Entanglement dynamics of qubits in a common environment

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The entanglement dynamics of a quantum register with two or three two-level atoms interacting with a common environment is analytically studied by the quantum jump method. In contrast to the usual belief that the environment plays a role of destroying the entanglement, it is found that the environment can also produce stable entanglement between the qubits that are prepared initially in a separable state. Our study indicates how the environment noise produces the entanglement in manner of incoherence and emphasizes the constructive role played by the environment in certain tasks of quantum information processing.

I. INTRODUCTION

Entanglement is a remarkable feature of quantum mechanics and plays a fundamental role in quantum computation and quantum information processing. Since the creation of the entanglement between qubits has fundamental importance for further quantum information processing, it has attracted much attention in recent years [1]. The entanglement can be produced either by a direct interaction between qubits [2], or by an indirect interaction between the qubits through a third party [3]. However, both of the above processes are confined in closed system, i.e., the influences of environment are neglected. In a real situation, a quantum system is never isolated and inevitably interacts with its environment. A severe effect of this unwanted interaction is decoherence, which generally leads to the degradation of quantum coherence and entanglement. Till today, decoherence remains to be the bottleneck to the practical implement of quantum information processing. Recently, the influence of the environment to the entanglement have been investigated [4-7]. It was found that in certain situations the environment played a mediation role can be constructive to the production of the entanglement, in contrast to its destructive role. Here we consider that the environment acts as a third party to induce the entanglement between the qubits. On the one hand, due to the decoherence effect of the environment, the qubits will loss their unitarity under the time evolution. Correspondingly, the initial entanglement will loss together with the decoherence. On the other hand, the third party role of the environment will induce a stable entanglement between the qubits. We emphasize that the entanglement production induced by the environment is in manner of incoherence.

Entanglement dynamics behavior (i.e. the time-evolution of the entanglement quantity) of a quantum system coupled to its environment can reflect the details of the decoherence effect and is thus helpful to the study of the decoherence effect. However, few papers in the literature addressed the influence of the environment to the entanglement dynamics [8, 9]. Further studies of the entanglement dynamics are desirable if one wants to control the influences of the environment on the entanglement [10].

In the present work we model a quantum register by two or three dipole-dipole interacted two-level atoms coupling to a common environment and study exactly its entanglement dynamics behavior. The environment we used has infinite degrees of freedom, in contrast to the single-mode model used by Kim et al. [4]. By introducing a collective mode consisting of all the qubit modes and eliminating the enormous irrelevant degrees of freedom of the environment, a quantum master equation of the atomic system can be obtained by the standard Born-Markovian approximation. We employ the quantum jump method to study the master equation obtained. We present the time evolution behavior of bi- and tri-partite entanglement defined by the concurrence [11] or the negativity [12]. In contrast to the simple oscillating behavior of the entanglement between the qubits obtained by Kim et al [4], our results indicate that for some specific states a stable entanglement between the qubits can be created through the environment in manner of incoherence. This is in contrast to the behavior of the entanglement induced by the dipole-dipole interactions, in which the environment plays a usual role to suppress the entanglement induced coherently.

The paper is organized as follows. In Section II the model is presented and the quantum jump method to the master equation is given. In Section III we study the entanglement dynamics of a two-qubit system. The discussion of the entanglement dynamics in three-qubit system is presented in Section IV. Finally, Section V is devoted to a brief summary.

II. THE QUANTUM JUMP METHOD TO THE MASTER EQUATION

We consider a quantum register composed of $N$ identical two-level atoms interacting collectively with an environment. The time evolution of the density operator

