ABSTRACT

The velocity dispersion, or eccentricity distribution, of protoplanets interacting with planetesimals is set by a balance between dynamical friction and viscous stirring. We calculate analytically the eccentricity distribution function of protoplanets embedded in a cold, shear-dominated planetesimal swarm. We find a distinctly non-Rayleigh distribution with a simple analytical form. The peak of the distribution lies much lower than the root-mean-square value, indicating that while most of the bodies have similarly small eccentricities, a small subset of the population contains most of the thermal energy. We also measure the shear-dominated eccentricity distribution using numerical simulations. The numerical code treats each protoplanet explicitly and adds an additional force term to each body to represent the dynamical friction of the planetesimals. Without fitting any parameters, the eccentricity distribution of protoplanets in the N-body simulation agrees with the analytical results. This distribution function provides a useful tool for testing hybrid numerical simulations of late-stage planet formation.

Subject headings: planets and satellites: formation — solar system: formation

1. INTRODUCTION

Terrestrial planets, ice giants, and the cores of the gas giants are thought to form by accretion of planetesimals into protoplanets. The protoplanets emerge from the swarm of planetesimals after an epoch of runaway accretion. The subsequent dynamics of the protoplanets set several important features of the final planetary configuration, such as the mass and number of planets or cores. It is difficult to constrain this evolution, however, without constraining the properties of the disk in which they are embedded.

One important yet uncertain parameter is the size of the planetesimals, the building blocks. The outer Solar System and the later stages of formation in the inner Solar System likely lack gas, allowing the formation of kilometer-size bodies through gravitational instabilities. Those bodies collide and grind each other down to even smaller sizes in a collisional cascade. The existence of bodies small enough to damp their own velocity dispersion is an inevitable conclusion from the existence of Uranus and Neptune (Goldreich et al. 2004b). Without such small bodies, the ability of a growing protoplanet to gravitationally focus the planetesimals becomes inefficient, and the growth timescale becomes too long, of order $10^{12}$ years in the outer solar system.

The unavoidable influence of the planetesimals make numerical studies of planet formation difficult to carry out accurately. Despite modern computational power, an integration of the equations of motion for each body in a protoplanet and planetesimal swarm is impossible. Even without allowing planetesimal fragmentation, the number of kilometer-size bodies needed to comprise a Neptune size mass is humongous, of order $10^{12}$. Kokubo & Ida (1996) performed numerically feasible but physically less appropriate N-body simulations of a proto-planetary disk in which the size of planetesimals is larger than the value required to form the ice giants of our Solar System. Although interesting from a dynamical viewpoint, the results of such simulations can not be extrapolated to the scenario of smaller planetesimals since they lack collisional damping.

An alternative numerical approach to studying these systems is a coagulation code (Lee 2000; Kenyon & Lu 1998) in which the bodies are divided into size bins and the interaction of each pair of bins is calculated statistically. This approach fails once the number of bodies in any bin is not sufficiently large. Kenyon & Bromley (2006) have developed a hybrid code that treats planetesimals statistically while a small number of large bodies are integrated individually.

In this paper, we examine the processes that shape the eccentricity distribution of the large bodies. We assume, simply, that the planetesimals constitute a cold disk due to sufficiently frequent collisions. As a first step, we include the dynamical friction that the planetesimals exert on the large bodies but ignore the much slower process of their accretion onto those bodies. The rates of cooling from dynamical friction and heating from mutual excitations are discussed in §2. We write a Boltzmann equation to show the change in the distribution function of eccentricities due to each process in §3 and discuss the solution to that equation. In §4 we present the results of complementary N-body simulations designed to measure the eccentricity distribution directly. A discussion of the results follows in §5.

