y e^{2ia\theta}) \quad (18)$$

where θ is the scattering angle and

$$\begin{aligned} X &= 2(p_0 + m_A)(q_0 + p_0 - m_A) \\ Y &= 2(p_0 - m_A)(q_0 + p_0 + m_A) \end{aligned} \quad (19)$$

with

$$p_0 = (m_A^2 + k^2)^{1/2} \quad q_0 = (m_B^2 + k^2)^{1/2}$$

Note that X and Y are positive in the physical region; we would not have obtained positive quantities at this point if the factor $-(i\gamma q + i\gamma q')$ had been omitted in (15).

Now

$$|\vec{k}' a\rangle = e^{-i\theta(J_z - a)} |\vec{k} a\rangle$$

where J_z is angular momentum along the z axis, and the $\exp(-ia\theta)$ arises from phase conventions already employed in (18). Then by expanding the states in eigenvectors of J_z we have the representation

$$\langle \vec{k}', a | T^\dagger \Delta T | \vec{k} a \rangle = \sum_n C_n^a e^{in\theta} \quad (20)$$

with $C_n^a \gg 0$, and the sum is over integral values between $\pm\infty$. It follows from (12), (13) and (14) that

$$2N g_m F = \sum_n C_n' e^{in\theta} \quad (21)$$

with

$$C_n' = \sum_a \{ X C_n^a + Y C_{n-2a}^a \} \quad (22)$$

Since $\text{Im } F$ is real, we can rewrite (21)

$$2N g_m F = C_0' + 2 \sum_1^\infty C_n' \cos n\theta \quad (23)$$

Since $^{7)} (-d/d \cos \theta)^m \cos n\theta$ is positive at $\theta=0$, and takes there its maximum value for real θ , the desired positivity properties follow for $\text{Im } F$.

Because of the factor $(i\gamma q + i\gamma q')$ in (10) one finds that it is $-F(u, t)$ which stands in the same relation to the crossed reaction as $F(s, t)$ does to the original. So we define

$$\begin{aligned} F'(s, t) &= (s-u) F(s, t) \\ \bar{F}'(s, t) &= F'(u, t) \end{aligned} \quad (24)$$

and conclude that these functions have Martin's positivity. Subject then to the given analyticity assumptions, the continuation of $F'(s, t)$ to positive t is analytic at least as far as some t_1 fixed independently of s . This requires the convergence of the series (18) at the corresponding imaginary value $i\theta_1$ of θ . Since $\cosh n\theta_1$ is positive, and since from (17)

$$X^{-1} C'_n \gg \sum_a C_n^a$$

it follows that the series

$$\left(\sum_a C_0^a\right) + 2 \sum_1^{\infty} \left(\sum_a C_n^a\right) \cos n\theta$$

also converges at $\theta = i\theta_1$. But from (20), (3) and (2), this is essentially the series for $\text{Im} f(\theta)$, whose convergence at $i\theta_1$ gives the Froissart bounds.

To generalize for arbitrary spin the Wigner-Bargmann formalism ⁸⁾ is convenient. The wave function of a particle of spin $\frac{1}{2}\varrho$ transforms like the product of ϱ Dirac wave functions. In such a product space the fully symmetric subspace describes the states of the spin ϱ particle, and the remainder of the space is redundant. The generalized spinor amplitude will then have ϱ pairs of Dirac indices for particle A of spin $\frac{1}{2}\varrho$, and σ pairs for particle B of spin $\frac{1}{2}\sigma$:

$$W_{(l'_1 l_1)(l'_2 l_2) \dots (m'_1 m_1)(m'_2 m_2) \dots}$$

W is supposed to be fully symmetric separately in each of the four sets of indices l, l', m, m' , so that there are no transitions into or out of the redundant parts of the spin spaces.

Each pair of indices can now be treated separately as was the original single pair. Thus scattering matrix elements

$$\langle \vec{k}' a'_1 a'_2 \dots b'_1 b'_2 \dots | T | \vec{k} a_1 a_2 \dots b_1 b_2 \dots \rangle$$

are obtained (apart from a normalization factor) by saturating the indices of W with wave functions

$$(\bar{u}'_{l'_1}(a'_1) u_{l_1}(a_1)) \dots (\bar{u}'_{m'_1}(b'_1) u_{m_1}(b_1)) \dots$$

The invariant amplitude $F(s, t)$ is formed by saturating the indices of W with factors

$$\begin{aligned} & [(m_A - irp)(-irq - irq')(m_A - irp')]_{l'l} \\ & [(m_B - irq)(-irp - irp')(m_B - irq')]_{m'm} \end{aligned}$$

for each pair of indices $(l'l)$ and $(m'm)$.

