Strong monotonicity in mixed-state entanglement manipulation

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A strong entanglement monotone, which never increases under local operations and classical communications (LOCC), restricts quantum entanglement manipulation more strongly than the usual monotone since the usual one does not increase on average under LOCC. We propose new strong monotones in mixed-state entanglement manipulation under LOCC. These are related to the decomposability and 1-positivity of an operator constructed from a quantum state, and reveal geometrical characteristics of entangled states. These are lower bounded by the negativity or generalized robustness of entanglement.

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I. INTRODUCTION

It is a key concept for quantum information science that distant parties can manipulate quantum entanglement by local operations and classical communication (LOCC). However, the entanglement manipulation suffers some fundamental restrictions: LOCC cannot create entanglement \( \mathcal{I} \) and LOCC cannot increase the total amount of entanglement. This monotonicity is characterized by mathematical functions \( E(\sigma) \), called entanglement monotones \( \mathcal{M} \). When a quantum state \( \rho_i \) is obtained from \( \sigma \) with probability \( p_i > 0 \) by LOCC (i indexes the multiple outcomes), the functions satisfy

\[
E(\sigma) \geq \sum_i p_i E(\rho_i),
\]

and therefore \( E(\sigma) \) does not increase on average under LOCC. Many such monotones have been proposed such as entanglement measures (e.g. entanglement cost \( \mathcal{E} \), distillable entanglement \( \mathcal{D} \), and relative entropy of entanglement \( \mathcal{R} \)), negativity \( \mathcal{N} \), robustness of entanglement \( \mathcal{R} \), best separable approximation (BSA) measure \( \mathcal{B} \), and so on \cite{footnote1}.

On the other hand, there exists a much stronger restriction in the entanglement manipulation: the Schmidt number \( \mathcal{S} \), which is a general extension of the Schmidt rank to mixed states, cannot increase even with an infinitesimally small probability. This type of restriction, called strong monotonicity here, may be characterized by strong monotone functions \( M(\sigma) \) which satisfy

\[
M(\sigma) \geq M(\rho_i) \quad \text{for all } \rho_i.
\]

Namely \( M(\sigma) \) never increases under LOCC. Note that the strong monotones are generally discontinuous functions as explicitly shown later. Concerning the conversion between bipartite pure states, the Schmidt number is a unique strong monotone (the conversion is impossible when the Schmidt rank of the target state is larger than that of the initial state, but otherwise the conversion is possible with nonzero probability \( \mathcal{M} \)). Positive partial transpose preserving (PPT-preserving) operations can overcome the monotonicity of the Schmidt number, and therefore all pure entangled states become convertible under PPT-preserving operations \( \mathcal{M} \). Here, a map \( \Lambda \) is called PPT-preserving \( \mathcal{P} \) when both \( \Lambda \) and \( \Gamma = \sigma \Lambda \) is a completely positive (CP) map with \( \Gamma \) being a map of the partial transpose \( \mathcal{T} \). On the other hand, the manipulation of mixed-state entanglement still suffers some restriction even under PPT-preserving operations in single-copy settings \( \mathcal{T} \). This implies that there certainly exists strong monotonicity independent on the Schmidt number in mixed-state entanglement manipulation.

In this paper, we propose new strong monotones which are related to the decomposability and 1-positivity of an operator constructed from a quantum state. Here, an operator \( Z \) is called decomposable when \( Z = X + Y^T \) with \( X, Y \geq 0 \), and \( Z \) is called 1-positive when \( \langle ef | Z | ef \rangle \geq 0 \) for every product states \( |ef \rangle \) (see e.g. \cite{footnote2,footnote3,footnote4,footnote5,footnote6,footnote7} for the relation between entanglement and decomposability or 1-positivity). The singlet fraction and negativity maximized over (stochastic) LOCC are also strong monotones, but strong monotones studied in this paper are slightly different from those and reveal geometrical characteristics of entangled states. These are lower bounded by the negativity or generalized robustness of entanglement.

