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THE TOP POSITION OF THE LOOPS IN REGGE PARTIAL WAVES

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A B S T R A C T

It is shown that the "top positions" for loops in the partial waves of a Regge amplitude depend sensitively on the presence of zeros in the amplitude near $t = 0$. For example, for an input Regge trajectory with slope 1 GeV^{-2} , the slope of the "top trajectory" (J vs s_{top}) for an odd signature amplitude is found to be $\sim 1 \text{ GeV}^{-2}$ with nonsense choosing mechanism at $\alpha = 0$, but is $\sim 0.5 \text{ GeV}^{-2}$ with sense-choosing mechanism. The assumption that high J low mass loops are resonances is discarded since these high J waves are expected to be unreliable. Also this assumption can lead to the violation of both the s channel unitarity condition and the Froissart bound. From the consideration of phase factor, a top trajectory formula for the amplitude with nonsense-choosing mechanism is derived. The formula gives reasonable values for the top position over a wide range of s_0 values. For $\pi\pi \rightarrow \pi\omega$ case, using strong bootstrap assumptions together with this formula, one finds $\alpha_\zeta(0) = 0.45$. This zero intercept is in a good agreement with linear extrapolation of physical meson masses assuming exchange degeneracy. It is argued, however, that this agreement is rather accidental.

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I. - INTRODUCTION

Recently much attention has been drawn on the loop behaviour in the partial waves of Regge amplitudes. These loops appear for the amplitudes containing a phase factor $\exp(-i\pi\alpha(t))$, where the function falls off linearly with t near the forward direction. Some of the loops, especially those in the intermediate and low energy regions are relatively large in magnitude, while others such as for high J at low energies or all those at high energies are much smaller. Many of them are tiny.

In studying the partial waves for πN spin-flip amplitude B_ξ , Schmid ¹⁾ found that some features of B_ξ^s (e.g., the position of the top of the loop) are similar to that of the known resonances in the region for $1.5 < P_{\text{lab}} < 2.5$ GeV/c. Collins, Johnson and Squires ²⁾ found that πN partial waves such as P_{11} and D_{13} obtained from the extrapolation of Regge amplitudes of $P + P' + \rho + N + \Delta$ exchange resemble again to the experimental phase shifts, even down to a few hundred MeV. It seems that the extrapolation of the asymptotic amplitude could play an important role in describing the partial waves in the Argand diagram at the intermediate and, more optimistically, perhaps even at relatively low energies (cf. Alessandrini, Amati and Squires ³⁾).

In the spin $0-\frac{1}{2}$ scattering, the Regge amplitude (either spin-flip or non-flip) can be written as

$$A(s, t) = \frac{\beta(t) \cdot \eta(t)}{\Gamma(\alpha) \sin \pi \alpha} \left(\frac{s}{s_0} \right)^{\alpha-1} \quad (1)$$

where $\beta(t)$ is a smooth residue function and $\eta(t)$ depends on the trajectory coupling at $\alpha = 0$ (cf. Table I, also Ref. ⁴⁾).

Thus Regge amplitudes can be divided into three groups (called here types A_1 , A_2 and A_3) each of them containing one of the following factors :

$$\eta_1 = 1 \pm e^{-i\pi\alpha}, \quad (2)$$

$$\eta_2 = \frac{1}{\alpha} (1 - e^{-i\pi\alpha}), \quad (3)$$

$$\eta_3 = \alpha (1 + e^{-i\pi\alpha}). \quad (4)$$

In Section 2, we present the "top-position" as a function of s_0 for these three types of amplitudes. We also show that some features in the behaviour of the top position can be readily understood from simple formulae derived from the relevant phase factors alone. As we shall see only the "top trajectory" formula for A_1 amplitude gives reasonable representation for the actual top position over a wide range of s_0 . The corresponding formula for A_2 although cannot be used to predict the actual top position, explains the difference in slope between top trajectory for A_1 and that for A_2 .