2. SHEAR-DOMINATED COOLING AND HEATING RATES

The eccentricities of the protoplanets represent a kind of “thermal” energy in their orbits, relative to perfectly circular motion. The extra non-circular velocity itself varies in magnitude and direction over an orbital period; it is
simpler to use the eccentricity, a constant of motion for the two-body problem. Specifically, we calculate the vector eccentricity,

\[ \mathbf{e} = \frac{\mathbf{v} \times \mathbf{H}}{GM_p} - \frac{\mathbf{r}}{r}. \]  

This expression relates the eccentricity of the particle, \( \mathbf{e} \), to the particle’s position, \( \mathbf{r} \), its velocity, \( \mathbf{v} \), its orbital angular momentum vector, \( \mathbf{H} \), and its mass, \( M_p \). In general, a protoplanet can have an inclination relative to the disk plane, and the eccentricity vector can have three components. However, we show in §2.4 that the shear-dominated regime strongly inhibits the growth of inclinations. Two dimensions then suffice to describe the configuration space of \( \mathbf{e} \).

We use the quantity of the Hill radius repeatedly in this work; for reference we define its value as

\[ R_H = \left( \frac{M_p}{3M_\odot} \right)^{1/3} a = \frac{R}{\alpha}, \]  

where \( M_p \) is the mass of a particle, \( a \) is its semi-major axis, \( R \) is its radius, and

\[ \alpha = \left( \frac{M_p}{3M_\odot} \right)^{1/3} \frac{R}{a} = \left( \frac{\rho_p}{3 \rho_\odot} \right)^{1/3} \frac{R_\odot}{a}. \]  

The Hill radius in turn specifies an eccentricity, the Hill eccentricity,

\[ e_H = R_H/a. \]  

For most of this paper, we employ the “two groups” approximation \( \{ \text{Wetherill} \& \text{Stewart} 1989; \text{Goldreich} \text{et al.} 2004b \} \) and split the disk into two uniform populations. One group is the numerous smaller bodies, or “planetesimals.” We denote their surface mass density as \( \sigma \). The other group, the “protoplanets”, consists of the bodies that dominate the excitations of the disk particles. Each protoplanet has a radius \( R \), mass \( M \), and eccentricity \( e \). We write the total surface mass density in protoplanets as \( \Sigma \). We assume \( \sigma > \Sigma \) although this is not always true in the final stages of planet formation \( \{ \text{Goldreich et al.} 2004a \} \).

### 2.1. Eccentricity Excitation of Protoplanets

We analyze the interaction of two protoplanets from a frame rotating with a reference orbit at a semi-major axis \( a \). The difference between the Keplerian angular velocity at each radius induces a shearing motion between particles on nearby circular orbits. For an orbit interior to \( a \) by a distance \( b \),

\[ \Omega_{rel}(b) = \Omega(a + b) - \Omega(a) \approx \frac{3}{2} \Omega \frac{b}{a}, \]  

in the limit of \( b \ll a \). This angular frequency also specifies the rate of conjunctions for the two bodies with orbits separated by \( b \).

The change in their eccentricity from each conjunction can be calculated analytically for two nearly circular orbits when \( b \gg R_H \):

\[ e_k = A_k \; e_H \left( \frac{b}{R_H} \right)^{-2}, \]  

\[ A_k = \frac{16}{3} \left[ K_0 \left( \frac{2}{3} \right) + \frac{1}{2} K_1 \left( \frac{2}{3} \right) \right] \approx 6.7. \]  

\( K_0 \) and \( K_1 \) are modified Bessel functions of the second kind \( \{ \text{Goldreich} \& \text{Tremaine} 1980; \text{Petit} \& \text{Henon} 1986; \text{Duncan et al.} 1989 \} \). We note that \( e_k \) refers to the perturbed body, while \( e_H \) and \( R_H \) refer to the Hill parameters of the perturbing one. The kick, viewed as a change in the eccentricity vector, is independent of the original eccentricity of the particle. Its orientation is perpendicular to the line connecting the two protoplanets and the sun at conjunction; we therefore assume it is random.