II. STRONG MONOTONE \( M_1 \)

The first strong monotone we propose is the following:

Theorem 1: Let \( \sigma^T = P - Q \) be the Jordan decomposition (orthogonal decomposition) of \( \sigma^T \) and hence \( P, Q \geq 0 \), \( PQ = 0 \). The function \( M_1(\sigma) \), which is defined as the minimal \( x \) subject that \( \sigma \) is decomposable, is a strong monotone, i.e. \( M_1(\sigma) \) never increases under LOCC (and even under PPT-preserving operations).

Note that \( \sigma - (1 - x) P^T \) is always decomposable for \( x = 1 \), and hence \( M_1(\sigma) \leq 1 \). Before proving the above theorem, let us show explicit examples of \( M_1(\sigma) \) for several important classes of states.

(i) Separable states: For every PPT states \( \sigma^T \geq 0 \),
we have $P^R = \sigma$ and hence $\sigma - (1 - x)P^R = x\sigma$ which is decomposable only for $x \geq 0$. Since all separable states are PPT states, $M_1(\sigma) = 0$ for every separable $\sigma$.

(ii) Entangled pure states: Let

$$|\phi^+_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$$

be a maximally entangled state on $\mathbb{C}^d \otimes \mathbb{C}^d$ and $P^+_d = \langle \phi^+_d | \langle \phi^+_d |$. When $\sigma = P^+_d$, $P = P^S_d/d$ where $P^S_d$ is the projector onto the symmetric subspace on $\mathbb{C}^d \otimes \mathbb{C}^d$. Any decomposable operator $Z$ must satisfy $\langle ef | Z | ef \rangle \geq 0$ for every product states $|ef\rangle$, i.e. 1-positive, but

$$\langle 01 | P^+_d - (1 - x)(P^S_d/d)^R | 01 \rangle = -\frac{x - 1}{2d}.$$  

Therefore $P^+_d - (1 - x)(P^S_d/d)^R$ cannot be decomposable for $x < 1$ and $M_1(P^+_d) = 1$ independent on $d$. It has been shown that an entangled $|\psi\rangle$ can be converted to $P^+_d$ by LOCC where $r \geq 2$ is the Schmidt number of $|\psi\rangle$. Since $M_1$ never increases under LOCC and $M_1 \leq 1$, we have $M_1(|\psi\rangle) = 1$ for every entangled $|\psi\rangle$. Therefore, $M_1$ does not distinguish entangled pure states at all. This property however is desirable for the purpose of this paper that is to study strong monotonicity independent on the Schmidt number. As mentioned before, the Schmidt number is a unique strong monotone concerning the conversion between pure states. Therefore, the strong monotones independent on the Schmidt number should not distinguish entangled pure states.

It should be noted that $M_1(|ef\rangle) = 0$ as shown in (i) but in the close vicinity of $|ef\rangle$ there always exists a partially entangled pure state $|\psi\rangle$ for which $M_1(|\psi\rangle) = 1$. As a result, it is found that $M_1(\sigma)$ is a discontinuous function.

According to the above examples (i) and (ii), the following is concluded:

Corollary 1: If $\sigma$ is single-copy distillable under LOCC (or under PPT-preserving operations), then $M_1(\sigma) = 1$.

(iii) Antisymmetric Werner states: For an antisymmetric Werner state of $\sigma^A = 2/(d^2 - d)P^A_d$, $P = (1 - P^+_d)/(d^2 - d)$ where $1_d$ and $P^A_d$ is the identity on $\mathbb{C}^d \otimes \mathbb{C}^d$ and projector onto the antisymmetric subspace on $\mathbb{C}^d \otimes \mathbb{C}^d$, respectively. Then,

$$\langle 00 | P^A_d - (1 - x)(1_d - P^+_d)/(d^2 - d) | 00 \rangle = -\frac{1 - x}{d^2 - d},$$

which cannot be decomposable for $x < 1$, and $M_1(\sigma^A) = 1$.

