We generalize Hardy’s proof of nonlocality to the case of bipartite mixed statistical operators, and we exhibit a necessary condition which has to be satisfied by any given mixed state $\sigma$ in order that a local and realistic hidden variable model exists which accounts for the quantum mechanical predictions implied by $\sigma$. Failure of this condition will imply both the impossibility of any local explanation of certain joint probability distributions in terms of hidden variables and the nonseparability of the considered mixed statistical operator. Our result can be also used to determine the maximum amount of noise, arising from imperfect experimental implementations of the original Hardy’s proof of nonlocality, in presence of which it is still possible to put into evidence the nonlocal features of certain mixed states.

PACS numbers: 03.65.Ud

I. INTRODUCTION

Hardy’s proof on nonlocality [1] has been referred to as “the best version of Bell’s theorem” [2]. Such a proof establishes, by resorting to very simple arguments which do not involve the consideration of any violation of Bell-type inequalities [3, 4], a direct incompatibility between any local realistic model for almost any bipartite pure entangled state and the quantum mechanical predictions concerning properly chosen observables. However, the most widely used method to deny the existence of a local realistic model for composite states consists in the identification of a Bell’s inequality which is violated by the state under consideration. Contrary to the case of pure states in which any entangled vector implies the violation of a precise Bell’s inequality [5], the question of which mixed states do violate a Bell’s inequality and, as a consequence, do not admit a description in terms of a local hidden variable model is more complicated. In fact, within this wider scenario, there exist non-separable mixed states (that is, statistical operators which cannot be expressed as a convex sum of product states) which nonetheless admit a local realistic description and do not violate any Bell’s inequality [6, 7]. In brief, for mixed states the occurrence of entanglement does not in general rule out a local deterministic description. Equivalently, no general procedure is known to ascertain whether a statistical operator leads to the violation of at least a Bell’s inequality [8, 9, 10, 11].

In this paper we exhibit an alternative argument not resorting to Bell’s inequalities to reject the possibility of a local realistic description for certain mixed states. The argument is based on a reformulation and a generalization [12] of Hardy’s proof of nonlocality [1], leading, via simple set theoretic arguments, to an algebraic inequality whose violation by a certain mixed state implies the impossibility of a local hidden variable model for it. The novelty of the proof derives, on one side, from the fact that it applies to a large class of mixed states, contrary to Hardy’s proof which was restricted to pure states only, and, on the other side, that it holds for bipartite systems whose constituents belong to Hilbert spaces of arbitrary dimensions. The proof involves both the consideration of the trace distance of the considered mixed state from pure Hardy’s states (that is, pure entangled states having at least two different weights in their Schmidt decomposition) and of the probability for precise joint measurement outcomes. The idea underlying our method is simply that in the vicinity (with respect to the topology induced by the trace distance) of a Hardy’s state there exist uncountable many (non-separable) mixed states which do not admit any local realistic model. The usefulness of this new nonlocality argument is twofold. First, the possibility of deciding whether a given mixed state does exhibit genuine nonlocal features which cannot be reproduced by local classical models is extremely important for the theory of bipartite entangled mixed states. In fact, when this occurs one can implement efficient quantum communication protocols which cannot be locally reproduced by any classical mean [13]. Second, our result gives clear indications about the amount of noise which can be tolerated when performing an experimental check of nonlocality along the lines indicated by Hardy [14, 15]. More precisely, we can estimate the amount of noise (of the most general
kind) affecting the preparation of a pure Hardy’s state so as to give a mixed statistical operator which still exhibits nonlocal features.

II. THE GENERALIZED HARDY’S ARGUMENT

Given a bipartite state vector $|\psi\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, where $d_1$ and $d_2$ are arbitrary positive integers, let us consider its Schmidt decomposition in terms of appropriate orthonormal sets of states $\{|\alpha_i(1)\rangle\}$ belonging to $\mathbb{C}^{d_1}$ and $\{|\beta_i(2)\rangle\}$ belonging to $\mathbb{C}^{d_2}$, respectively,

$$|\psi(1,2)\rangle = \sum_{i=1}^{\min(d_1,d_2)} p_i|\alpha_i(1)\rangle \otimes |\beta_i(2)\rangle,$$

where the weights $p_i$ are positive real numbers satisfying the normalization condition $\sum_i p_i^2 = 1$. Suppose that there exist at least two such weights which are different from each other, e.g., $p_1 \neq p_2$, and let us denote any state displaying this property as a “Hardy state”. This is the only hypothesis which, in Hardy’s proof of nonlocality [1], is necessary to exhibit a contradiction between the existence of a local hidden variable model for a Hardy’s state and the quantum mechanical predictions for appropriate measurement outcomes. In our extension of Hardy’s proof the mixed states we are going to consider will belong to a neighborhood of a Hardy’s state, whose size will depend crucially on the values of $p_1$ and $p_2$ of such a state.