In Section 3, we extend our top trajectory formula for A_1 given in Section 2 to the general mass case and apply it to the B amplitude for πN charge exchange, the $\pi\pi \rightarrow \pi\omega$ and $\pi\pi \rightarrow \pi H$ amplitudes. We do not believe that there should be a simple correlation between Regge loops and the resonance loops. Nevertheless, we study the consequences that will follow if one associates the top trajectory with the direct channel Regge trajectory. Our formula also indicates that the loop structures present at low energies in the high partial waves (so-called "ancestors")⁵⁾ should be on the top trajectories with similar slopes as that for the prominent loops. Arguments that these high J waves are expected to be unreliable are presented. We observe that, should the ancestors be identified as resonances, they can lead to the violation of the Froissart bound. Finally in Section 4, we summarize the main points of this work.

2. - THE MAXIMA OF THE LOOPS FOR A_1, A_2 AND A_3 AMPLITUDES

In Fig. 1, we illustrate the top position of the loops for A_1 type of non-flip amplitude ⁶⁾ as a function of the parameter s_0 for $J = 2, 3, 4$ and 5 . Although the top position depends somewhat on the s_0 value, the general behaviour of the curve can be understood in terms of the behaviour of the phase factor $e^{-i\pi\alpha}$ alone. We recall ⁷⁾ that for πN scattering in the spinless case the partial wave expansion for this phase factor is given by

$$e^{-i\pi\alpha} = \sum_{J=0}^{\infty} (2J+1) P_J(\cos\vartheta) a_J \quad (5)$$

with

$$a_J = e^{-i\pi\alpha_0} (-i)^J e^{2\pi\alpha_1 q^2 i} j_J(2\pi\alpha_1 q^2)$$

and for the spin-flip case is given by

$$e^{-i\pi\alpha} = \sum_{J=1}^{\infty} (2J+1) P'_J(\cos\vartheta) a_J^F \quad (6)$$

where

$$a_J^F = e^{-i\pi\alpha_0} (-i)^{J-1} e^{2\pi\alpha_1 q^2 i} j_J(2\pi\alpha_1 q^2) / (2\pi\alpha_1 q^2)$$

Thus the expressions for the tops of the loops are

$$q_{top}^2 = \frac{J + 2\alpha_0 - (4n+1)}{4\alpha_1} \quad \text{for } a_J \quad (7)$$

and

$$q_{top}^2 = \frac{J + 2\alpha_0 - (4n+2)}{4\alpha_1} \quad \text{for } a_J^F. \quad (8)$$

It turns out that the relatively prominent loops are usually associated with $n = 0$ while the "ancestor" loops correspond to $n > 0$. In this Section we consider only the case $n = 0$. The top positions predicted by Eq. (7) are illustrated in Fig. 1 as straight lines. For s_0 between 0.3 and 10.0 GeV^2 the top position given by Eq. (7) is within 0.6 units of J . In the same region of s_0 , the average slope $\Delta J / \Delta s$ for the top trajectory varies from 0.9 to 1.1 near $J = 3$. This is in a rough agreement with Eq. (7), where the relevant slope is $\sim 1 \text{ GeV}^{-2}$.

Corresponding top position for A_2 amplitudes is shown in Fig. 2. Unlike in the case of A_1 , the top position here shows a strong dependence on s_0 . The slope $\Delta J / \Delta s$ in the interval $0.3 \lesssim s_0 \lesssim 10.0 \text{ GeV}^2$ varies from 0.35 to 0.6 near $J = 3$. Also the top position bifurcates at large values of s_0 for $J = 4$ and 5. The fact that $\Delta J / \Delta s = \frac{1}{2}$ follows from the particular form of the function η_2 . This can be seen by writing

$$A_2 \sim \frac{1 - e^{-i\pi\alpha}}{\Gamma(\alpha+1) \sin \pi\alpha} \left(\frac{s}{s_0}\right)^{\alpha-1} \beta(t) = i e^{-i\frac{\pi}{2}\alpha} \frac{\beta(t)}{\Gamma(\alpha+1) \cos \frac{\pi\alpha}{2}} \left(\frac{s}{s_0}\right)^{\alpha-1} \quad (9)$$

Since for $-2 < \alpha < 1$ the factor $\beta(t) / [\Gamma(\alpha+1) \cos(\pi\alpha/2)]$ is a smooth function of t and does not contain any known zeros, one expects that the controlling phase factor should be here $i e^{-i\pi\alpha/2}$, as compared to the $e^{-i\pi\alpha}$ factor for A_1 . From Eqs. (5) and (7) one sees that it is precisely this difference between the two factors which causes the corresponding difference in the slope by a factor of two.