(iv) Convex combination of $\sigma_0$ and $P^S_0$: Let $\sigma_0 = P_0 - Q_0$ be the Jordan decomposition. For the state of

$$\sigma = \sigma_0 + \lambda P^R_0,$$

$$\sigma^R = (1 + \lambda)P_0 - Q_0 \quad \text{(here $\sigma$ is not normalized but $M_1(\sigma)$ does not depend on the normalization)}.$$

As a result,

$$\sigma - (1 - x)P^R = \sigma_0 + \lambda P^R_0 - (1 - x)(1 + \lambda)P^R_0 = \sigma_0 - [1 - x(1 + \lambda)]P^R_0,$$

and therefore

$$M_1(\sigma) = \frac{M_1(\sigma_0)}{1 + x}$$

for the state of Eq. (6). An entangled isotropic state

$$\sigma_I = \eta P^+_d + (1 - \eta)\frac{1_d - P^+_d}{d^2 - 1},$$

where $\eta > 1/d$, can be rewritten as $\sigma_I \propto \sigma_0 + \frac{2(d - 1)}{(d - 1)(d + 1)}P^R_0$ with $\sigma_0 = P^+_d$ [correspondingly $P^0_0 = (1_d + dP^+_d)/(2d)$], and therefore

$$M_1(\sigma_I) = \frac{(d\eta - 1)(d + 1)}{d\eta + 1}(d - 1).$$

Similarly, for an entangled Werner state

$$\sigma_W = \mu \frac{2}{d^2 - d}P^A_d + (1 - \mu)\frac{2}{d^2 + 2}P^S_d,$$

where $\mu > 1/2$, putting $\sigma_0 = \sigma^A_d$ we have

$$M_1(\sigma_W) = \frac{(2\mu - 1)(d + 1)}{2\mu + d - 1}.$$

It has been mentioned that $\eta$ of an isotropic state and $\mu$ of a Werner state cannot increase under LOCC. This can be confirmed by $M_1(\sigma_I)$ and $M_1(\sigma_W)$, and moreover it is found that this is the case even under PPT-preserving operations, since these functions are monotonic with respect to $\eta$ and $\mu$, respectively, and these never increase under LOCC and even under PPT-preserving operations.

Equating Eq. (10) and Eq. (12), we have a relation:

$$\mu = \frac{(d - 1)\eta}{(d - 2)\eta + 1},$$

and therefore the reversible conversion of $\sigma_W \leftrightarrow \sigma_I$ is not prohibited by $M_1$ if $\mu$ and $\eta$ satisfy Eq. (13). Indeed, this reversible conversion is possible by PPT-preserving operations, whose trace non-preserving maps are

$$\sigma_W \rightarrow \sigma_I : \Lambda(X) = (\text{tr}XP^A_d)P^+_d + (\text{tr}XP^S_d)\frac{1_d - P^+_d}{d + 1},$$

$$\sigma_I \rightarrow \sigma_W : \Lambda(X) = (\text{tr}XP^+_d)P^A_d + (\text{tr}X(1_d - P^+_d))\frac{P^S_d}{d + 1}.$$
and therefore if \( \sigma = (|00\rangle + |11\rangle)/\sqrt{2} \) and \( |\psi^\pm \rangle = (|01\rangle \pm |10\rangle)/\sqrt{2} \), which is the magic basis and hence \( |\tilde{e}_i \rangle = (\sigma_2 \otimes \sigma_2)|e_i^+ \rangle = |e_i \rangle \) \( \mathbb{R} \leq 2 \).

Then we have

\[
\sigma_B - (1 - x)P = \left[ p_0 - \frac{(1 - x)(2p_0 - 1)}{4} \right]|e\rangle \langle e| + \sum_{i=1}^{3} \left[ p_i - \frac{(1 - x)(2p_0 - 1 + 4p_i)}{4} \right]|e_i \rangle \langle e_i |.
\]

(15)

For the decomposability of such a Bell diagonal operator, the following is useful:

**Lemma 1:** An operator \( A = \sum_{i=0}^{3} a_i |e_i \rangle \langle e_i | \) on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \), where \( a_i \geq a_{i+1}, \) is decomposable (and 1-positive) if and only if \( a_2 + a_3 \geq 0 \).