To begin with, let us consider an arbitrary bipartite statistical operator $\sigma \in \mathcal{B}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ (that is, a positive semidefinite, trace class, trace-one bounded operator) whose trace distance from the Hardy state $|\psi\rangle$ denoted as $D(\sigma, |\psi\rangle\langle\psi|)$ is equal to a positive number $\varepsilon$:

$$D(\sigma, |\psi\rangle\langle\psi|) \equiv \frac{1}{2} \text{Tr} |\sigma - |\psi\rangle\langle\psi|| = \varepsilon,$$

(2)

where $|A| \equiv \sqrt{A^\dagger A}$ is the positive square root of $A^\dagger A$. The trace distance between two arbitrary statistical operators $\sigma_1$ and $\sigma_2$ represents a good measure to quantify how close are the probability distributions of any measurement outcome associated to the two quantum states. In fact, a well known property of the trace distance is that

$$|\text{Tr}[P\sigma_1] - \text{Tr}[P\sigma_2]| \leq D(\sigma_1, \sigma_2)$$

(3)

for any projection operator $P$, the expression $\text{Tr}[P\sigma_1]$ representing the probability for the occurrence of a certain measurement outcome when the system is associated with the state $\sigma$. Since we already know that for any state vector $|\psi\rangle$, whose Schmidt decomposition involves at least two different Schmidt coefficients, a Hardy’s proof of nonlocality can be exhibited, our idea is that of trying to determine a neighborhood of $|\psi\rangle$ [measured in terms of the trace distance $D(\sigma, |\psi\rangle\langle\psi|) = \varepsilon$] such that all mixed states $\sigma$ belonging to it will exhibit nonlocal features. In this way, we will identify a whole class of bipartite mixed states, belonging to an (arbitrary) finite dimensional Hilbert space, which do not admit a local realistic description and which are, as a consequence, nonseparable.

To achieve this goal, let us recall the basic steps of Hardy’s argument [1] by resorting to the notation we used in our reformulation of that argument [12]. First of all, we define the following two $2 \times 2$ unitary matrices $U$ and $V$ whose entries depend on the weights $p_1$ and $p_2$:

$$U = \frac{1}{\sqrt{p_1 + p_2}} \begin{bmatrix} \sqrt{p_2} & -i\sqrt{p_1} \\ -i\sqrt{p_1} & \sqrt{p_2} \end{bmatrix} \quad V = \frac{1}{\sqrt{p_1^2 + p_2^2 - p_1p_2}} \begin{bmatrix} -i(p_2 - p_1) & \sqrt{p_1p_2} \\ \sqrt{p_1p_2} & -i(p_2 - p_1) \end{bmatrix}. \quad (4)$$

Subsequently we consider two orthonormal bases $\{|x_+(1)\rangle, |x_-(1)\rangle\}$ and $\{|y_+(1)\rangle, |y_-(1)\rangle\}$ belonging to the two-dimensional linear manifold of the first subsystem spanned by the vectors $\{|\alpha_1(1)\rangle, |\alpha_2(1)\rangle\}$, and two bases $\{|x_+(2)\rangle, |x_-(2)\rangle\}$ and $\{|y_+(2)\rangle, |y_-(2)\rangle\}$ for the two-dimensional linear manifold of the second subsystem spanned by the vectors $\{|\beta_1(2)\rangle, |\beta_2(2)\rangle\}$, according to:

$$\begin{bmatrix} |x_+(1)\rangle \\ |x_-(1)\rangle \end{bmatrix} = U \begin{bmatrix} |\alpha_1(1)\rangle \\ |\alpha_2(1)\rangle \end{bmatrix} \quad \begin{bmatrix} |y_+(1)\rangle \\ |y_-(1)\rangle \end{bmatrix} = UV \begin{bmatrix} |\alpha_1(1)\rangle \\ |\alpha_2(1)\rangle \end{bmatrix} \quad (5)$$

$$\begin{bmatrix} |x_+(2)\rangle \\ |x_-(2)\rangle \end{bmatrix} = U \begin{bmatrix} |\beta_1(2)\rangle \\ |\beta_2(2)\rangle \end{bmatrix} \quad \begin{bmatrix} |y_+(2)\rangle \\ |y_-(2)\rangle \end{bmatrix} = UV \begin{bmatrix} |\beta_1(2)\rangle \\ |\beta_2(2)\rangle \end{bmatrix}. \quad (6)$$
The state $|\psi\rangle$ of Eq. (1) can then be expressed in three equivalent forms in terms of the basis vectors defined in Eqs. (5-6), as:

$$
|\psi\rangle = i\sqrt{p_1p_2} [ |x_+(1)\rangle|x_-(2)\rangle + |x_-(1)\rangle|x_+(2)\rangle ] + (p_2 - p_1) |x_-(1)\rangle|x_-(2)\rangle + \sum_{i>2} p_i |x_i(1)\rangle|x_i(2)\rangle
$$

$$
= i\sqrt{p_1^2 + p_2^2 - p_1p_2} |y_-(1)\rangle|x_-(2)\rangle + i\sqrt{p_1p_2} |x_-(1)\rangle|x_+(2)\rangle + \sum_{i>2} p_i |x_i(1)\rangle|x_i(2)\rangle
$$

$$
= i\sqrt{p_1p_2} |x_+(1)\rangle|x_-(2)\rangle + i\sqrt{p_1^2 + p_2^2 - p_1p_2} |x_-(1)\rangle|x_-(2)\rangle + \sum_{i>2} p_i |x_i(1)\rangle|x_i(2)\rangle.
$$

With the aim of displaying the particular set of joint probability distributions which conflict with any local hidden variable model, we consider the four operators $X_1, Y_1, X_2,$ and $Y_2$ having as eigenstates associated to the eigenvalues $+1$ and $-1$ the orthonormal vectors $\{ |x_+(1)\rangle, |x_-(1)\rangle \}$, $\{ |y_+(1)\rangle, |y_-(1)\rangle \}$, $\{ |x_+(2)\rangle, |x_-(2)\rangle \}$, and $\{ |y_+(2)\rangle, |y_-(2)\rangle \}$, respectively, while they act as the null operator in the manifolds orthogonal to the bidimensional ones corresponding to the nonzero eigenvalues. According to Eq. (7) the quantum joint probabilities concerning the set of observables $X_1, Y_1, X_2$ and $Y_2$, when the system is in the state $|\psi\rangle$, satisfy the following relations:

$$
P_{\psi}(X_1 = +1, X_2 = +1) = 0,
$$

$$
P_{\psi}(Y_1 = +1, X_2 = -1) = 0,
$$

$$
P_{\psi}(X_1 = -1, Y_2 = +1) = 0,
$$

$$
P_{\psi}(Y_1 = +1, X_2 = 0) = 0,
$$

$$
P_{\psi}(X_1 = 0, Y_2 = +1) = 0,
$$

$$
P_{\psi}(Y_1 = +1, Y_2 = +1) = \frac{p_1^2p_2^2(p_1 - p_2)^2}{(p_1^2 + p_2^2 - p_1p_2)^2} \equiv a.
$$

Since by hypothesis, the weights $p_1$ and $p_2$ are strictly positive and different from each other [thus implying that the parameter $a$ we have defined in Eq. (13) is strictly positive], one is able to set up a Hardy-like proof of nonlocality [1] by resorting to a set theoretic argument leading to a contradiction between the considered probability distributions of Eqs. (8)-(13) and the possibility of accounting for them by means of a local realistic model where additional hidden variables predetermine the outcomes of any conceivable measurement.