Top positions for A_3 are illustrated in Fig. 3. The slope $\Delta J / \Delta s$ varies from 1.1 to 1.5 GeV^{-2} in the range $0.3 \lesssim s_0 \lesssim 10.0$ for $J = 3$. There are two families of tops, one of which disappears for small values of s_0 . The expression for the top position associated with the relevant phase factor is more complicated than in the previous cases and will not be discussed here.

3. - TOP TRAJECTORY FORMULA AND ITS APPLICATIONS

A - Top trajectory formula for the general mass case

In the previous Section we have seen that Eq. (7) gives reasonable approximation for the actual top position. We consider further the top trajectory formula here, restricting ourselves to the A_1 type only. In a general mass case, Eq. (7) should be changed to

$$q_i q_f \Big|_{top} = \frac{J + 2(\alpha_0 + \alpha_1 t_{max}) - (4n+1)}{4\alpha_1}. \quad (10)$$

Here we have replaced q^2 by $q_i q_f$, the product of initial and final cms momenta and also α_0 by $\alpha_0 + \alpha_1 t_{max}$. If either in the initial or in the final state the two particles have equal masses, the equation is simplified to

$$\Delta_{top} = \sum + \frac{J + 2\alpha_0 - (4n+1)}{\alpha_1} \quad (11)$$

with $\sum$ denoting the sum of four particle masses squared. By comparing Eqs. (7) and (8), one sees that corresponding formula for the flip case can be obtained by replacing J in Eqs. (10) and (11) by $J-1$. For example, Eq. (11) for the flip amplitude reads

$$\Delta_{top} = \sum + \frac{J + 2\alpha_0 - (4n+2)}{\alpha_1}. \quad (11a)$$

Again for $n = 0$ one gets the top positions of the prominent loops and for $n \geq 1$, the positions of ancestors. The situation for the ancestors is more complicated and is discussed in Section 3D.

B - Applications of the top trajectory formula

In Section 2, we have already compared the top trajectory formula for the non-flip amplitude [Eq. (7), with $n=0$] with the actual top positions for $J = 2-5$. In Fig. 4, a comparison for the top trajectory formula for the flip amplitude [Eq. (8), with $n=0$] and the actual top positions for the Regge amplitude $B^{(-)}$ used in Ref. 7) is presented up to $J = 12$. Note that for $J < 10$ the agreement is reasonable. The maximum difference between two loops is 0.6, which occurs near $J = 5$. In the same Figure, we also show the top trajectory which is obtained by taking for each energy the maximum of $\text{Im } B_J$ versus J . We see that for a range of J the maximum in J does not coincide with the maximum in s . As s increases, the difference becomes larger and larger. In this paper, whenever not explicitly stated, maximum in s is always understood.

In the Regge parametrization one can replace $(s/s_0)^{\alpha-1}$ factor by $[(s-u)/s_0]^{\alpha-1}$ factor. For $\pi\pi \rightarrow \pi\omega$ case using this amplitude as suggested by earlier FESR analysis, Rubinstein, Schwimmer, Veneziano and Virasoro 8) found that for $J = 3$ and 5 the top positions of the prominent loop occur at 1.66 and 2.17 GeV, respectively. Equation (11a) predicts it to be at 1.63 and 2.16 GeV, respectively. For $\pi\pi \rightarrow \pi H$ case 8) the comparison is again quite favourable. We refer the reader to Table II in the Appendix for the detailed numbers.

C - Bootstrap conditions implied by the trajectory formula

If one correlates these "prominent loops" with resonances and identifies the energy for the top of the loop to be the resonance mass, then expressions such as (7), (8), (10), (11) and (11a) can be thought of as direct channel trajectory formulae. It is, however, hard to believe that there is such a simple correlation. Nevertheless we discuss here some consequences following from this strong assumption. For the reaction $\pi\pi \rightarrow \pi\omega$, assuming the ρ trajectory is dominating in all three channels, one has from Eq. (11a)

$$\Delta_{top} = \sum + \frac{J + 2\alpha_0 - 2}{\alpha_1} \quad (12)$$

After substituting $J = \alpha_0 + \alpha_1 s_{top}$ in the expression one finds

$$\alpha_g(0) = 0.45 .$$

This value is in a good agreement with the zero intercept obtained from the linear extrapolation of the physical meson masses together with the assumption of exchange degeneracy⁹⁾. Similarly, for reaction $\pi\pi \rightarrow \pi\pi$ assuming the exchange degenerate trajectories $\rho + f^0$ are dominating, one obtains from Eq. (11)