**Proof:** When \( A \) on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) is expressed as \( A = (I \otimes \Theta)P_+^2 \) with \( \Theta \) being a map, the following four statements are equivalent \( \mathbb{R} \leq 2 \geq 26 \):

(a) \( A \) is decomposable, (b) \( \Theta \) is a decomposable positive map, (c) \( \Theta \) is a positive map, and (d) \( A \) is 1-positive. Since \( |e_i \rangle \) is an orthonormal set, any pure state is expanded as \( |\psi \rangle = \sum_{i=0}^{3} \lambda_i |e_i \rangle \) with \( \sum_{i=0}^{3} |\lambda_i|^2 = 1 \). So that \( |\psi \rangle \) is a product state, \( |\psi \rangle |\psi \rangle = \sum_{i=0}^{3} \lambda_i^2 = 0 \) \( \mathbb{R} \leq 0 \). When the real and imaginary part of \( \lambda_i \) are \( r_i \) and \( c_i \), respectively, the above two conditions are written as \( \sum_{i=0}^{3} r_i^2 = 1 \) and \( \sum_{i=0}^{3} c_i = 1 \). Therefore, the four dimensional real vectors \( \tilde{r} \) and \( \tilde{c} \), whose elements are \( r_i \) and \( c_i \), respectively, satisfy \( r_i^2 + c_i^2 = 1/2 \) and \( r_i c_i = 0 \), and it is easy to see \( r_i^2 + c_i^2 \leq 1/2 \).

Then,

\[
\langle \psi | A | \psi \rangle = \sum_{i=0}^{3} |\lambda_i|^2 a_i \geq \sum_{i=0}^{3} |\lambda_i|^2 a_2 + |\lambda_3|^2 a_3
\]

\[
= a_2 - (a_2 - a_3)(r_i^2 + c_i^2) \geq \frac{1}{4} (a_2 + a_3),
\]

and therefore if \( a_2 + a_3 \geq 0 \) then \( \langle \psi | A | \psi \rangle \geq 0 \) for every \( |e \rangle \). Conversely, if \( a_2 + a_3 < 0 \) then \( \langle 00 | A | 00 \rangle < 0 \).

Using this lemma, it is found that Eq. (15) is decomposable if and only if \( x \geq (2p_0 - 1)/(1 - 2p_1) \), and

\[
M_1(\sigma_B) = \frac{2p_0 - 1}{2p_1}.
\]

(16)

for an entangled \( \sigma_B \). It has been shown that \( p_0 \) of the entangled Bell diagonal state cannot increase under LOCC \( \mathbb{R} \leq 29 \). By Eq. (16), it is found that \( p_1 \) also cannot increase unless \( p_0 \) is decreased. It has been shown further that almost all entangled two-qubit states can be converted to Bell diagonal states by LOCC in a reversible fashion \( \mathbb{R} \leq 30 \). Such reversible LOCC does not change \( M_1 \), and hence \( M_1(\sigma) \) for such the entangled two-qubit state \( \sigma \) agrees with Eq. (16) of the converted \( \sigma_B \).

Note that Eq. (16) is equal to 1 for \( p_1 = 1 - p_0 \), and therefore \( M_1(\sigma) = 1 \) for every entangled two-qubit states of rank-2. However, two-qubit states of rank-2 are not single-copy distillable \( \mathbb{R} \leq 31 \) (though a special state is quasi distillable \( \mathbb{R} \leq 32 \)) under LOCC and even under PPT-preserving operations \( \mathbb{R} \leq 13 \leq 14 \). Therefore the converse of the corollary 1 does not hold in general.

### III. A FAMILY OF STRONG MONOTONES

Let us now prove the theorem 1. Our starting point is the following function:

\[
M(\sigma) = \min_{\sigma \geq 0} \sup_{|\psi \rangle \in \mathcal{L}_\sigma} \frac{\text{tr}(\Omega |\psi \rangle \langle \psi |)}{\text{tr}(\Omega)},
\]