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of the system can be described by the following master equation

$$\frac{d\rho(t)}{dt} = -i[H_1, \rho(t)] + \gamma \frac{1}{2} \sum_{j=1}^{N} (2\sigma_+^j \rho(t) \sigma_-^j - \sigma_-^j \sigma_+^j \rho(t) - \rho(t) \sigma_-^j \sigma_+^j),$$

where $\gamma$ is the decay constant of the collective mode represented by

$$J_\alpha = \sum_{i=1}^{N} \sigma_i^\dagger (\alpha = \pm, z)$$

and

$$H_I = \sum_{i>j} (g_{ij} \sigma_i^+ \sigma_j^- + h.c.)$$

describes the dipole-dipole interactions of the qubits. The first term on the right-hand side of Eq. (1) generates a coherent unitary evolution of density matrix, while the second term represents the decoherence effect of the environment on the system and generates an incoherent dynamics of system. For simplicity, here we take the coupling constant $g_{i,j} = g$. Thus the interaction Hamiltonian becomes

$$H_I = g \left( J_+ J_- - J_z \right).$$

By using individual mode representation Eq. (1) can be rewritten as

$$\frac{d\rho(t)}{dt} = -i[H_I, \rho(t)] + \gamma \frac{1}{2} \sum_{i,j=1}^{N} (2\sigma_+^i \rho(t) \sigma_-^j - \sigma_-^j \sigma_+^i \rho(t) - \rho(t) \sigma_-^j \sigma_+^i).$$

It should be pointed out that the terms with $i = j$ in the sum on the right-hand side of Eq. (2) denote the individual dissipations of each atoms due to the environment, while the $i \neq j$ terms describe the couplings between the atoms induced by the common environment.

To get the analytical solution of Eq. (1), we make use of the quantum jump method [13, 18]. The reduced density matrix $\rho(t)$ can be written as the summation of the conditional density matrix $\rho^{(n)}$ (t) defined by

$$\rho(t) = \sum_n \rho^{(n)} (t) = \sum_n Tr_{F}\{Q_n \rho_{tot}(t)\},$$

where $\rho_{tot}$ is the total density matrix of the system and the quantized environment, $Tr_{F}\{\cdot\}$ is the partial trace over the modes of the environment, and $Q_n$ is the projection operator on the state of the quantized environment field that contains $n$ photons. The $\rho^{(n)}$ (t) denotes the reduced density matrix of the system with $n$ photons detected in the environment. The time evolution of the $\rho^{(n)}$ (t) can be derived readily,

$$\frac{d\rho^{(n)}(t)}{dt} = -i[H_{\text{eff}}^{(n)}(t) - \rho^{(n)}(t) H_{\text{eff}}^{\dagger}(t)]$$

$$- \gamma J_- \rho^{(n-1)}(t) J_+(1 - \delta_{n0}),$$

where $H_{\text{eff}} = H_I - \frac{1}{2} J_+ J_-$. Due to the dissipative effect of the environment on the system, the effective Hamiltonian $H_{\text{eff}}$ is a non-Hermitian operator. It is straightforward to write down the formal solution of Eq. (3)

$$\rho^{(n)}(t) = e^{-iH_{\text{eff}} t} \rho^{(0)}(0) e^{iH_{\text{eff}} t},$$

$$\rho^{(n)}(t) = \int_0^t dt' e^{-iH_{\text{eff}}(t-t')} \gamma J_- \rho^{(n-1)}(t') J_+ e^{iH_{\text{eff}}(t-t')},$$

where $n = 1, 2, \ldots$ and $\rho^{(0)}(0)$ is the initial state. Thus, the total reduced density matrix of the system can be obtained by the summation over all the conditional density operators $\rho^{(n)}(t)$. In the following we explicitly consider a two-qubit and a three-qubit systems.

### III. ENTANGLEMENT DYNAMICS IN A TWO-QUBIT SYSTEM

To simplify the following calculations, we use the collective state representation [14]. The space of this representation is spanned by the common eigenstates of the complete set of commuting operators $\{J_\pm, J_z\}$ for two-qubit system. This collective state representation can also be obtained by the angular momentum additive rules and the Clebsch-Gordan coefficients are obtained from the product state representation of the individual ones [15]. The two-qubit basis denoted by $|j, m\rangle$ includes the spin singlet and triplet

$$|0, 0\rangle = \frac{1}{\sqrt{2}}(|++\rangle - |--\rangle),$$

$$|1, 1\rangle = |++\rangle,$$

$$|1, 0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |--\rangle),$$

$$|1, -1\rangle = |--\rangle.$$

The $J_\pm$ act on the collective basis by

$$J_{\pm} |j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)}|j, m \pm 1\rangle.$$