$$\Delta_{top} = \sum + \frac{J + 2\alpha_0 - 1}{\alpha_1} \quad (13)$$

and

$$\alpha_g(0) = 0.31 ,$$

In general for the spinless meson-meson scattering, if only one exchange-degenerate trajectory function is dominating in all three channels one has

$$\alpha_0 = \frac{1 - \alpha_1 \sum}{3} .$$

Here the zero intercept is a function of external masses. For instance, for KK scattering, the bootstrap with exchange degenerate f^0 , ω^0 , ρ and A_2 trajectories would give $\alpha_0 \approx 0$. It is possible that the corresponding s_0 value for $KK \rightarrow KK$ differs considerably from that in $\pi\pi \rightarrow \pi\pi$, and in $\pi\pi \rightarrow \pi\pi$ from that in $\pi\pi \rightarrow \pi\omega$, so Eqs. (11) and (11a) might break here and the paradox would disappear.

D - Ancestors

For $n > 0$, loops are in general smaller than those for $n = 0$. For $\pi\pi \rightarrow \pi\pi$ scattering with the exchange of degenerate trajectories $\alpha = 0.5 + t$ and with $n = 1$ in the region of $0.3 \lesssim s_0 \lesssim 5.0$, the slope of ancestor trajectories remains $\Delta J / \Delta s_{\text{top}} \approx 1 \text{ GeV}^{-2}$ for the A_1 type of amplitude.

The quoted slope is the same both with and without the factor $\Gamma(\alpha) \sin \pi\alpha$ in the denominator of Eq. (1). In contrast to the prominent loops, the ancestor trajectories are sensitive to s_0 value. Only the slope of these trajectories agrees with that predicted from the phase factor alone.

It has been argued ¹⁰⁾ that, with conventional Regge parametrization, partial waves for Regge amplitudes only up to certain J_{max} value can be trusted. In particular, the J_{max} value should crudely correspond to the partial wave where the first zero in its contribution to the amplitude coincides with the first zero in the same amplitude. The J values for the ancestors are usually larger than this J_{max} value and therefore unreliable. Also this situation is aggravated here by the fact that the extrapolation of an asymptotic amplitude to the low energy region is usually not well defined, in particular for high J values. Therefore, one should definitely not take these high J loops seriously.

We mention here, if the ancestor loops occur below inelastic threshold, the assumption that these ancestors are physical can lead to the violation of unitarity condition ¹⁰⁾. This is because, below inelastic threshold, the physical partial waves should have elasticity equal to unity, but the elasticity for ancestor loops is smaller. Also, since these loops correspond to high spin low mass objects, the assumption that they lie on the direct channel Regge trajectories can lead to the violation of the Froissart bound - some ancestor trajectories have intercepts even above $J = 2$.

4. - DISCUSSION

In Section 2, we presented some features of the top trajectories associated with three types of parametrizations. The top trajectory for A_1 can be approximately described by expressions such as Eqs. (7), (8) (10), (11) and (11a). On the other hand, the slope of the top trajectory for A_2 is found to be smaller than that for A_1 . This can be understood through the inspection on the relevant phase factor [see Eq. (9)]. The main difference between A_1 and A_2 is that there is a zero in the residue function at $\alpha = 0$ for A_1 , while this zero is removed for A_2 . This demonstrates that the partial wave behaviour is strongly coupled to the presence of zeros in the residue near $t = 0$. We have investigated the effect of s_0 only for the specific amplitudes considered. For instance, if Regge residues happen to have other parametrization (e.g., introducing a cross-over zero ¹¹⁾ in the residue near $t = 0$), this could complicate the present picture.

