Spurious detection of phase synchronization in coupled nonlinear oscillators

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Coupled nonlinear systems under certain conditions exhibit phase synchronization, which may change for different frequency bands or with presence of additive system noise. In both cases, Fourier filtering is traditionally used to preprocess data. We investigate to what extent the phase synchronization of two coupled Rössler oscillators depends on (1) the breadth of their power spectrum, (2) the width of the band-pass filter, and (3) the level of added noise. We find that for identical coupling strengths, oscillators with broader power spectra exhibit weaker synchronization. Further, we find that within a broad band width range, band-pass filtering reduces the effect of noise but can lead to a spurious increase in the degree of synchronization with narrowing band width, even when the coupling between the two oscillators remains the same.

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In recent years both theoretical and experimental studies of coupled nonlinear oscillators has demonstrated that such oscillators can exhibit phase synchronization [1–5]. Analysis of experimental data has also indicated the presence of phase synchronization in a range of coupled physical, biological and physiological systems [6–17]. In many of these studies, an important practical question is how multi-variate time series characterized by relatively broad power spectrum are phase synchronized in a specific frequency range [18–24]. The presence of internal or external noise may also be an obstacle when quantifying phase synchronization from experimental data [18, 19, 25–27]. In both cases a band-pass filter is traditionally applied either to reduce the noise effect or to extract the frequency range of interest. Thus, it is important to know to what extent the width of the band-pass filter influences the results of the phase synchronization analysis, as well as what is the range of the index values obtained from the analysis that indicate a statistically significant phase synchronization.

To address these questions, we consider a system of two coupled Rössler oscillators (1,2) defined as

\[
\begin{align*}
\dot{x}_{1,2} &= -\omega_1 y_{1,2} - z_{1,2} + C(x_{2,1} - x_{1,2}), \\
\dot{y}_{1,2} &= \omega_1 x_{1,2} + ay_{1,2}, \\
\dot{z}_{1,2} &= f + z_{1,2}(x_{1,2} - b) 
\end{align*}
\]

(1)

with parameter values $a = 0.165$, $f = 0.2$, and $b = 10$. For the mismatch of natural frequencies, we choose $\omega_{1,2} = \omega_0 \pm \Delta \omega$, with $\omega_0 = 0.6$ and $\Delta \omega = 0.005$ [Fig. 1(a)]. The time step in our simulation is $\Delta t = 2\pi/10^3$, and the signal length $n = \text{int}[t/\Delta t]$ with $t = 10^4$, where $\text{int}[x]$ denotes the integer part of $x$.

We first investigate the characteristics of the system defined in Eq. (1) by comparing them with the characteristics of a second set of two coupled Rössler oscillators (3,4) studied in [3]. The system (3,4) is also described by Eq.(1), and has the same values for parameters $a$, $f$, and $b$ as system (1,2). The only differences are the natural frequency $\omega_0 = 1$ and the frequency mismatch $\Delta \omega = 0.015$ [Fig. 1(b)]. We observe a significantly broader power spectrum for system (1,2) with $\omega_0 = 0.6$ and frequency mismatch $\Delta \omega = 0.005$ [Fig. 1(c)]. Further, we observe that the instantaneous phase differences $\Delta \psi_{1,1} = [\phi_{x_1}(t) - \phi_{x_2}(t)] \mod (2\pi)$ for system (1,2) exhibits larger fluctuations [Fig 1(d)], described by a broader distribution [Fig. 1(e)], compared to system (3,4), suggesting a weaker 1:1 phase synchronization for system (1,2). To quantify the degree of phase synchronization in the two Rössler systems we use the synchronization index $\rho = (S_{\text{max}} - S)/S_{\text{max}}$ [18], where $S = -\sum_{k=1}^{N} P_k \ln P_k$ is the Shannon entropy [28] of the distribution $P(\Delta \psi_{1,1} | \Delta \psi_{1,1})$, and $S_{\text{max}} = \ln N$, where $N = \text{int}[\exp(0.626 + 0.4 \ln(n - 1.0))]$ is the optimized number of bins over which the distribution is obtained [29]. For system (3,4) with a narrow power spectrum we obtain a significantly larger value of $\rho$ compared to the system (1,2) characterized by a broader power spectrum [Fig 1(f)]. Varying the values of the coupling strength $C$, we find that the phase synchronization index $\rho$ is consistently higher for system (3,4) characterized by the narrower power spectrum. Thus, for the same coupling strength $C$ and for identical other parameters, system (1,2) with $\omega_0 = 0.6$, which has a broader power spectrum, exhibits weaker synchronization compared to system (3,4) with $\omega_0 = 1$, which has a narrow power spectrum. These findings are complementary to a recent study indicating a different degree of phase synchronization for the spectral components of coupled chaotic oscillators [30].