Thus the problem essentially factorizes. Proceeding as before, we come to a representation with positive C 's :

$$\begin{aligned} g_m F(s, t) = \sum_n e^{in\theta} \sum_{ab} \{ C_n^{a_1 a_2 \dots b_1 b_2 \dots} \\ (X_A + Y_A e^{2ia_1\theta})(X_A + Y_A e^{2ia_2\theta}) \dots (X_B + Y_B e^{2ib_1\theta}) \dots \} \end{aligned} \quad (25)$$

where the coefficients X_A and Y_A are given by (19), and X_B and Y_B are analogous positive expressions in which the roles of the two particles are exchanged. Thus one obtains again a representation

$$g_m F(s, t) = \sum C'_n \cos n\theta$$

with positive coefficients. In the crossing relation a factor (-1) arises for each pair of Dirac indices. So finally

$$\begin{aligned} F'(s, t) &= (s-u)^\xi F(s, t) \\ \bar{F}'(s, t) &= F(u, t) \end{aligned} \quad (26)$$

(where $\xi = 1$ if only one of the two spins is half integral, and zero otherwise) have the correct positivity properties. Once again the convergence of

$$\sum C'_n \cos n\theta$$

for imaginary argument implies the convergence of the sub-series

$$\sum_n \cos n\theta \left(\sum_{a,b} C_n^{a_1, a_2, \dots, b_1, b_2, \dots} \right)$$

which is again essentially $\text{Im} f(\theta)$.

3. THE GENERAL AMPLITUDE

To bound amplitudes other than the spin average, a more general partial wave expansion is required. For this the helicity formalism⁹⁾, as used by Martin and Mahoux, is undoubtedly the most efficient. However, it may be of some interest to exhibit another, which, while not fully exploiting angular momentum conservation, suffices to give Froissart bounds.

We return now to the notation of the introduction, without reference to redundant spin states. The plane wave state $|\vec{k}ab\rangle$ can be expanded in terms of suitably normalized states $|l m ab\rangle$ of orbital angular momentum l and projection m :

$$|\vec{k}ab\rangle = k^{-\frac{1}{2}} \sum_{lm} Y_{lm}^*(\hat{k}) |l m ab\rangle \quad (27)$$

where $\hat{k}$ is a unit vector along $\vec{k}$, and the Y_{lm} are essentially the usual spherical harmonics, except that we normalize so that

$$\int d\hat{k} |Y_{lm}(\hat{k})|^2 = 4\pi$$

Summing this from $m = -l$ to $m = +l$, and recalling that $\sum_m |Y_{lm}(\hat{k})|^2$ is independent of $\hat{k}$ (because the Y_{lm} transform unitarily into one another under rotation)

$$\sum_m |Y_{lm}(\hat{k})|^2 = 2l+1$$

whence

$$|Y_{\ell m}(\hat{k})| \leq (2\ell+1)^{1/2} \quad (28)$$

Using (27)

$$\langle \vec{k}' a' b' | T | k a b \rangle = k^{-1} \sum_{\ell \ell'} \langle \ell' m' a' b' | T | \ell m a b \rangle Y_{\ell' m'}(\hat{k}') Y_{\ell m}^*(\hat{k})$$

where $m' + a' + b' = m + a + b$, and for particles of spin $\frac{1}{2} \rho$ and $\frac{1}{2} \sigma$, $|\ell' - \ell| \leq \rho + \sigma$. This expansion simplifies if $\hat{k}$ is taken along the axis of spin quantization. Then

$$Y_{\ell m}(\hat{k}) = \delta_{m0} (2\ell+1)^{1/2}$$

and

$$\langle \vec{k}' a' b' | T | \vec{k} a b \rangle = k^{-1} \sum_{\ell' \ell} (2\ell+1)^{1/2} \langle \ell' m' a' b' | T | \ell 0 a b \rangle Y_{\ell' m'}(\hat{k}') \quad (29)$$

Taking $a' = a$, $b' = b$, and averaging over spins (so that terms with $\ell' \neq \ell$ or $m' \neq m$ must cancel), the original expansion (1) is recovered, since

$$Y_{\ell 0}(\hat{k}') = (2\ell+1)^{1/2} P_{\ell}(\cos \theta)$$

From (28) and (29)

$$|\langle \vec{k}' a' b' | T | \vec{k} a b \rangle| \leq k^{-1} \sum_{\ell} (2\ell+2\sigma+2\rho+1) \sum_{\ell'} |\langle \ell' m' a' b' | T | \ell 0 a b \rangle|$$