When a total system is composed of two subsystems described by a two-dimensional Hilbert state, the natural measure of the entanglement between these two subsystems is the entanglement of formation [16]. This quantity can be analytically calculated as a function of the concurrence $C$ [11], which can also be taken as a measure of the entanglement. The $C$ is defined by

$$C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4),$$

where $\lambda_i (i = 1, \ldots, 4)$ are eigenvalues of the matrix $[\rho^{1/2}(\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y) \rho^{1/2}]^{1/2}$. Here $\rho^*$ is the complex conjugation of $\rho$. The concurrence varies from $C = 0$ for a disentangled state to $C = 1$ for a maximally entangled state. In the following we shall use the concurrence to quantify the degree of entanglement in the two-qubit system.
In order to study explicitly the entanglement dynamics of the two-qubit system, the initial state is taken as \( \rho(0) = |+\rangle\langle+| + |-\rangle\langle-| \) for \( g = 1.0 \) (solid line), \( 0.2 \) (dashed line), and \( 0 \) (dotted line) and \( \gamma = 0.1 \). (b) The time evolution of \( \Delta C = C_{g=1.0} - C_{g=0} \) with \( \gamma = 0.1 \) (solid line) and \( \gamma = 0 \) (dashed line).

FIG. 1: (a) The time evolution of the concurrence \( C(t) \). The pair of atoms is initially prepared in the state \( \rho(0) = |+\rangle\langle+| \) for \( g = 1.0 \) (solid line), \( 0.2 \) (dashed line), and \( 0 \) (dotted line) and \( \gamma = 0.1 \). (b) The time evolution of \( \Delta C = C_{g=1.0} - C_{g=0} \) with \( \gamma = 0.1 \) (solid line) and \( \gamma = 0 \) (dashed line).

In order to study explicitly the entanglement dynamics of the two-qubit system, the initial state is taken as \( \rho(0) = |+\rangle\langle+| + |-\rangle\langle-| \). Expanding this state in the collective basis of Eqs. (5) and substituting it into Eqs. (6), we can obtain analytically the solution of the master equation (5) after some straightforward calculations

\[
\rho(t) = \frac{e^{-2\gamma t}}{2} |1,0\rangle\langle1,0| + \frac{1 - e^{-2\gamma t}}{2} |1,-1\rangle\langle1,-1| \\
+ \frac{1}{2} |0,0\rangle\langle0,0| + \frac{e^{-i2\gamma t}}{2} |1,0\rangle\langle0,0| \\
+ \frac{e^{i2\gamma t}}{2} |0,0\rangle\langle1,0|.
\]  

(7)

Then the time evolution behavior of the entanglement measure, i.e. the concurrence, can be determined by

\[
C(t) = \frac{1}{2} \sqrt{(e^{-2\gamma t} - 1)^2 + e^{-2\gamma t}(\sin 2\gamma t)^2}.
\]  

(8)

From Eq. (7) and Eq. (8) we can see that the steady solution of the atomic system expressed in the collective representation is

\[
\rho_s = \frac{1}{2} |1,-1\rangle\langle1,-1| + \frac{1}{2} |0,0\rangle\langle0,0|,
\]

which is a probabilistic mixture of the extreme states \( |j,-j\rangle \) and an entangled mixed state with \( C_s = 0.5 \). The extreme states \( |j,-j\rangle \) is decoherence-free states, as shown in Ref. 17. One of the decoherence-free states, i.e. \( |1,-1\rangle \) is maximal entangled pure state. This means that the system induces a collective entangled state which is immune to the decoherence when the qubits interact collectively to the environment. The entangled decoherence-free state is the main physical reason for the present entanglement production. From the purification scheme proposed in Ref. 18 one can get the maximal entangled state \( |1,-1\rangle \) from a collective pair of the states with probability 1/16. Thus it is shown that the maximal entangled state \( |1,-1\rangle \) between the two atoms can be produced from a product state by the environment.