In order to generalize such a result to mixed states, let us consider an arbitrary statistical operator $\sigma$ having trace distance from $|\psi\rangle\langle\psi|$, as defined in Eq. (2), equal to $\varepsilon > 0$. As a consequence of Eq. (9), which gives an upper bound to the difference of the probability distributions associated to the different quantum states $\sigma_1 = |\psi\rangle\langle\psi|$ and $\sigma_2 = \sigma$, respectively, and taking into account Eqs. (8)-(13), one obtains:

$$
P_{\sigma}(X_1 = +1, X_2 = +1) \leq \varepsilon,
$$

$$
P_{\sigma}(Y_1 = +1, X_2 = -1) \leq \varepsilon,
$$

$$
P_{\sigma}(X_1 = -1, Y_2 = +1) \leq \varepsilon,
$$

$$
P_{\sigma}(Y_1 = +1, X_2 = 0) \leq \varepsilon,
$$

$$
P_{\sigma}(X_1 = 0, Y_2 = +1) \leq \varepsilon,
$$

$$
P_{\sigma}(Y_1 = +1, Y_2 = +1) \in [a - \varepsilon, a + \varepsilon].
$$

Now, suppose that there exists a more complete description of quantum systems than the one characterized by the simple assignment of the statistical operator $\sigma$. This description is called a stochastic hidden variable model and it consists of (i) a set $\Lambda$ whose elements $\lambda$ are called hidden variables; (ii) a normalized probability distribution $\rho: \Lambda \rightarrow [0, 1]$; (iii) a set of probability distributions $P_{\lambda}(A_1 = a, B_2 = b)$ for the measurement outcomes of any pair of observables $A_1$ and $B_2$ associated to the first and to the second subsystem respectively, defined for any value $\lambda \in \Lambda$, such that

$$
P_{\sigma}(A_1 = a, B_2 = b) = \int_{\Lambda} d\lambda \rho(\lambda) P_{\lambda}(A_1 = a, B_2 = b).
$$

The left hand side of Eq. (20) gives simply the quantum probability distributions concerning the outcomes $\{a, b\}$ for the joint measurement of the observables $A_1$ and $B_2$, when the system is associated with the statistical operator $\sigma$. A deterministic hidden variable model (also known as a realistic model) is a particular instance of a stochastic model in which the probabilities $P_{\lambda}$ can take only the values 0 or 1. A hidden variable model is called local [10] if the following
factorizability condition holds for any conceivable joint probability distribution $P_{\lambda}(A_1 = a, B_2 = b)$ and for any value of the hidden variable $\lambda \in \Lambda$

$$P_{\lambda}(A_1 = a, B_2 = b) = P_{\lambda}(A_1 = a)P_{\lambda}(B_2 = b),$$

(21)
in all cases in which the measurement processes for the observables $A_1$ and $B_2$ occur at spacelike separated locations. The locality condition imposes that no causal influence can exist between spacelike separated events. It is worth noticing that it has been proved \cite{17} that deterministic and stochastic hidden variable models are completely equivalent when one imposes to them the locality request. For this reason, in what follows, we will deal, without any loss of generality, only with local realistic models reproducing the quantum mechanical predictions for the state $\sigma$ in terms of probability distributions $P_{\lambda}$ assuming only the values 0 or 1. Finally, we will denote as $\mu(\Sigma)$ the measure of any subset $\Sigma$ of $\Lambda$ with respect to the weight function $\rho(\lambda)$, i.e.,

$$\mu(\Sigma) = \int_{\Sigma} d\lambda \rho(\lambda).$$

(22)

To begin with, let us define the following subsets $A, B, C, D$ of the set $\Lambda$ of the hidden variables:

$$A = \{ \lambda \in \Lambda \mid P_{\lambda}(X_1 = 1) = 1 \},$$

(23)

$$B = \{ \lambda \in \Lambda \mid P_{\lambda}(X_2 = 1) = 1 \},$$

(24)

$$C = \{ \lambda \in \Lambda \mid P_{\lambda}(Y_1 = 1) = 1 \},$$

(25)

$$D = \{ \lambda \in \Lambda \mid P_{\lambda}(Y_2 = 1) = 1 \}.$$  

(26)

Suppose now that a local and realistic description exists for the mixed state $\sigma$ and let us consider the joint probability distribution $P_{\sigma}(X_1 = 1, X_2 = 1)$. With our assumptions, we have

$$P_{\sigma}(X_1 = 1, X_2 = 1) = \int_{\Lambda} d\lambda \rho(\lambda)P_{\lambda}(X_1 = 1, X_2 = 1)$$

$$= \int_{\Lambda} d\lambda \rho(\lambda)P_{\lambda}(X_1 = 1)P_{\lambda}(X_2 = 1) = \mu[A \cap B],$$