But, invoking in turn the superiority of the mean square to the squared mean, and the unitarity bound (4)

$$\sum_{e'} |\langle e'm'a'b' | T | l o a b \rangle| \leq (2\sigma + 2\vartheta + 1) \left[\sum_{e'} |\langle e'm'a'b' | T | l o a b \rangle|^2 \right]^{1/2}$$

$$\leq (2\sigma + 2\vartheta + 1) \left[g_m \langle l o a b | T | l o a b \rangle \right]^{1/2}$$

The quantity in the last square bracket is less than unity, Eq. (5). The exponential decrease implied by (6) is then sufficient to secure the Froissart bound for large k

$$|\langle \vec{k}' a' b' | T | \vec{k} a b \rangle| < C' k \log^2 k \quad (30)$$

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A P P E N D I X

Kinematical singularities are usually supposed absent ^{3),10)} in the context of wave functions transforming as $(0,j)$ or $(j,0)$. Such a wave function is obtained from the Wigner-Bargmann by contracting every Dirac index with a projection operator $1 + \gamma_5$. In a representation with γ_5 diagonal and upper eigenvalues $+1$ this just suppresses all components for which any Dirac index has values other than 1 and 2 . Conversely, the suppressed components can be reconstructed using the Wigner-Bargmann equations. So one passes from Wigner-Bargmann to $(j,0)$ wave functions, or vice versa, by a linear transformation whose components are polynomials of finite degree in components of momenta. It is then a matter of indifference in which version the sesquilinear form giving the scattering amplitude is supposed to have coefficients free from kinematic singularities.

Actually this kinematic equivalence holds for any pair of wave functions representing a massive particle of given spin. For either may be obtained from the other by a linear transformation with polynomial coefficients. This is well known. My colleagues were unable to supply a reference, but F. Yndurain kindly supplied the following proof.

We may consider separately the parts of the wave function transforming as different irreducible representations $(a/2, b/2)$. Such a part can be written in spinor notation

$$\psi_{\alpha_1 \dots \alpha_a \beta_1 \dots \beta_b}$$

with complete symmetry in the α 's and complete symmetry in the β 's. Now dots can be created or annihilated by contraction with matrices $(\vec{\sigma} \cdot \vec{p} \pm p_0)/m$. So we may consider instead a wave function

$$\psi_{\alpha_1 \dots \alpha_a \beta_1 \dots \beta_a}$$

or simplifying notation

$$\psi_{\alpha_1 \alpha_2 \dots}$$

This gives a reducible representation containing only those of type $(j,0)$. Subsidiary conditions having been used to make the spin unique, only a single value of j will appear. But it may appear several times, which we enumerate by an index n . Then

$$\psi_{\alpha_1 \alpha_2 \dots} = \sum_{nm} C_{\alpha_1 \alpha_2 \dots}^{n j m} \psi_{n j m} \quad (A.1)$$

$$\psi_{n j m} = \sum \bar{C}_{n j m}^{\alpha_1 \alpha_2 \dots} \psi_{\alpha_1 \alpha_2 \dots} \quad (A.2)$$

where the mutually reciprocal matrices C and $\bar{C}$ have constant coefficients. Because of the common $(j,0)$ transformation character $\psi_{n j m}$ must have the form

$$\psi_{n j m} = a_n \psi_{j m} \quad (A.3)$$

where the a_n are constants. Let now $\psi'_{n' j m}$ be analogous quantities arising in the reduction of some other representation of the same state. Then

$$\psi'_{n' j m} = b_{n'} \psi_{j m}$$

where the $b_{n'}$ are constants. Thus we can use for example

$$\psi'_{n' j m} = (b_{n'}/a_n) \psi_{n j m}$$

to complete the chain of transformations from the first to the second representation. Only at the points where dots are added or removed do momentum dependent coefficients appear in the sequence of linear transformations, and the dependence is polynomial.

R E F E R E N C E S

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- 7) See Ref. 1), or note that the property holds for $m=0$ and that the reduction formula

$$\frac{d \cos n\theta}{d \cos \theta} = \frac{n \sin n\theta}{\sin \theta} = \frac{n \sin (n-2)\theta}{\sin \theta} + 2 \cos (n-1)\theta$$

allows the first derivative, and then any derivative, to be expressed as a superposition of terms $\cos n'\theta$ with positive coefficients.

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