In the discussion of the loop behaviour, we would like to distinguish between the prominent loops and the second order loops. The prominent loops are defined to be the minimum number of waves such that when one sums up their contributions, one essentially recovers the Regge amplitude. The crude resemblance between the behaviour of the prominent loops and the physical partial waves has been demonstrated phenomenologically by Schmid ¹⁾ and also by Collins et al. ²⁾. On the other hand, it is hard to believe that similar resemblance could still preserve on the level of the second order effect. We suggest one should not consider these two effects on the same footing. We mention here some speculative examples of such second order loops :

- a - The ancestors : In Ref. ⁵⁾ it was found that it is unlikely that these objects could be identified as resonances. We agree with their point of view. Identifying them as resonances could lead to the violation of the Froissart bound, as the slope of their top trajectories appears to be standard. These loops are of second order. Their existence in the Argand diagram for the physical amplitude seems to be doubtful. In particular, the extraction from a Regge amplitude for such high J values is unreliable.

- b - Since exchange degeneracy seems to be satisfied approximately ^{12), 9)}, we suggest loops caused by the breaking of exchange degeneracy should be classified as second order effect also.
- c - For πN charge-exchange reaction, the B amplitude is much more important than the A' amplitude, and the B amplitude approximately attains its asymptotic behaviour at considerably lower energy than the A' amplitude. For practical purpose, below $P_L \sim 2 \text{ GeV}/c$, we suggest loops from A' should be thought of as of second order effect and should not be treated in the same footing as those obtained from B amplitude ⁷⁾.

Now let us turn to the prominent loops. It is unlikely that these loops should be identified as resonances even in the approximate sense. If one still wants to associate these loops as resonances, the following remarks become relevant.

- a - A_2 amplitude cannot play an important role in bootstrapping trajectories with nominal slopes.
- b - In $\pi\pi \rightarrow \pi\omega$ calculation, bootstrap of the A_1 type leads to

$$\alpha_g(0) = 0.45.$$

This is in good agreement with linear extrapolation to the meson physical masses with the assumption of exchange degeneracy. However, the result seems to be accidental. The reason is as follows. Using similar assumption on $\pi\pi \rightarrow \pi\pi$, $KK \rightarrow KK$ reactions, one obtains different $\alpha_g(0)$ zero intercepts here. Of course, one can assume for the latter cases the relevant residue could be sufficiently different such that formula (13) actually breaks down. However, this in turn makes the success of using Eq. (13) for $\pi\pi \rightarrow \pi\omega$ case to be accidental.

- c - We recall the resonance mass, or the position of the top of the loop is defined most unambiguously in the narrow width approximation, where maximum of $\text{Im } a_\ell(s)$ in s coincides with that in ℓ . In the region where maximum in s is far from that in ℓ , a priori one does not know which top position should be correlated with the resonance mass. Therefore, simple prescription of identifying top position with resonance mass breaks down here.

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A P P E N D I X

PARTIAL WAVE AMPLITUDES FOR $\pi\pi \rightarrow \pi H$ AND THE TOP POSITIONS
FOR THE LOOPS

In this case there are two independent helicity amplitudes.
They can be written as

$$A_{\lambda=1}^J = \frac{\sqrt{2J(J+1)}}{2J+1} \frac{S_{\pi\pi}}{8\sqrt{\lambda}} \int_{-1}^1 dx [P_{J-1} - P_{J+1}] F_1,$$

and

$$A_{\lambda=0}^J = \frac{S_{\pi H}}{8m_H} \int_{-1}^1 F_2 P_J dx + \frac{S_{\pi\pi}}{8m_H} (1+m_H^2-m_\pi^2) \int_{-1}^1 F_1 x P_J dx$$

where

$$F_1 = \frac{1}{\Gamma(\alpha+1)} \left\{ \alpha \beta_A + 2 \beta_B \nu \right\} \left(\frac{\nu}{.5} \right)^{\alpha-1} \left(\operatorname{tg} \frac{\pi\alpha}{2} + i \right) + (t \leftrightarrow u)$$

and

$$F_2 = \frac{1}{\Gamma(\alpha+1)} \left\{ 3\alpha \beta_A - 2 \beta_B \nu \right\} \left(\frac{\nu}{.5} \right)^{\alpha-1} \left(\operatorname{tg} \frac{\pi\alpha}{2} + i \right) + (t \leftrightarrow u),$$

$$\nu = \frac{1-u}{2}.$$

Partial wave amplitudes with definite values of orbital angular momentum L are given as

$$A_{L=J+1}^J = \sqrt{\frac{J}{2J+1}} A_{\lambda=0}^J + \sqrt{\frac{2(J+1)}{2J+1}} A_{\lambda=1}^J,$$

$$A_{L=J-1}^J = -\sqrt{\frac{J+1}{2J+1}} A_{\lambda=0}^J + \sqrt{\frac{2J}{2J+1}} A_{\lambda=1}^J.$$