(17)

where the minimization is performed over all possible decompositions of \( \sigma = \sigma_+ - \sigma_- \), such as \( \sigma_+ \in \mathcal{C}_+ \) and \( \sigma_- \in \mathcal{C}_- \). Note that \( \sigma_\pm \) (unnormalized) are not necessarily positive, and the sets \( \mathcal{C}_\pm \) are specified later. The supremum is taken for all possible operations \( \Omega \) that belong to some operational class \( \mathcal{L}_\sigma \), which may depend on \( \sigma \). Then, suppose that \( \sigma \) is converted to \( \varrho \) by LOCC or PPT-preserving operations with nonzero probability \( p \), and hence there exists a PPT-preserving map \( \Lambda_{\sigma \rightarrow \varrho} \) such that \( \Lambda_{\sigma \rightarrow \varrho}(\sigma) = p\varrho \) (LOCC is also PPT-preserving). Moreover suppose that the sets \( \mathcal{C}_\pm \) have been chosen such that \( (1/p)\Lambda_{\sigma \rightarrow \varrho}(\sigma_\pm) = \varrho \in \mathcal{C}_\pm \), and suppose that \( \mathcal{L}_\sigma \) has been chosen such that \( \Omega \in \mathcal{L}_\sigma \) for every \( \Omega \in \mathcal{L}_\varrho \). Under these assumptions, the function \( M(\sigma) \) is indeed a strong monotone because

\[
M(\sigma) = \min_{\sigma \in \mathcal{C}_+} \sup_{\Omega \in \mathcal{L}_\sigma} \frac{\text{tr}(\Omega - \sigma)}{\text{tr}(\Omega + \sigma)},
\]

and \( M \) never increases under the conversion of \( \sigma \rightarrow \varrho \) if it is possible with nonzero probability. Note that the function \( M \) is neither convex nor concave in general.

Then, let us consider the case where \( \mathcal{C}_\pm = \{ A | A^T \geq 0 \} \). Namely, the minimization in Eq. (17) is performed over all possible decomposition such as \( \sigma^T = a_+ - a_- \), with \( a_+ \geq 0 \) and \( a_- \geq 0 \). This is the same decomposition introduced in Eq. (13) as the minimization problem for the negativity (see also [11]). Since \( \Lambda_{\sigma \rightarrow \varrho} \) is PPT-preserving, \( \Gamma \circ \Lambda_{\sigma \rightarrow \varrho} \circ \Gamma \) is a CP map. Therefore \( \varrho = (1/p)\Gamma \circ \Lambda_{\sigma \rightarrow \varrho} \circ \Gamma (\varrho) \geq 0 \) and \( \varrho \in \mathcal{C}_\pm \) is satisfied \( \mathbb{R} \leq 12 \). The explicit form of the function \( M(\sigma) \) in this case, denoted by \( M_1(\sigma) \), is

\[
M_1(\sigma) = \min_{\sigma \geq 0} \sup_{\varrho \in \mathcal{C}_\pm} \frac{\text{tr}(\Omega \varrho)}{\text{tr}(\varrho)},
\]

(19)

where \( \mathcal{L}_\sigma \) is chosen as a set of PPT-preserving operations restricted to \( \text{tr}(\sigma) > 0 \) (note that the optimization is supremum considering \( \text{tr}(\sigma) \rightarrow 0 \)). For this choice of \( \mathcal{L}_\sigma \), \( \text{tr}(\Omega \varrho) \in \mathcal{L}_\sigma \) for every \( \Omega \in \mathcal{L}_\varrho \) and hence the strong monotonicity Eq. (13) holds. Here, suppose that there exists a nonzero positive operator \( a_\pm \geq 0 \) such that \( a_\pm = a_\pm - a_- > 0 \) for some decomposition of \( \sigma^T = a_+ - a_- \).

Therefore,

\[
\max_{\varrho} \frac{\text{tr}(\varrho)}{\text{tr}(\varrho)} = \max_{\varrho} \frac{\text{tr}(\varrho) + \text{tr}(\varrho)}{\text{tr}(\varrho) + \text{tr}(\varrho)} \geq \max_{\varrho} \frac{\text{tr}(\varrho)}{\text{tr}(\varrho)}.
\]


where \( \text{tr}\Omega(a^F) = \text{tr}[\Gamma\Omega_\delta\Gamma(a)] \geq 0 \) and \( \text{tr}\Omega(a^F_\delta) = \text{tr}\Omega(\sigma) + \text{tr}\Omega(a^F) \geq \text{tr}\Omega(a^F_\delta) \) were used. As a result, it is found that the minimization in Eq. (19) is reached when \( a_x \) are orthogonal to each other and hence