The time evolution behavior of the concurrence is showed in Fig. 1(a). We consider three cases, i.e., the absence of the dipole-dipole interaction (\( g = 0 \), dotted line), weak (\( g = 0.2 \), dashed line) and strong (\( g = 1.0 \), solid line) dipole-dipole interaction. In the case of \( g = 0 \), the entanglement increases rapidly, then approaches to a stable value. Since there is no dipole-dipole interaction between the qubits, it is very clear that this entanglement is induced purely by the environment. This is a typical example that the environment plays a constructive role in the entanglement formation between qubits. When the dipole-dipole interaction is switched on, the entanglement induced by the environment is not affected but the total entanglement oscillates. More stronger the dipole-dipole interaction is, more faster the entanglement does oscillate, which is easy to see from Eq. (8). At the large time scale, the stable entanglement is independent of the dipole-dipole interaction, which indicates the entanglement induced by the dipole-dipole interaction decays exponentially, as shown in Fig. 1(b) by solid line. For the entanglement induced by the coherent interaction, the environment plays a destructive role, as usual. To confirm this, we switch off the environment, i.e., \( \gamma = 0 \). In this case, the entanglement introduced by the coherent interaction oscillates without dissipation, as showed in Fig. 1(b)(dashed line). In a word, the environment can act as a third party to induce the entanglement between the qubits in manner of incoherence. It also plays a usual role in destroying the entanglement induced by the dipole-dipole interaction in manner of coherence. The dual nature of the environment can be seen more clearly by using Werner state (8) as the initial state.

The Werner state reads

\[
W_\pm = (1 - p) \frac{I_4}{4} + p |\Psi_\pm\rangle\langle\Psi_\pm|,
\]

(9)

where \( I_4 \) is a 4 \times 4 identity matrix, \( |\Psi_+\rangle = |1,0\rangle, |\Psi_-\rangle = |0,0\rangle \), as in Eqs. (5), and \( p \) denotes the fidelity of \( W_\pm \) to \( |\Psi_\pm\rangle \). It is known that \( W_\pm \) are entangled states for \( p > 1/3 \) and \( C_\pm = \frac{3p-1}{2} \). By the similar way, one can obtain the time evolutions behavior of \( \rho_\pm \), as follows.
\[
\rho_\pm(t) = \frac{(1 - p)e^{-2\gamma t}}{4}|1, 1\rangle\langle 1, 1| + \left\{ \frac{(1 + 3p)e^{-2\gamma t}}{4} + \frac{(1 - p)\gamma e^{-2\gamma t}}{2} \right\} |1, 0\rangle\langle 1, 0| + \frac{1 - p}{4}|0, 0\rangle\langle 0, 0|
\]

\[
+ \left\{ \frac{1 - p}{4} + \frac{(1 + 3p)(1 - e^{-2\gamma t})}{4} + \frac{(1 - p)[1 - e^{-2\gamma t}(1 + 2\gamma t)]}{4} \right\} |1, -1\rangle\langle 1, -1|,
\]

\[
\rho_- (t) = \frac{(1 - p)e^{-2\gamma t}}{4}|1, 1\rangle\langle 1, 1| + \frac{(1 - p)e^{-2\gamma t}(1 + 2\gamma t)}{4}|1, 0\rangle\langle 1, 0| + \frac{1 - p}{4}|0, 0\rangle\langle 0, 0|.
\]

Because only the diagonal forms of collective states is involved in the Werner states, the coherent interaction \(H_I\) of Eq. (1) has no actions on the Werner states.