Note that here F_1^{Regge} contains a sum of two amplitudes of type A_1 and A_2 [cf. Eq. (2)]. Since A_2 type generates trajectories in the direct channel with the slope twice as small as the slope of the input trajectory, we argue that with this type of parametrization the coupling β_B/β_A has to be relatively small. In other words both F_1 and F_2 have to be dominated by the A_1 type of amplitude. Choosing ⁸⁾ $\beta_B/\beta_A = -1/6$, we get the results shown in Table II.

Note that the top positions given by Eqs. (11) and (11a) are in a good agreement with the numerical results. We observe that since it is the same resonance which couples to $A_{L=J-1}^J$ and $A_{L=J+1}^J$ and in fact also to $A_{L=J}^J$ for the $\pi\pi \rightarrow \pi\omega$ case, the resonance loop should have the same top for all these three cases. However, for $J = 3$, our results are, respectively, 2.05, 1.85 and 1.63 GeV, and for $J = 5$ they are 2.47, 2.40 and 2.16 GeV.

- TABLE I -

Function (t)

Signature	Mechanism	Non-flip amplitude	Flip amplitude
Even	Chew	$1+e^{-i\pi\alpha}$	$\alpha(1+e^{-i\pi\alpha})$
	Gell-Mann	$1+e^{-i\pi\alpha}$	$1+e^{-i\pi\alpha}$
	Non-compensating	$\alpha(1+e^{-i\pi\alpha})$	$\alpha(1+e^{-i\pi\alpha})$
Odd	Choosing sense	$\frac{1}{\alpha}(1-e^{-i\pi\alpha})$	$1-e^{-i\pi\alpha}$
	Choosing nonsense	$1-e^{-i\pi\alpha}$	$1-e^{-i\pi\alpha}$

- TABLE II -

Top positions for $\pi\pi \rightarrow \pi H$ partial waves ¹³⁾

		$A_{\lambda=1}^J$	$A_{\lambda=0}^J$ First term	$A_{\lambda=0}^J$ Second term	$A_{\lambda=0}^J$	$A_L^J = J+1$	$A_L^J = J-1$
$J = 3$	Eqs. (11a), (11)	1.74	2.01	1.74	-	-	-
	Our numerical results	1.70	2.00	1.70	2.00	1.85	2.05
$J = 5$	Eqs. (11a), (11)	2.24	2.46	2.24	-	-	-
	Our numerical results	2.20	2.45	2.15	2.45	2.40	2.47

R E F E R E N C E S

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$$A_i = \eta_i(t) \left(\frac{s}{s_0}\right)^{\alpha-1}$$

where $\alpha = 0.5 + t$ and even signature is used for the A_1 amplitude. We have repeated our calculation with the inclusion of smooth $\Gamma(\alpha)\sin\pi\alpha$ factor in the denominator as in Eq. (1). The main features remain essentially the same.
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FIGURE CAPTIONS

- Figure 1 : The top position of the partial waves for the amplitude A_1 vs s_0 for $J = 2$ to 5 (see Ref. ⁶). The curves are the top positions evaluated numerically. The straight lines are the corresponding top positions given by Eq. (7).
- Figure 2 : The top position of the partial waves for the amplitude A_2 vs s_0 for $J = 2$ to 5 (see Ref. ⁶). Curves are the top positions evaluated numerically. Note the bifurcations for $J = 4$ and 5 at large s_0 values.
- Figure 3 : The top position of the partial waves for the amplitude A_3 vs s_0 for $J = 2$ to 5 (see Ref. ⁶). The solid curves and the dashed curves correspond to two families of tops ; as s_0 value decreases, the tops corresponding to dashed curves eventually disappear.
- Figure 4 : The top trajectories : I is given by Eq. (8) with $n = 0$, II and III are obtained numerically for maximum of $\text{Im} B^{(-)}$ respectively in s and in ℓ . The amplitude $B^{(-)}$ is given by Eqs. (2.9) and (2.10) of Ref. ⁷.

FIG.1

FIG.2

FIG.3

FIG.4