(27)

where the second equality is implied by the locality condition Eq. (21), and the third is a consequence of the fact that the product $P_{\lambda}(X_1 = 1)P_{\lambda}(X_2 = 1)$ does not vanish only within the subset $A \cap B$, where it takes the value one, so that the whole integral gives the measure of such a set. Finally, by resorting to Eq. (14), we can conclude that $\varepsilon$ is an upper bound for the measure of the subset $A \cap B$, that is, $\mu[A \cap B] \leq \varepsilon$. The situation becomes slightly more complicated when we consider Eq. (15) and impose that there exists a local and realistic model also for such a probability distribution. In fact, by noticing that the only outcomes for the observable $X_2$ are $-1, 0$ and $+1$ and, as a consequence, that the relation $P_{\lambda}(X_2 = -1) + P_{\lambda}(X_2 = 0) + P_{\lambda}(X_2 = +1) = 1$ holds for any $\lambda \in \Lambda$, we have

$$P_{\sigma}(Y_1 = 1, X_2 = -1) = \int_{\Lambda} d\lambda \rho(\lambda)P_{\lambda}(Y_1 = 1)P_{\lambda}(X_2 = -1)$$

$$= \int_{\Lambda} d\lambda \rho(\lambda)P_{\lambda}(Y_1 = 1)[1 - P_{\lambda}(X_2 = 1) - P_{\lambda}(X_2 = 0)]$$

$$= \mu[C] - \mu[B \cap C] - \int_{\Lambda} d\lambda P_{\lambda}(Y_1 = 1)P_{\lambda}(X_2 = 0)$$

(29)

(30)

Using Eqs. (15) and (17), we obtain an upper bound for the difference of the measures of the sets $C$ and $B \cap C$

$$\mu[C] - \mu[B \cap C] \leq 2\varepsilon.$$  

(31)

We can now repeat our argument for all Eqs. (13)-(19) obtaining in this way two other relations, $\mu[D] - \mu[A \cap D] \leq 2\varepsilon$ and $\mu[C \cap D] \in [a - \varepsilon, a + \varepsilon]$. Concluding, the following set of constraints on the measure of the considered subsets of $\Lambda$ have to be satisfied by any local and realistic model accounting for the quantum mechanical predictions implied by any state $\sigma$ satisfying Eq. (2):

$$\mu[A \cap B] \leq \varepsilon,$$

(32)

$$\mu[C] - \mu[B \cap C] \leq 2\varepsilon,$$

(33)

$$\mu[D] - \mu[A \cap D] \leq 2\varepsilon,$$

(34)

$$\mu[C \cap D] \in [a - \varepsilon, a + \varepsilon].$$  

(35)
Up to now, no constraint has been imposed on the two parameters $\varepsilon$, quantifying the distance between an arbitrary mixed state $\sigma$ and the precise pure state $|\psi\rangle$ of Eq. (1) and $a$, which specifies the non-zero probability of Eq. (13). Now we will show that the very assumption that a local realistic description of the implications of the state $\sigma$ is possible, implies a precise relation between such parameters. As a consequence, all states for which such a relation is violated do not admit any local realistic description.

In order to find out the relation constraining the values of $\varepsilon$ and $a$ it is useful to resort to the consideration of the complements $\bar{A}, \bar{B}, \bar{C}$, and $\bar{D}$ in $\Lambda$ of the subsets $A, B, C$, and $D$ (that is, $\bar{A} \equiv \Lambda - A, \bar{B} \equiv \Lambda - B, \bar{C} \equiv \Lambda - C$, and $\bar{D} \equiv \Lambda - D$). We begin by taking into account Eq. (32) and the fact that the measure of the whole set $\Lambda$ equals 1, thus getting:

$$\mu[(A - A) \cup (A - B)] = \mu[(A - (A \cap B)], = 1 - \mu[A \cap B] \geq 1 - \varepsilon$$  \hspace{1cm} (36)

In the same way, using Eq. (33), we obtain:

$$\mu[(A - C) \cup (A - D)] = \mu[(A - (C \cap D)] = 1 - \mu[C \cap D] \in [1 - a - \varepsilon, 1 - a + \varepsilon].$$  \hspace{1cm} (37)

Finally, by considering Eqs. (33) and (34) and resorting to simple set manipulations, we find that $\mu[(A - B) \cup (A - C)] - \mu[A - C] \leq 2\varepsilon$ and $\mu[(A - A) \cup (A - D)] - \mu[A - D] \leq 2\varepsilon$.