The eccentricity kick given by one protoplanet is strongest for the particles that approach with an impact parameter on the order of the Hill radius. Interactions from a greater distance, however, occur more often. In a shear-dominated disk, eccentricities are small, \( e \ll e_H \); to change that eccentricity significantly only requires small perturbations. These frequent but weaker perturbations dictate the overall velocity evolution of the protoplanets \( \{ \text{Goldreich et al.} 2004b; \text{Rafikov} 2004 \} \).

Specifically, the average differential rate that one protoplanet receives eccentricity kicks of strength \( e \) from other protoplanets is given by

\[ d\mathcal{R}_{ex}(e) = 2 \; n_{\text{big}} \; \frac{3}{2} \Omega(b(e)) \frac{db}{de} \; de, \]  

where \( n_{\text{big}} \) is the number of large bodies, \( \Omega(b(e)) \) is the orbital angular velocity at distance \( b(e) \), and \( db/dde \) is the derivative of the impact parameter with respect to the eccentricity.
where \( n_{big} \) is the number surface density of protoplanets and \( \frac{2}{\pi} \Omega b(e) \) is the velocity of encounters at those separations (given by eq. 3). The factor of two accounts for the combination of interior and exterior encounters. The excitation rate of a protoplanet with eccentricity \( e \) is then the rate of kicks comparable in magnitude to its current eccentricity:

\[
\frac{1}{e} \frac{de}{dt}_{ex} \sim e \left| \frac{dR_{ex}(e)}{de} \right| \sim \frac{\Sigma}{\rho R \alpha^2} \frac{1}{e}. \tag{9}
\]

The inverse of this rate can be interpreted as the timescale for a protoplanet’s eccentricity to change by an amount \( e \).

### 2.2. Dynamical Friction

As each protoplanet moves through the disk, it scatters and excites the eccentricities of the planetesimals that surround it. Cold planetesimals that approach a protoplanet with impact parameters of about a Hill sphere leave with \( \sim m \epsilon_H \) of additional momentum. This can either add to or subtract from the eccentricity of the protoplanet depending on the relative orientation between the pre-encounter eccentricities of the protoplanet and planetesimal. We write the net effect (Goldreich et al. 2004b)

\[
M \frac{de}{dt} \sim -n_s R_H \epsilon_H m(\epsilon_H + e) + n_s R_H \epsilon_H m(\epsilon_H - e), \tag{10}
\]

for a number surface density of planetesimals \( n_s \). This formula yields the damping rate, or the inverse damping time,

\[
\tau_d^{-1} \equiv \frac{1}{e} \frac{de}{dt}_{d} \bigg|_{d,e} = C_d \sigma \Omega \frac{1}{\rho R \alpha^2}. \tag{11}
\]

Calculating the coefficient \( C_d \) requires a more precise analysis of planetesimal scattering. We adopt the value \( C_d \approx 10 \) found by Ohtsuki et al. (2002), who measure the coefficient numerically.

### 2.3. Planetesimal Interactions

The distribution of planetesimal eccentricities does not affect our results, as neither the excitation nor the damping rates depend on their eccentricity as long as the planetesimals remain in a shear-dominated state. In this work, we focus on a range of parameters such that collisional cooling keeps the eccentricities of planetesimals below \( \epsilon_H \) and enforces the condition of shear domination.

### 2.4. Inclinations

An orbit with a small inclination angle \( i \) carries its particle out of the disk plane on vertical excursions of size \( \sim \alpha a \). An interaction that excites that particle’s eccentricity also affects its inclination, but with a magnitude inhibited by the geometry of the distant encounter:

\[
\frac{i_k}{\epsilon_k} \sim \frac{a}{b} \Rightarrow \frac{i_k}{i} \sim \left( \frac{\epsilon}{\epsilon_H} \right)^{3/2} \frac{\epsilon_k}{\epsilon} \tag{12},
\]

where \( b \) is the impact parameter of the perturber and \( i_k \) is the resulting change in inclination from an encounter. In contrast, planetesimals just entering the Hill sphere of a protoplanet damp the protoplanet’s non-circular velocity in all dimensions; no equivalent geometric factor inhibits the damping of inclinations. With the growth of inclinations suppressed, shear-dominated protoplanet disks are effectively two-dimensional (Wetherill & Stewart 1993; Goldreich et al. 2004b; Rafikov 2003).