\[ M_1(\sigma) = \max_{Q\in\mathcal{L}_\sigma} \frac{\text{tr}\Omega(Q^F)}{\text{tr}\Omega(P^F)} \tag{20} \]

with \( \sigma^F = P - Q \) being the Jordan decomposition. Moreover, it can be assumed that \( \Omega = R\Omega \) where

\[ \mathcal{R}(X) = \int dUdV(U \otimes V)X(U \otimes V)^\dagger \tag{21} \]

is the random application of local unitary transformations. All such PPT-preserving operations have the form of

\[ \Omega(X) = (\text{tr}XA)\mathbb{1} \tag{22} \]

with \( A \geq 0 \) and \( A^F \geq 0 \). Moreover, since \( \Omega \) is restricted to \( \text{tr}\Omega(\sigma) \geq 0 \) and hence \( \text{tr}\Omega(P^F) = \text{tr}\Omega(\sigma) + \text{tr}\Omega(Q^F) \geq 0 \), we can assume \( \text{tr}P^FA = 1 \) without loss of generality. Then, \( M_1(\sigma) \) is reduced to \( M_1(\sigma) = \max_A \text{tr}(Q^F)A \) subject that

\[ A \geq 0, \quad A^F \geq 0, \quad \text{tr}P^FA = 1. \tag{23} \]

This is a convex optimization problem, whose optimal value coincides with the optimal value of the dual problem (33). Since

\[ \text{tr}Q^FA = -\text{tr}YA^F - x(1 - \text{tr}P^FA) - \text{tr}A(-Q^F + xP^F - Y^F) + x, \tag{24} \]

where \( x \) is a Lagrange multiplier, the dual problem is \( M_1(\sigma) = \min_x \text{tr}A^F \) subject to the constraints of \( Y \geq 0 \) and

\[ X = -Q^F + xP^F - Y^F = \sigma - (1 - x)P^F - Y^F \geq 0. \tag{25} \]

These constraints can be read as \( \sigma - (1 - x)P^F = X + Y^F \) with \( X, Y \geq 0 \), and consequently the theorem 1 is obtained.

It is obvious that \( M_1(\sigma) \) is a strong monotone under PPT-preserving operations because \( \mathcal{L}_\sigma \) was chosen as a set of PPT-preserving operations. Then, let us consider another function \( M^\text{sep}_1(\sigma) \) for which \( \mathcal{L}_\sigma \) in Eq. (19) is replaced by the set of LOCC restricted to \( \text{tr}\Omega(\sigma) > 0 \). By this, since the set of stochastic LOCC coincides with the set of stochastic separable operations, the constraints of Eq. (28) become (33)

\[ A \geq 0, \quad A \text{ is separable,} \quad \text{tr}P^FA = 1. \tag{26} \]

Following an idea of (33) and putting

\[ A = \sum_i q_i|ef^{(i)}\rangle\langle ef^{(i)}|, \tag{27} \]

with \( q_i \geq 0 \), we have

\[ \text{tr}Q^FA = -x\left(1 - \sum_i q_i\langle ef^{(i)}|P^F|ef^{(i)}\rangle\right) \]

\[ -\sum_i q_i\langle ef^{(i)}| - Q^F + xP^F|ef^{(i)}\rangle + x. \tag{28} \]

Therefore, it is found that the corresponding dual problem becomes \( M^\text{sep}_1(\sigma) = \min_x A \) subject to

\[ \langle ef|\sigma - (1 - x)P^F|ef\rangle \geq 0 \quad \text{for every } |ef\rangle, \tag{29} \]

i.e. subject that \( \sigma - (1 - x)P^F \) is 1-positive. This dual problem for \( M^\text{sep}_1(\sigma) \) has a simple geometrical meaning as well as \( M_1(\sigma) \) as shown in Fig. 1.