The concurrence can be calculated by the similar way. Fig. 2 shows the concurrences of \(\rho_\pm(t)\) as a function of \(p\) and \(t\). When \(p < 1/3\), a common characteristic of \(\rho_\pm(t)\) is that they have no entanglement initially. With time, a stable entanglement is formed through the environment. For large \(p\), the concurrences of \(\rho_\pm(t)\) show apparently different behaviors. For \(\rho_+(t)\), the initial entanglement decays rapidly. Then the entanglement is reconstructed by the environment. For \(\rho_-(t)\), the initial entanglement does not decay. The entanglement induced by the environment is superposed to the initial entanglement. At large \(p\) near 1, the environment plays a minor role in reconstructing the entanglement in both \(\rho_\pm(t)\). This is due to that the weight of the decoherence-free state \(|\Psi_\perp\rangle\) in the first term of the Werner state \(W_\pm\) decreases with increasing \(p\), as indicated by Eq. (9). But for the initial entanglement, the environment plays different roles for \(\rho_\pm(t)\), which can be seen clearly from the case of \(p = 1\). For the initial state \(W_+\), which is maximal entangled pure state \(|\Psi_+\rangle\), it evolves to a product state and loses all of its entanglement. For the initial state \(W_-\), i.e. the pure state \(|\Psi_-\rangle\), is independent of time and free from the decoherence. This is consistent with Ref. [17].

It is not difficult to prove that the concurrences \(C_\pm(t)\) asymptotically evolute to the steady values

\[
C_{+s} = \frac{1 - p}{4},
\]

\[
C_{-s} = \frac{1 + 3p}{4}.
\]

The stable entanglement here is produced by the incoherent action of environment on the system purely, because the coherent part \(H_I\) of Eq. (1) has a zero-action on the Werner states.

IV. ENTANGLEMENT DYNAMICS IN A THREE-QUBIT SYSTEM

The three-qubit basis can be constructed from the addition of angular momenta \(J_{1,2}\) and \(J_3\). The space of this representation is spanned by the common eigenstates of the complete set of commuting operators \(\{J_{1,2}^2, J^2, J_z\}\). The collective basis \(|j_{1,2}, j, m\rangle\) can be obtained by the corresponding Clebsch-Gordan coefficients [15].

\[
|1, 3/2, 3/2\rangle = |++++\rangle,
\]

\[
|1, 3/2, -3/2\rangle = |---\rangle,
\]

\[
|1, 3/2, 1/2\rangle = \frac{1}{\sqrt{3}}(|+++\rangle + |++-\rangle + |---\rangle),
\]

\[
|1, 3/2, -1/2\rangle = \frac{1}{\sqrt{3}}(|++-\rangle + |---\rangle + |--+\rangle),
\]

\[
|1, 1/2, 1/2\rangle = \frac{1}{\sqrt{6}}(|++-\rangle - |+++\rangle - |---\rangle),
\]

\[
|1, 1/2, -1/2\rangle = \frac{1}{\sqrt{6}}(|++-\rangle + |+++\rangle - 2|---\rangle),
\]

\[
|0, 1/2, 1/2\rangle = \frac{1}{\sqrt{2}}(|++-\rangle - |+++\rangle),
\]

\[
|0, 1/2, -1/2\rangle = \frac{1}{\sqrt{2}}(|++-\rangle - |+++\rangle).
\]

The \(J_{\pm}\) act on the collective basis by

\[
J_{\pm}|j_{1,2}, j, m\rangle = \sqrt{(j + 1) - m(m \pm 1)}|j_{1,2}, j, m \pm 1\rangle,
\]

which is independent of the quantum number \(j_{1,2}\).

To discuss the entanglement dynamics in the three-qubit system, we use the negativity proposed by Vidal and Werner to quantify the degree of entanglement [12]. The idea of this measure of the entanglement is the Peres-Horodecki criterion for the separability of bipartite systems [19]. The negativity was originally introduced to an arbitrary two-qubit state \(\rho\) and defined as [12, 20]

\[
N(\rho) = -2\sum_{i} \mu_i^-,
\]