### 2.5. The Eccentricity Distribution - a Qualitative Discussion

The dynamical friction rate sets a characteristic time over which the eccentricities of all of the bodies are changed significantly. In this sense, the eccentricity distribution of the proto-planetary swarm is reset every \( \tau_d \). The excitation rate, however, varies with \( \epsilon \). Equating the excitation rate, equation 9 and the damping rate, equation 11 yields an important reference value, \( \epsilon_{eq} \):

\[
\epsilon_{eq} \sim \frac{\Sigma}{\sigma} \epsilon_H. \tag{13}
\]

Statistically, each protoplanet receives one kick of this magnitude every damping timescale.

We deduce the distribution of eccentricities on each side of \( \epsilon_{eq} \) by examining the dependence of the kicking rate on eccentricity. Excitation to \( \epsilon \gg \epsilon_{eq} \) requires a kick \( \epsilon_k \gg \epsilon_{eq} \). Such strong kicks occur less often in one damping timescale than kicks of strength \( \epsilon_{eq} \) by a factor of \( \epsilon_{eq}/\epsilon_k \). With fewer kicks to populate the high eccentricity distribution, the number of bodies with such eccentricities echoes the rate of kicks and falls off with eccentricity as \( \epsilon^{-1} \) (Goldreich et al. 2004b).

Kicks of order \( \epsilon \ll \epsilon_{eq} \) occur frequently in each damping timescale, thereby overwhelming the effects of dynamical friction on the lowest eccentricity bodies. A dynamic equilibrium dominated by only the stirring mechanism implies that kicks to and from every eccentricity vector occur at the same rate. For this to be true the distribution must be constant over the configuration space. The number of bodies with an eccentricity of order \( \epsilon \ll \epsilon_{eq} \) then scales as the area of configuration space, \( \sim \epsilon^2 \).
3. A BOLTZMANN EQUATION

In the following section we develop a differential equation to describe analytically the distribution function of protoplanet eccentricities. We construct this equation in the spirit of the Boltzmann equation, examining the change in the number of bodies with a particular eccentricity due to the effects of dynamical friction and viscous stirring.

The space of possible eccentricities is inherently two-dimensional (eq. 11), since inclinations can be neglected (eq. 12). Additionally, the interaction rates depend only on the magnitude of the protoplanet eccentricity, forcing the distribution function to share this dependence: \( f(e) = f(|e|) \). The two-dimensional \( f(e) \) is related to the number of bodies with velocity on the order of \( e \) by its integral, roughly \( e^2 f(e) \).

Dynamical friction lowers the eccentricities of all bodies proportionally to their eccentricity. Equivalently, the number of bodies with a certain \( e \) changes as the protoplanets with that value are damped to lower eccentricities and replaced by bodies from a higher eccentricity. We write this as:

\[
\frac{\partial f(e)}{\partial t} \bigg|_{d.f.} = - \text{div} \left( f(e) \frac{\partial e}{\partial t} \right) = \frac{\partial f(e)}{\partial e} \frac{e}{\tau_d} + \frac{2f(e)}{\tau_d},
\]

where we have used \( \frac{\partial e}{\partial t} = -e/\tau_d \) for the effects of dynamical friction.