Recent work has shown that coupled Rössler oscillators may exhibit different degrees of synchronization for different ranges of time scales obtained via wavelet transform [31]. Here, we ask to what extent the width of a band-pass filter affects the degree of phase synchronization between two coupled Rössler oscillators. While the output observables $x_1$ and $x_2$ of system (1,2) are clearly not in phase [Fig.2(a)], after Fourier band-pass filtering in the range of $\Delta f = 0.01$ centered at the peak of the power spectrum $2\pi f \approx 0.54$ [Fig. 1(c)], the observables $x_1$ and $x_2$ appear 1:1 synchronized with well aligned peaks [Fig. 2(b)]. The effect of the band-pass filter can be clearly seen in the behavior of the instantaneous phase difference $\Delta \psi_{1,1}$ [Fig. 2(c)] and in the shape of the probabil-
FIG. 1: Differences in the synchronization of two Rössler systems with identical coupling strengths and different power spectra. Phase plot trajectories of the variables $x$ vs. their Hilbert transform $x^H$ for: (a) system (1,2), with $x_1$ corresponding to $\omega_1 = \omega_0 + \Delta \omega$, where $\omega_0 = 0.6$ and $\Delta \omega = 0.005$; (b) system (3,4), with $x_3$ corresponding to $\omega_3 = \omega_0 - \Delta \omega$, where now $\omega_0 = 1$ and $\Delta \omega = 0.015$. For both Rössler systems $C = 0.03$. (c) Power spectra of the time sequence $x_1$ (dashed line) and $x_3$ (solid line). A broader spectrum is observed for system (1,2) compared to system (3,4). (d) Instantaneous phase difference $\psi_{1,1} \equiv (\phi_{1}(t) - \phi_{2}(t)) \mod (2\pi)$ for system (1,2) (dashed line), and $\psi_{3,1} \equiv (\phi_{3}(t) - \phi_{4}(t)) \mod (2\pi)$ for system (3,4) (solid line), and (e) their corresponding distributions $P(\Delta \psi_{1,1})$. System (1,2) exhibits larger fluctuations in $\Delta \psi_{1,1}$ and is characterized by a broader distribution $P(\Delta \psi_{1,1})$. (f) Synchronization index $\rho$ as a function of the coupling strength $C$. For identical values of $C$, system (3,4) (solid line) which is characterized by a narrower power spectrum exhibits stronger synchronization (larger index $\rho$) compared to system (1,2) with a broader power spectrum. Specifically, for identical coupling strength $C = C_0 = 0.03$, the index $\rho = \rho_0$ (C) for system (1,2), while $\rho = 0.3 > \rho_0$ (c) for system (3,4) although the frequency mismatch for system (3,4) is much larger. The effect of a Fourier band-pass filter applied to the system (1,2) while keeping $C = 0.03$ fixed is equivalent to an increase of the coupling strength of the system leading to a larger index $\rho_1 > \rho_0 (\Delta)$ as also shown in Fig. 2(e).
FIG. 2: Effects of band-pass filtering on synchronization. Time sequence of the variables $x_1$ and $x_2$ of system (1,2): (a) before and (b) after applying a band-pass Fourier filter with band width $\Delta f = 0.01$. After band-pass filtering the sequences $x_1$ and $x_2$ are better aligned in time (with almost matching peaks). (c) Instantaneous phase difference $\Delta \psi_{1,1}$, and (d) the distribution $P(\Delta \psi_{1,1})$ before (dashed line) and after (solid line) the Fourier band-pass filtering. After filtering, $\Delta \psi_{1,1}$ is characterized by less fluctuations and a much narrower distribution $P(\Delta \psi_{1,1})$, indicating a stronger synchronization, although the coupling strength $C = 0.03$ remains constant. (e) Dependence of the index $\rho$ on the band width $2\pi \Delta f$ for fixed $C = 0.03$. A filter with a relatively broader band width ($2\pi \Delta f > 1$) leaves the synchronization index $\rho$ practically unchanged, $\rho = \rho_0$, where $\rho_0$ characterizes the synchronization between $x_1$ and $x_2$ before filtering. Narrowing $\Delta f$ leads to a sharp increase in $\rho$, which is an artifact of the Fourier filtering as the coupling $C$ and all other parameters remain unchanged, e.g. for $\Delta f = 0.005$, $\rho = \rho_1 \approx 4\rho_0$.