The previous relations can also be expressed in terms of the complements of the involved sets, in which case they take the form

$$\mu[\bar{A} \cup \bar{B}] \geq 1 - \varepsilon, \hspace{1cm} (38)$$

$$\mu[\bar{B} \cup \bar{C}] - \mu[\bar{C}] \leq 2\varepsilon, \hspace{1cm} (39)$$

$$\mu[\bar{A} \cup \bar{D}] - \mu[\bar{D}] \leq 2\varepsilon, \hspace{1cm} (40)$$

$$\mu[\bar{C} \cup \bar{D}] \in [1 - a - \varepsilon, 1 - a + \varepsilon]. \hspace{1cm} (41)$$

These new equations are more suited for deriving the desired constraint between $\varepsilon$ and $a$ since they involve only the union $\cup$ of subsets. In order to complete our argument, we have first of all to derive three useful relations RI-RIII.

**RI:** $\mu[\bar{A}] \leq \mu[\bar{A} \cap \bar{D}] + 2\varepsilon$.

*Proof.* This is easily proved by noticing that

$$\mu[\bar{A} \cup \bar{D}] - \mu[\bar{D}] = \mu[\bar{A}] - \mu[\bar{A} \cap \bar{D}] \leq 2\varepsilon$$  \hspace{1cm} (42)

due to Eq. (40).

Just in the same way, taking into account Eq. (39), we get

**RII:** $\mu[\bar{B}] \leq \mu[\bar{B} \cap \bar{C}] + 2\varepsilon$.

Finally, by resorting to elementary set manipulations, we can easily prove that

**RIII:** $\mu[\bar{A} \cap \bar{D}] + \mu[\bar{B} \cap \bar{C}] \leq \mu[\bar{C} \cup \bar{D}] + \mu[\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D}]$.

*Proof.* It is obvious that $\bar{A} \cap \bar{D} \subseteq \bar{D}$ and, similarly, that $\bar{B} \cap \bar{C} \subseteq \bar{C}$. As a consequence $(\bar{A} \cap \bar{D}) \cup (\bar{B} \cap \bar{C}) \subseteq (\bar{C} \cup \bar{D})$ and $\mu[(\bar{A} \cap \bar{D}) \cup (\bar{B} \cap \bar{C})] \leq \mu[\bar{C} \cup \bar{D}]$. By the properties of any measure defined on sets, we have:

$$\mu[(\bar{A} \cap \bar{D}) \cup (\bar{B} \cap \bar{C})] = \mu[\bar{A} \cap \bar{D}] + \mu[\bar{B} \cap \bar{C}] - \mu[\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D}] \leq \mu[\bar{C} \cup \bar{D}]$$  \hspace{1cm} (43)

from which our conclusion holds.

Now we have at our disposal all the relations we need in order to derive the desired constraint between $\varepsilon$ and $a$. In fact,

$$\mu[\bar{A} \cup \bar{B}] = \mu[\bar{A}] + \mu[\bar{B}] - \mu[\bar{A} \cap \bar{B}] \leq 4\varepsilon + \mu[\bar{A} \cap \bar{D}] + \mu[\bar{B} \cap \bar{C}] - \mu[\bar{A} \cap \bar{B}]$$  \hspace{1cm} (44)

$$\leq 4\varepsilon + \mu[\bar{C} \cup \bar{D}] + \mu[\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D}] - \mu[\bar{A} \cap \bar{B}]$$  \hspace{1cm} (45)

$$\leq 4\varepsilon + \mu[\bar{C} \cup \bar{D}], \hspace{1cm} (46)$$
where the first majorization is implied by $\mathbf{RI}$ and $\mathbf{RII}$, the second is implied by $\mathbf{RIII}$ and finally, the last inequality is a trivial consequence of the fact that $\mu[A \cap B \cap C \cap D] - \mu[A \cap B] \leq 0$ since $A \cap B \cap C \cap D \subseteq A \cap B$. At this point, using the inequalities (38) and (41), we end up with the desired relation

$$6\varepsilon - a \geq 0.$$  (47)