At a given \( e \), particles are kicked to a new eccentricity \( e_n \) at an average rate that depends on the magnitude of the kick, \( |e_n - e| \). Also, particles with an initial eccentricity \( e_n \) are kicked to \( e \) at the same rate. The total flux of particles to and from a given eccentricity is:

\[
\frac{\partial f(e)}{\partial t} \bigg|_{\text{kicks}} = \int \int p(|e_n - e|)[f(e_n) - f(e)] \, d^2e_n,
\]

where \( p(e) \) describes the rate at which bodies experience changes in their eccentricities by an amount \( e \). This is the two-dimensional analog of the excitation rate equation (eq. 9):

\[
p(e) = \frac{1}{2\pi |e|} \left| \frac{\partial R_{ex}}{\partial e} \right| = A_k \frac{9}{16\pi^2} \frac{\Sigma \Omega}{\rho R \alpha^2} e^{-H} \frac{1}{e^3}.
\]

The sum of the dynamical friction terms and the kicking integral describes the dynamics of shear-dominated protoplanets interacting with each other in a smooth disk of planetesimals. The combined influence of these two processes can bring the protoplanets into an equilibrium state, where the number of particles with eccentricity \( e \) remains constant in time:

\[
0 = \frac{\partial f(e)}{\partial e} \frac{e}{\tau_d} + \frac{2f(e)}{\tau_d} + \int \int p(|e_n - e|) [f(e_n) - f(e)] \, d^2e_n.
\]

3.1. The Solution

We show in Appendix A that

\[
f(e) = \frac{1}{2\pi e_s^2} \left[ 1 + \left( \frac{e}{e_s} \right)^2 \right]^{-3/2},
\]

\[
e_s = \frac{9A_k}{8\pi C_d} e^{-H} \approx 0.24 \frac{\Sigma}{\sigma} e^{-H}
\]
satisfies the equilibrium equation, equation 14 for all \( e \). This function is the equilibrium eccentricity distribution of shear-dominated protoplanets.

The solid line in Fig. 1 shows a distribution function for \( \Sigma \approx 0.002 \text{ g cm}^{-2} \) and \( \sigma = 0.1 \text{ g cm}^{-2} \). Although the function formally extends above \( e_H \), we stress that it is only accurate for eccentricities \( e \ll e_H \). Both the dynamical friction and the excitation rates (eqs. 14 & 16) are not valid for \( e \gtrsim e_H \).

Several moments of the distribution can be calculated in terms of the only free parameters:

\[
\frac{e_{\text{median}}}{e_H} = 0.41 \frac{\Sigma}{\sigma}, \quad \frac{\langle e \rangle}{e_H} = 0.24 \frac{\Sigma}{\sigma} \log(3 \frac{\sigma}{\Sigma}), \quad \frac{(1/e)^{-1}}{e_H} = 0.24 \frac{\Sigma}{\sigma}.
\]

According to equation 14, \( (e) \) is infinite. However, the single largest kick in eccentricity from an almost circular protoplanet encounter is of order \( e_H \). Truncating the integral at \( e_H \) produces the logarithmic term in the expression above. Moments higher than the mean also diverge; realistically, they are dominated by the bodies with the highest eccentricities, of order \( e_H \).

It is easy to see that this solution, in the high- and low-eccentricity limits, produces the same power-laws discussed in §2.3. In fact, it can be shown directly from equation 17 that any solution to this differential equation reduces to those limiting power-laws.
4. NUMERICAL SIMULATIONS

Here we describe a direct measurement of the eccentricity distribution from gravitational N-body simulations that include an additional force to represent dynamical friction.

The N-body part of our simulation uses Gauss’s equations to evolve a set of orbital constants chosen to vary slowly under small perturbations. A modified version of Kepler’s equation produces the orbital phase for each body at each time step. The IDA solver from the SUNDIALS software package (Hindmarsh et al. 2005) integrates the dynamical equations. During close encounters of two protoplanets, we integrate their motion relative to the center of mass of the pair.

We represent the planetesimal population in these simulations with an extra force term that damps the non-circular velocities of the protoplanets at the rate $R_d(e)$, given by equation 11. An ad-hoc transition between the damping rate for $e < e_H$ and the appropriate rate for $e > e_H$ prevents unphysical enhancements of the damping force during close encounters. The growth of protoplanets in mass due to planetesimal accretion is not included; the accretion rate is always lower than the dynamical friction rate and will not affect the eccentricity evolution (Goldreich et al. 2004).