![FIG. 1: The set of positive operators (unnormalized states) and set of PPT operators are schematically shown as two circles. The intersection of the two circles corresponds to the set of (unnormalized) PPT states. The set of decomposable operators corresponds to the convex cone of the two circles. Moreover, the set of 1-positive operators embodies the set of decomposable operators. When \( \sigma^F = P - Q \) is the Jordan decomposition, \( P^F \) is located on the edge of the set of PPT operators. \( P^F \) is an unnormalized PPT state for many classes of states, but sometimes \( P^F \) is not a state (see (37)). On the other hand, \( \delta \equiv -\sigma - (1 - x)P^F \) is located on the edge of the set of 1-positive operators. The ratio of the interior division \( x \) corresponds to the strong monotone \( M^\text{sep}_1(\sigma) \). When \( \delta \) is located on the edge of the set of decomposable operators, the ratio of the interior division \( x \) corresponds to the strong monotone \( M_1(\sigma) \).

\[ \frac{\text{tr}Q}{\text{tr}P} = \frac{N(\sigma)}{1 + N(\sigma)}. \tag{30} \]
IV. STRONG MONOTONE $M_2$

Let us return to Eq. (17) and consider the case where $C_+ = \{A|A \geq 0, A \text{ is separable}\}$ and $C_- = \{A|A \geq 0\}$, namely the same decomposition for the minimization problem of the generalized robustness of entanglement $R_G$ (see also (11)). Moreover, $\mathcal{L}_+$ is chosen as a set of LOCC restricted to $\text{tr}\Omega(\sigma) > 0$. When $\Lambda_{\sigma-\epsilon}$ is LOCC, $\sigma_+ = (1/p)\Lambda_{\sigma-\epsilon}(\sigma_+)$ is separable and $\sigma_\pm \in C_\pm$ holds. The explicit form of the function $M(\sigma)$ in this case, denoted by $M_2^{sep}(\sigma)$, is

$$M_2^{sep}(\sigma) = \min_{\sigma_\pm \in \mathcal{L}_\pm} \frac{\text{tr}\Omega(\sigma_-)}{\text{tr}\Omega(\sigma_+)}$$

(31)

with the constraints of $\sigma = \sigma_+ - \sigma_-$, $\sigma_\pm \geq 0$, and $\sigma_+$ is separable. Since $\text{tr}\Omega(\sigma_-)/\text{tr}\Omega(\sigma_+)$ is not necessarily optimal for every separable $\sigma$, $M_2^{sep}(\sigma)$ is not necessarily 1-positive. Moreover, as in the case of the example (iv) for $M_2$, $M_2^{sep}(\sigma) = 1$ only when $\sigma_+ = P_2^+$, which is not separable and not allowed as a decomposition of $\sigma$. When $F$ corresponds to a projection of rank-1, $Z$ is 1-positive only when $Z \geq 0$. However, $Z \geq 0$ for $x < 1$ only when $\sigma_+ = P_2^+$, which is not separable and not allowed as a decomposition of $\sigma$. When $F$ corresponds to a projection of rank-2, $Z$ is 1-positive if and only if $Z$ is decomposable, and hence $Z = X + Y^T$ must hold with $X, Y \geq 0$. For $x < 1$, $Z$ is not positive, and hence $Y^T$ must have a negative eigenvalue, $Y^T$ must have three positive (and one negative) eigenvalues (36, 37), and $Z = X + Y^T$ must have three positive eigenvalues. However, $Z$ cannot have three positive eigenvalues because of $\sigma_+ \geq 0$. After all, the constraints in Eq. (32) cannot be satisfied for $x < 1$, and hence $M_2^{sep}(P_2^+) = 1$. Since all pure entangled $|\psi\rangle$ can be converted to $P_2^+$ by LOCC and $M_2^{sep}$ never increases under LOCC, $M_2^{sep}(1) = 1$. Moreover, as in the case of the example (iv) for $M_1(\sigma)$, for the mixed state of

$$\sigma = \sigma_0 + \lambda \sigma_+,$$

(33)

where $\sigma_+$ constitutes an optimal decomposition for $\sigma_0$, it is found that

$$M_2^{sep}(\sigma) \leq \frac{M_2^{sep}(\sigma_0)}{1 + \lambda}.$$  

(34)

since $(1 + \lambda)\sigma_+^*$ is not necessarily optimal for $\sigma$. In this way, $M_2^{sep}(\sigma) \in [0, 1]$ certainly represents non-trivial strong monotonicity under LOCC, and we have