We summarize what we have just proved:

**Theorem:** Let us consider an arbitrary Hardy state $|\psi(1,2)\rangle = \sum p_i |\alpha_i(1)\rangle \otimes |\beta_i(2)\rangle$ having two different (non-zero) weights $p_1 \neq p_2$, and a statistical operator $\sigma$ such that its trace distance $D(\sigma, |\psi\rangle\langle\psi|)$ from the state $|\psi\rangle\langle\psi|$ equals $\varepsilon$. Then, if a local and deterministic hidden variable model exists for $\sigma$, the inequality $6\varepsilon - a \geq 0$ (where $a = \frac{p_1^2 p_2^2 (p_1 - p_2)^2}{(p_1^2 + p_2^2 - p_1 p_2)^2}$) has to be satisfied for any choice of the Hardy state $|\psi\rangle$.

The relevance of this theorem derives from the fact that it guarantees that, given a mixed state $\sigma$, if there exists a Hardy state $|\psi\rangle$ such that its trace distance $D(\sigma, |\psi\rangle\langle\psi|)$ is strictly less than $\frac{a}{2}$ then no local realistic description for $\sigma$ can be given. As a consequence, any such mixed state $\sigma$ cannot be separable (otherwise it would admit a local realistic description): thus, the relation of Eq. (47) also represents a new separability condition for any bipartite mixed state of the Hilbert space $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$.

Before concluding, let us underline two important aspects of our result. First of all, our nonlocality proof applies to a large class of (bipartite) mixed state. In fact, the set of pure entangled states with at least two different weights, i.e., the Hardy states, includes the overwhelming majority of all pure entangled states and we have just proved that every mixed state $\sigma$ belonging to an appropriate neighborhood of a Hardy state (i.e., one of size less than $a/6$ in the trace distance, where $a$ depends on the considered Hardy state) cannot be described by a local realistic model. Thus, our method allows us to determine a vast class of mixed states exhibiting nonlocal features and, as a by-product, a vast class of non-separable states as well. Secondly, our result might be relevant from a practical point of view. In fact, when implementing a Hardy-experiment [14, 15] aiming to put into evidence the truly nonlocal features of an entangled Hardy’s state, one has to face the problem of the unavoidably imperfect preparation of such a state. In general one will be dealing with mixtures of states rather than with the precise pure state one wants to study. Actually, due to unavoidable couplings with the environment, the Hardy’s state $|\psi\rangle$ one wishes to prepare will usually be corrupted by different kind of noises, thus resulting in a mixed state $\sigma$ such as

$$\sigma = p|\psi\rangle\langle\psi| + (1 - p)\tilde{\sigma},$$  (48)

where $\tilde{\sigma}$ is the (trace-one) statistical operator describing the noise affecting the pure state $|\psi\rangle$. Our previous argument shows that, notwithstanding the nonpure nature of the actual state $\sigma$ one is dealing with, there exists an interval of values for the parameter $p$ such that one can still put into evidence a contradiction between the quantum mechanical predictions and those of any local realistic model. In fact, for the state of Eq. (47) one can repeat our earlier arguments, evaluate the trace distance $D(\sigma, |\psi\rangle\langle\psi|)$ and determine the precise interval of values for the parameter $p \in (0, 1)$ (which quantifies the amount of noise in the state) for which a contradiction with locality condition is still present.

### III. CONCLUSIONS

We have derived a generalization of the original Hardy’s proof of nonlocality which works for a particular class of bipartite mixed states. More precisely, we have exhibited a necessary condition which has to be satisfied whenever a local and deterministic hidden variable model for a mixed state exists, the condition being that the state itself has to lay outside appropriate neighborhoods of all conceivable (entangled) Hardy’s states. As a consequence, all states which violate the above condition do not admit any local realistic description, this in turn implying that they are not separable.

### IV. ACKNOWLEDGMENTS

Work supported in part by Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Italy. We thank Professor R. Horodecki and M. Piani for their useful comments.