Each simulation begins with the protoplanets on circular orbits with random phases and random semi-major axes, within a chosen annulus. The average spacing between bodies, $M/(\Sigma 2\pi a)$, is several Hill radii. The protoplanets interact for several damping timescales $\tau_d$ before the distribution reaches equilibrium.

We record the eccentricity of the protoplanets every $\Delta t \approx 0.1 \tau_d$ starting at about 100 $\tau_d$. Several hundred orbits produces a well-populated histogram of eccentricities. The bodies in the inner and outer edges of the disk are not measured, to avoid artificial boundary effects that inhibit excitations. We bin the resulting eccentricities logarithmically. Errors are assigned to each bin according to a Poisson distribution with the sample size defined as the product of the number of bodies measured and the sampling time in units of the damping timescale $\tau_d$. Since each protoplanet suffers a significant change in eccentricity every $\tau_d$, one measurement of the eccentricity distribution is independent from a previous measurement if they are separated in time by $\tau_d$. We sample faster than $\tau_d$ to increase the resolution of the histogram slightly.

The statistical error bars do not take into account the inhomogeneity of the protoplanet disk on small length scales. Given a surface density, the mass of a single protoplanet sets a typical radial separation between bodies. This length scale corresponds to an eccentricity scale through equation 6 (in the simulations presented here, this value is slightly below $e_H$). As the disk evolves, the viscous stirring causes migrations in the semi-major axes of the particles that smooth the average radial distribution. If measured only over intervals shorter than the migration timescale, the eccentricity distribution may vary for eccentricities above the eccentricity set by the typical separation. Fluctuations from this effect are visible in Figs. 1 and 2.

Several simulations of disks with different protoplanet mass distributions are presented below.

4.1. Equal Mass Protoplanets

Figure 1 shows the eccentricity distribution measured from a simulated disk of 120 equal mass protoplanets ($M = 2.5 \times 10^{-9} M_\odot$) with surface densities $\Sigma \approx 0.002 g cm^{-2}$ and $\sigma = 0.1 g cm^{-2}$. A single population of protoplanets best reflects the “two groups” approximation we use to derive equations 9 and 14. The analytic solution, equation 15, for the same parameters in the simulation is superposed on Figure 1. While the overall match is not perfect, the shape of each curve is strikingly similar. The two curves match extremely well if one is shifted by around 15 percent in the $e$ direction. This difference is attributable to the difficulty of assigning a correct value of $\Sigma$ to the simulation given a finite number of protoplanets.

We note that there are no free parameters in this comparison. The numerical distribution is a direct counting of the number of bodies within each eccentricity bin, while a choice of $\Sigma$, $\sigma$, and $M$ completely specifies the analytical curve.

4.2. Mass Distributions

Naturally occurring protoplanet populations exhibit non-trivial distributions in mass. Before describing such a disk in the framework we have developed, we clarify several points.

Protoplanets with different masses, or equivalently, different radii, experience different viscosities stirring rates. We decompose the total surface density in protoplanets, $\Sigma$, into a differential quantity, $d\Sigma/dR$, and write the excitation rate of a body with radius $R$ as

$$R_x(e, R) \sim \int \frac{d\Sigma}{dR'} \frac{\Omega}{\rho R' \alpha^2} \frac{1}{e} \frac{e_H(R')}{e} dR'.$$  (20)

The identity $e_H(R') = (R'/R)e_H(R)$ when substituted into equation 20 yields

$$R_x(e, R) \sim \frac{\Omega}{\rho R \alpha^2} \frac{1}{e} \frac{e_H(R)}{e} \int \frac{d\Sigma}{dR'} dR'.$$  (21)