Theorem 2: The function $M_2^{sep}(\sigma)$, which is defined by Eq. (31), is a strong monotone under LOCC. If $\sigma$ is single-copy distillable under LOCC, then $M_2^{sep}(\sigma) = 1$. Note that $M_2^{sep}(\sigma)$ and $M_2(\sigma)$ (defined below) are lower bounded by the generalized robustness of entanglement $R_G$ as

$$M_2(\sigma) \geq M_2^{sep}(\sigma) \geq \min_{\sigma_-} \frac{\text{tr}\sigma_+}{\text{tr}\sigma_-} = \frac{R_G(\sigma)}{1 + R_G(\sigma)}.$$  

(35)

Here, $M_2(\sigma)$ is a strong monotone for which $\mathcal{L}_\sigma$ in Eq. (11) is chosen as the set of PPT-preserving operations, and hence $M_2(\sigma)$ is defined such that the constraint of the 1-positivity in Eq. (32) is replaced by the constraint of the decomposability.

V. SUMMARY

We proposed four strong entanglement monotones, $M_1$, $M_1^{sep}$, $M_2$, and $M_2^{sep}$, and studied those properties. All these strong monotones take 1 for every pure entangled states and hence represent the strong monotonicity independent on the Schmidt number in mixed-state entanglement manipulation. Moreover, these strong monotones provide necessary conditions for single-copy distillability. These are lower bounded by the negativity or generalized robustness of entanglement. All these strong monotones are derived from the strong monotone function $M(\sigma)$ given by Eq. (17), whose optimization problem of the supremum concerning LOCC or PPT-preserving operations can be reduced to a simple convex optimization problem. The constraint of the dual optimization problem is described by the decomposability and 1-positivity of an operator constructed from a given quantum state, and it clearly reveals the geometrical characteristics of entangled state as shown in Fig. 4. In this paper, we concentrated our attention on the bipartite systems but it is obvious that these strong monotones can be applicable to every bipartite partitioning of multipartite systems.

Finally, let us briefly discuss the relation between the strong monotones studied in this paper and asymptotic distillability. If $\sigma$ is asymptotically distillable, there must exist a trace-preserving LOCC (ended by twirling) which produces an isotropic state close to a maximally entangled state, namely there must exist LOCC $\Lambda$ such that $\Lambda(\sigma^{\otimes n}) = \sigma_I$ with $n \rightarrow \infty$. Here $\sigma_I$ is an isotropic state given by Eq. (10) and the distillable entanglement $E_D$ is given by $(\log d)/n \rightarrow E_D(\sigma)$ with $d$ being the dimension of $\sigma_I$. On the other hand, $M_1(\sigma_I) - 1$ for $n \rightarrow \infty$ as explicitly shown in Eq. (10) [$M_1(\sigma_I)$ is a discontinuous function but it is continuous on an isotropic state]. This implies that

$$1 \geq M_1(\sigma^{\otimes n}) \geq M_1(\Lambda(\sigma^{\otimes n})) = M_1(\sigma_I) - 1,$$

(36)

and therefore $M_1(\sigma^{\otimes n})$ must satisfy $M_1(\sigma^{\otimes n}) \rightarrow 1$ for $n \rightarrow \infty$ so that $\sigma$ is asymptotically distillable. In this way, the asymptotic behavior of $M_1(\sigma^{\otimes n})$ provides a condition necessary to the asymptotic distillability. However, since the negativity satisfies $N(\sigma^{\otimes n}) \rightarrow \infty$ for every non-PPT (NPT) states, it is found using Eq. (30)
that $M_1(\sigma^{\otimes n}) \rightarrow 1$ for every NPT states. Therefore, $M_1$ does not provide any non-trivial result concerning asymptotic distillability unfortunately. This is the case for $M_{sep}^1(\sigma^{\otimes n})$ also. On the other hand, it is open whether $M_{sep}^2(\sigma^{\otimes n}) \rightarrow 1$ for every NPT states or not [as far as we know the asymptotic behavior of $R_G(\sigma^{\otimes n})$ for every NPT states has not been shown yet]. Note that the continuity of $M_{sep}^2$ on an isotropic state is also open.

The problem obtaining the tractable criterion for single-copy distillability in higher dimensional systems is still open as well as the asymptotic distillability. We wish the results in this paper could shed some light on these open problems.

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