where \(\mu_i^-\) is the negative eigenvalue of the partial transpose of \(\rho\) with respect to the \(i\)-th qubit, i.e. \(\rho^{T_i}\). Given a bipartite state one can calculate the partial transpose \(\rho^{T_i}\) of the density operator. The state is exactly separable if \(\rho^{T_i}\) is again a positive operator. However, if one of the eigenvalues of \(\rho^{T_i}\) is negative then the state is entangled [19]. In term of this viewpoint, the negativity is used to quantify the degree that \(\rho^{T_i}\) fails to
be positive and to represent the strength of quantum correlation between the two subsystems. It was proved that the relation between $C(\rho)$ and $N(\rho)$ is $N(\rho) \geq \sqrt{1 - C(\rho)^2 + C^2(\rho)} - |1 - C(\rho)|$ for any two-qubit state [20]. The merit of using the negativity to quantify the entanglement is that it allows us to investigate the entanglement properties between part $i$ and the sum of other components in the multipartite system.

Consider the initial state $\rho(0) = |+ + +\rangle\langle + + +|$, which is a product state. Expanding this initial state in the coupled basis of Eqs. (11) and using the quantum jump method, the time-dependent solution of the master equation can be obtained analytically. It reads

$$
\rho(t) = \frac{e^{-\gamma t}}{3} |1, \frac{3}{2}, \frac{1}{2}\rangle\langle 1, \frac{3}{2}, \frac{1}{2}| + \frac{e^{-\gamma t}}{6} |1, \frac{1}{2}, \frac{1}{2}\rangle\langle 1, \frac{1}{2}, \frac{1}{2}| + \frac{e^{-\gamma t}}{2} |0, \frac{1}{2}, \frac{1}{2}\rangle\langle 0, \frac{1}{2}, \frac{1}{2}| - \frac{e^{-i3\gamma t - \frac{5\gamma}{6}t}}{\sqrt{18}} |1, \frac{3}{2}, \frac{1}{2}\rangle\langle 1, \frac{1}{2}, \frac{1}{2}|
$$

\[
- \frac{e^{i3\gamma t - \frac{5\gamma}{6}t}}{\sqrt{18}} |1, \frac{1}{2}, \frac{1}{2}\rangle\langle 1, \frac{3}{2}, \frac{1}{2}| + \frac{e^{-i3\gamma t - \frac{1}{2}\gamma t}}{\sqrt{6}} |1, \frac{1}{2}, \frac{1}{2}\rangle\langle 0, \frac{1}{2}, \frac{1}{2}| + \frac{e^{i3\gamma t - \frac{5\gamma}{6}t}}{\sqrt{6}} |0, \frac{1}{2}, \frac{1}{2}\rangle\langle 1, \frac{1}{2}, \frac{1}{2}|
\]

\[
- \frac{e^{-\gamma t}}{\sqrt{12}} |1, \frac{1}{2}, \frac{1}{2}\rangle\langle 0, \frac{1}{2}, \frac{1}{2}| - \frac{e^{-\gamma t}}{2} |0, \frac{1}{2}, \frac{1}{2}\rangle\langle 0, \frac{1}{2}, \frac{1}{2}| + \frac{4(e^{-\gamma t} - 1)}{3e^{3\gamma t}} |1, \frac{3}{2}, \frac{1}{2}\rangle\langle 1, \frac{3}{2}, \frac{1}{2}|
\]

\[
+ \frac{1 - e^{-\gamma t}}{6} |1, \frac{1}{2}, \frac{1}{2}\rangle\langle 1, \frac{1}{2}, \frac{1}{2}| + \frac{1 - e^{-\gamma t}}{2} |0, \frac{1}{2}, \frac{1}{2}\rangle\langle 0, \frac{1}{2}, \frac{1}{2}| - \frac{2e^{-i3\gamma t - \frac{5\gamma}{6}t}(e^{\gamma t} - 1)}{\sqrt{18}} |1, \frac{3}{2}, \frac{1}{2}\rangle\langle 1, \frac{1}{2}, \frac{1}{2}|
\]