In words, the excitation rate of one body only depends on the total surface density of all other bodies, regardless of the specific mass distribution. This differs from the assertion by Goldreich et al. (2004) that only the most massive bodies contribute to the viscous stirring rate. Equation 21 seems to indicate that there should be no distinction between big bodies and small bodies since every body contributes to the viscous stirring. A closer investigation uncovers the mass range of bodies that provide significant stirring.
Eccentricity kicks of strength $e_k$ can occur at any combination of $M$ and $b$ that satisfies the inverse square law of gravitation: $e_k \sim M(R') b(R')^{-2}$. However, the smallest impact parameter that contributes to a body’s excitation is about $R_H$. A minimum $b(R') \sim R_H$ sets a minimum mass for bodies to kick a body with mass $M$ by an amount $e_k$:

$$M_{\text{min}}(e_k, R) \sim \frac{e_k}{e_H} M \tag{22}$$

Likewise, a body can only be as far away as its radial position in the disk, $a$. This sets a maximum mass,

$$M_{\text{max}}(e_k, R) \sim \left( \frac{e_k}{e_H} M \right) \frac{a^2}{R_H^2}. \tag{23}$$

For a choice of the most relevant kick strength, $e_k$, these limits define the sizes of bodies that participate in the excitation of a body with size $R$.

As a numerical confirmation of these results, we simulate a disk of planetesimals with a surface mass density $\sigma = 0.2 \text{ g cm}^{-2}$ and 120 protoplanets. In this case, we divide the protoplanets into two groups of different mass: sixty of mass $m_1 = 2 \times 10^{24}$ g, and sixty of mass $m_2 = 3.8 \times 10^{25}$ g. These masses are within the limits set by equations 22 & 23. We plot the absolute eccentricity distribution of each mass group binned separately in Figure 2. Additionally, we plot the analytic distributions given by $\sigma$, as specified above, and $\Sigma$, the sum of the surface densities of both groups.

It is clear that each group of protoplanets with the same mass matches the analytic distribution well. The offset between the peak of each group is due to the dependence of the distribution on the Hill eccentricity of each body. In general, the distribution for a swarm of protoplanets with a mass distribution is merely the sum of individual distributions for protoplanets of each mass.

5. CONCLUSIONS

We presented an analytic model for the distribution function of the eccentricities of a protoplanet population embedded in a shear-dominated planetesimal disk. The eccentricity distribution measured with numerical simulations matches the analytic result very well.

Since we have manually inserted the dynamical friction rate that we expect into the numerical simulations, this work does not test our prescription of dynamical friction. However, the numerical and analytic representations of viscous stirring are completely independent. Equation 17 uses a viscous stirring rate involving only distant encounters. In our numerical simulations, Newton’s laws dictate the protoplanet interactions directly without any simplifying assumptions.

The consistency of the two calculations proves that in a two-dimensional shear-dominated disk, interactions between
FIGURE 2: A comparison of the results of a numerical simulation of protoplanets in a perfectly bimodal mass distribution ($m_1 = 2 \times 10^{24} \text{ g}, m_2 = 3.8 \times 10^{25} \text{ g}$). We simulate sixty bodies of each mass, for a total surface density in protoplanets of $\Sigma \approx 0.003 \text{ g cm}^{-2}$ and a planetesimal surface density of $\sigma = 0.2 \text{ g cm}^{-2}$. The eccentricities of each mass group are binned separately; each distribution is a good match to equation 18 when scaled to the appropriate Hill eccentricity. The error bars are assigned following the same algorithm as Figure 1.

non-crossing orbits are entirely responsible for setting the eccentricities of the protoplanets. The analytic form of the distribution function provides a simple way to test similar hybrid numerical codes.

Several features of the distribution highlight interesting properties of the dynamics. We reason in §2.5 that most protoplanets have eccentricities $\sim (\Sigma/\sigma)e_H$, the value of $e$ where the excitation and damping timescales are equal. The distribution function shows this to be true: the median and mean (up to a logarithmic factor) of any distribution are on the order of this equilibrium eccentricity. Higher moments of the distribution, however, are dominated by the highest eccentricity bodies. This signals that different statistics of the distribution can reflect different subsets of the overall population. For example, the average “thermal” energy of the protoplanets is represented by the root-mean-squared eccentricity, $\langle e^2 \rangle$. The fractionally fewer bodies with eccentricities close to $e_H$ dominate $\langle e^2 \rangle$ and thus, contain most of the energy.