FIG. 3: Effect of external additive white noise on phase synchronization for system (1,2). (a) Dependence of the synchronization index $\rho$ on the noise strength $\sigma_n$ for fixed value of the coupling constant $C$. (b) Dependence of the synchronization index $\rho$ on the coupling strength $C$ for different levels of white noise which are defined through the standard deviation $\sigma_n$.

For very strong noise ($\sigma_n = \sigma = 8.3$), the two Rössler oscillators in Eq.(1) appear not to be synchronized, characterized by low values for the index $\rho$, even for very large values of the coupling constant $C$ [Fig. 3(b)]. We note, that with increasing noise strength $\sigma_n$ the position of the crossover to the plateau of maximum synchronization shifts to smaller values of $C$ in Fig.3(b), indicating that with increasing $\sigma_n$ the level of the plateau drops faster compared to the decline in the growth of $\rho$ with increasing coupling $C$.

To reduce the effect of noise in data analysis, a common approach is to apply a band-pass filter. In the case of the coupled Rössler oscillators defined in Eq.(1), we ask to what extent a band-pass filter can reduce the effect of external noise while preserving the expected “true” phase synchronization as presented by $\rho_0$ in Fig. 1(e). To answer this question, we first need to determine what are the limits to which spurious phase synchronization can be obtained purely as a result of band-pass filtering of two uncorrelated and not coupled Gaussian noise signals. Our results for the synchronization index $\rho$ obtained from multiple realizations of pairs of uncoupled white noise signals show that the synchronization index $\rho$ can reach different maximum values $\rho_{\text{max}}$, indicated by arrows in Fig. 4(a), for different band width $\Delta f$ — with decreasing the band width $\rho_{\text{max}}$ increases. The values of $\rho_{\text{max}}$ provide an estimate of the maximum possible effect additive noise may have on the spurious “detection” of phase synchronization in coupled oscillators. Thus, empirical observations of synchronization index $\rho > \rho_{\text{max}}$ may indicate presence
of a genuine phase synchronization between the outputs of two coupled oscillators, which is not an artifact of external noise. Our simulations show that the value of $\rho_{\text{max}}$ does not change significantly with the length of the uncorrelated noise signals. In Fig. 4(b) we show how the synchronization index $\rho$ for system (1,2) depends on the strength of the added noise and on the width $\Delta f$ of the band-pass filter. For very broad band width $\Delta f$ the noise is not sufficiently filtered, and the synchronization between the two oscillators decreases ($\rho$ decreases) with increasing noise strength $\sigma_n$. With decreasing band width $\Delta f$, i.e., applying a stronger filter, the effect of the noise is reduced, and correspondingly the index $\rho$ increases — approaching the value $\rho_0$ expected for the system (1,2) without noise. On the other hand, applying a filter with too narrow band width $\Delta f$ leads to a spurious synchronization effects with $\rho > \rho_0$ [Fig.4(b)], following closely the dependence of $\rho$ on $\Delta f$ shown in Fig. 2(e) for a Rössler system without noise.

In summary, our results indicate that phase synchronization between coupled nonlinear oscillators may strongly depend on the width of the power spectrum of these oscillators. Further, we find that while external noise can affect the degree of phase synchronization, band-pass filtering can reduce noise effects but can also lead to a spurious overestimation of the actual degree of phase synchronization in the system. This is of importance when analyzing empirical data in specific narrow frequency ranges, for which the coupling strength may not be known a-priori.

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