\[
- \frac{2e^{i3\gamma t - \frac{5\gamma}{6}t}(e^{\gamma t} - 1)}{\sqrt{18}} |1, \frac{1}{2}, \frac{1}{2}\rangle\langle 1, \frac{3}{2}, \frac{1}{2}| + \frac{2e^{i3\gamma t - \frac{5\gamma}{6}t}(e^{\gamma t} - 1)}{\sqrt{6}} |1, \frac{3}{2}, \frac{1}{2}\rangle\langle 0, \frac{1}{2}, \frac{1}{2}|
\]

\[
+ \frac{2e^{i3\gamma t - \frac{5\gamma}{6}t}(e^{\gamma t} - 1)}{\sqrt{6}} |0, \frac{3}{2}, \frac{1}{2}\rangle\langle 0, \frac{3}{2}, \frac{1}{2}| - \frac{1 - e^{-\gamma t}}{12} |1, \frac{1}{2}, \frac{1}{2}\rangle\langle 0, \frac{1}{2}, \frac{1}{2}| - \frac{1 - e^{-\gamma t}}{12} |0, \frac{1}{2}, \frac{1}{2}\rangle\langle 1, \frac{1}{2}, \frac{1}{2}|
\]

\[
+ \left(1 + e^{-4\gamma t} - \frac{4e^{-3\gamma t}}{3}\right) |1, \frac{3}{2}, \frac{3}{2}\rangle\langle 1, \frac{3}{2}, \frac{3}{2}|.
\]

(12)

In Fig. 3 we show the time evolution behavior of the negativities $N_A$ and $N_B$ corresponding to the partial transpose with respect to the first atom $A$ and second atom $B$, respectively. In the present case $N_A = N_C$, where $C$ denotes the third atom. From Fig. 3 we can see that the system shows the similar property of the entanglement production as the two-qubit system. In the absence of the interaction, the entanglement induced by the environment increases smoothly, then approaches to a constant value at large time scale for both $N_A$ and $N_B$. After the interaction is switched on, the oscillation begins to occur. Moreover, the oscillations have different frequencies. Different oscillation behaviors observed at short time scale ($t \lesssim 11$) and at long time scale ($t \gtrsim 11$) is due to that some components decay rapidly. From Eq. (12) we can derive that the time-dependent solution asymptotically tends to the steady state

$$
\rho_s = \frac{1}{3} |1, \frac{3}{2}, -\frac{3}{2}\rangle\langle 1, \frac{3}{2}, -\frac{3}{2}| + \frac{2}{3} |\psi_+\rangle\langle \psi_+|,
$$

\[
|\psi_+\rangle = \sqrt{\frac{2}{3}} |0, \frac{1}{2}, -\frac{1}{2}\rangle - \frac{1}{\sqrt{2}} |1, \frac{1}{2}, -\frac{1}{2}\rangle,
\]

(13)

where all of the extreme state $|j_{1,2}, j, -j\rangle$ are decoherence-free state [17]. Furthermore the two extreme states with $j = 1/2$ are entangled states. From Eqs. (13) it is obviously to see that the steady state is the probabilistic mixture and coherent superposition of the decoherence-free state. The reason for the existence of the coherent superposition is that both the coherent interactions $H_I$ and the dissipative operators $J^{\pm}_z$ don't distinguish the subspaces of $|j_{1,2} = 0, j = \frac{1}{2}, j_2\rangle$ and $|j_{1,2} = 1, j = \frac{1}{2}, j_2\rangle$, so they preserve their coherence.
during the time evolution as its initial value. Once again we verify our result that the decoherence mechanism in our cases considered is helpful for the entanglement production.

V. CONCLUSIONS

In conclusion, we have investigated the model of quantum register, which is composed of two or three atoms, coupled to an environment. Using the quantum jump method, the time-dependent solution of the master equation and then the entanglement dynamics of the system are studied analytically. The dual nature of the environment on the entanglement is showed. On the one hand, the environment destroys the entanglement induced coherently by the dipole-dipole interactions. On the other hand, the environment can induce incoherently the entanglement between the qubits. The entanglement dynamics of the two- or three-qubits studied in the present paper show clearly the constructive role of the environment in the entanglement production. This suggests that one can use the decoherence to construct environment-assisted quantum information processing.

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