The shape of the distribution also merits discussion. $N$-body integrations of a group of single mass bodies show that their eccentricities follow a Rayleigh distribution (Ida & Makino 1992). For reference, we plot a Rayleigh distribution in Figure 1. It is entirely inconsistent with our calculations. This is not surprising. In addition to simulating bodies in the regime of eccentricities that are large compared to the Hill eccentricity, Ida & Makino (1992) do not include any effects that can balance the mutual excitations of their particles. The dynamical friction in our simulations balances the viscous stirring and establishes the equilibrium distribution we derive.

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THE ANALYTIC DISTRIBUTION FUNCTION

Here we outline the evaluation of the equilibrium equation, equation (17), using the distribution function, equation (18). To simplify the notation, we rescale all eccentricities by \( e^* \) and algebraically manipulate the coefficients of each term in equation (17). We are left with the equivalent burden of proving that

\[
g(e) = (1 + e^2)^{-3/2} \quad (A1)
\]

satisfies

\[
2\pi \frac{\partial g(e)}{\partial e} e + 4\pi g(e) = \int \int \frac{g(e) - g(e_n)}{|e_n - e|^3} d^2 e_n. \quad (A2)
\]

The left-hand side is easy to compute:

\[
\text{L.H.S.} = \frac{4\pi}{(1 + e^2)^{3/2}} - \frac{6\pi e^2}{(1 + e^2)^{3/2}}. \quad (A3)
\]

To integrate of the right-hand side, we translate the origin of the integration variables by \( e \) and rotate them to align \( e \) with one of the coordinate axes. In those coordinates:

\[
I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(k^2 + h^2)^{3/2}} \left[ \frac{1}{(1 + e^2)^{3/2}} - \frac{1}{(1 + k^2 + (h + e)^2)^{3/2}} \right] dk \ dh, \quad (A4)
\]

with \( e_n = \{k, h\} \).

After the integration over \( k \), we rewrite the integral in terms of a new variable \( h' = (1 + e^2)/(eh) \),

\[
I = \int_{-\infty}^{\infty} \left( \frac{2e}{(1 + e^2)^{5/2}} + \frac{8}{(1 + e^2)^2} \frac{\partial^2 E(e^2 z/(1 + e^2))}{\partial z^2} |h'| h'^2 \right) dh', \quad (A5)
\]

where, \( z = -2h' - h'^2 \), and \( E(e^2 z/(1 + e^2)) \) is the complete elliptic integral of the second kind.

We change the integration variable to \( z \), taking care to evaluate the integrand with the appropriate branch of the double-valued relation \( h'(z) \). The integral evaluates to

\[
I = \frac{4\pi}{(1 + e^2)^3} + \frac{8}{(1 + e^2)^2} \int_0^1 \left[ \frac{4 - 3z}{\sqrt{1 - z}} \right] \frac{\partial^2 E(e^2 z/(1 + e^2))}{\partial z^2} \ dz. \quad (A6)
\]

With the second derivative of the elliptic function expressed as a power series, each term can be integrated over \( z \). The remaining power series in \( e^2/(1 + e^2) \) equals

\[
I = \frac{4\pi}{(1 + e^2)^3} - \frac{3\pi e^4}{(1 + e^2)^4} \left[ 2 F_1 \left( \frac{5}{2}; 1; 3; \frac{e^2}{1 + e^2} \right) - \frac{1}{2} 2 F_1 \left( \frac{3}{2}, 2; 3; \frac{e^2}{1 + e^2} \right) \right]. \quad (A7)
\]

After additional algebraic manipulation, this result equals the left hand side of the original equation (eq. A3).