Abstract
A charged particle beam travelling across perfectly conducting structures whose boundaries do not have constant cross section, such as an RF cavity or bellows, generates longitudinal and transverse wake fields. We discuss in this lecture the general features of wake fields, and show a few simple examples in cylindrical geometry: perfectly conducting pipe and the resonant modes of an RF cavity. We then study the effect of wake fields on the dynamics of a beam in a linac, such as beam break-up instabilities and how to cure them.

1 INTRODUCTION
Parasitic forces, also called wake fields, are generated by a charged particle beam interacting with the vacuum chamber components. These components may have a complex geometry: kickers, bellows, RF cavity, diagnostics components, special devices, etc. To solve Maxwell’s equations in a given structure with the beam current as the source of fields, a study of the field is required. For this complicated task, dedicated computer codes were developed to solve the electromagnetic problem in the frequency or in the time domain. There are several useful codes for the design of accelerator devices, like MAFIA, ABCI, URME, etc., as reported in Ref. [1].

In this lecture we discuss the general features of the parasitic fields [2-10], and then show a few simple examples of them in cylindrical geometry: a perfectly conducting pipe and the resonant modes of an RF cavity. Although the space charge forces have been studied separately [11], they can be seen as a particular case of wake fields, see Appendix B for a simple example [12].

We then study the effect of the wake fields on the dynamics of a beam in a linac such as beam break-up instabilities, assuming a high-energy beam where the motion is ‘frozen’ in longitudinal space, and the way to cure it [13].

2 WAKE FIELDS AND POTENTIALS
2.1 Wake potentials
Parasitic fields depend on the particular charge distribution of the beam. It is therefore desirable to know what is the effect of a single charge (i.e. find the Green function) in order to reconstruct the fields produced by any charge distribution.

The electromagnetic fields created by a point charge act back on the charge itself and on any other charge of the beam. We therefore focus our attention on the source charge \( q_0 \), and on the test charge \( q \), assuming that both are moving with the same constant velocity \( v = \beta c \) on trajectories parallel to the axis.

Let \( E \) and \( B \) be the fields generated by \( q_0 \) inside the structure, \((s_0, r_0)\) be the position of the source charge, and \((s = s_0 + z, r)\) be the position of the test charge \( q \).

Since the velocity of both charges is along \( z \), the Lorentz force has the following components:
Thus, there can be two effects on the test charge: a longitudinal force which changes its energy, and a transverse force which deflects its trajectory. If we consider a device of length $L$, the energy gain [J] is:

$$U = \int_0^L F_\parallel ds$$

and the transverse deflecting kick [Nm] is:

$$M = \int_0^L F_\perp ds .$$

Note that the integration is performed over a given path of the trajectory. These quantities, normalized to the charges, are called wake potentials (volt/coulomb) and are both functions of the distance $z$:

Longitudinal wake potential [V/C]:

$$w_\parallel = -\frac{U}{q_0 q} .$$

Transverse wake potential [V/Cm]:

$$w_\perp = \frac{1}{r_0 q_0 q} M .$$

The minus sign in the longitudinal wake-potential means that the test charge loses energy when the wake is positive. Positive transverse wake means that the transverse force is defocusing.

As a first example, consider the longitudinal wake-potential of 'space charge'. The longitudinal force inside a relativistic cylinder of radius $a$ travelling inside a cylindrical pipe of radius $b$ is given by [11]:

$$F_\parallel(r,z) = -\frac{q}{4\pi\varepsilon_0 c^2}\left(1 - \frac{r^2}{a^2} + 2 \ln \frac{b}{a}\right)\frac{\partial}{\partial z} \lambda(z) .$$

Note that since the space charge forces move together with the beam, and the electric field is constant vs. $z$, we can derive the wake potential per unit length (volt/coulomb metre). To get the wake potential of a piece of pipe, we just multiply by the pipe length. Assuming $r \rightarrow 0$ and a charge line density given by $\lambda(z) = q_0 \delta(z)$ we obtain:

$$\frac{dw_\parallel(z)}{ds} = \frac{1}{4\pi\varepsilon_0 c^2}\left(1 + 2 \ln \frac{b}{a}\right)\frac{\partial}{\partial z} \delta(z) .$$

Another interesting case is the longitudinal wake-potential of a resonant higher order mode (HOM) in an RF cavity. When a charge crosses a resonant structure, it excites the fundamental and higher order modes. Each mode can be treated as an electric RLC circuit loaded by an impulsive current, as shown in Fig. 1.

Just after the charge passage, the capacitor is charged with a voltage \( V_0 = Cq_0 \) and the electric field is \( E_{00} = V_0/l_0 \). The time evolution of the electric field is governed by the same differential equation of the voltage:

\[
\ddot{V} + \frac{1}{RC} \dot{V} + \frac{1}{LC} V = \frac{1}{C} \dot{I}.
\] (8)

The passage of the impulsive current charges only the capacitor, which changes its potential by an amount \( V_c(0) \). This potential will oscillate and decay producing a current flow in the resistor and inductance. For \( t > 0 \) the potential satisfies the following equation and boundary conditions:

\[
\ddot{V} + \frac{1}{RC} \dot{V} + \frac{1}{LC} V = 0
\]
\[
V(t = 0^-) = \frac{q}{C} \equiv V_0
\]
\[
V(t = 0^+) = \frac{\dot{q}}{C} = \frac{I(0^+)}{C} = \frac{V_0}{RC}
\] (9)

which has the following solution:

\[
V(t) = V_0 e^{-\frac{t}{\tau}} \left[ \cos(\omega t) - \frac{\Gamma}{\omega} \sin(\omega t) \right]
\] (10)

where \( \omega^2 = 1/\tau^2 \) and \( \Gamma = 1/2RC \).

Putting \( z = ct \) (\( z \) is positive behind the charge) we obtain (see Fig 2.):

\[
w(z) = \frac{-V(z)}{q_0} = \frac{V_0}{q_0} e^{-\frac{z}{\tau c}} \left[ \cos(\omega z/c) - \frac{\Gamma}{\omega} \sin(\omega z/c) \right].
\] (11)
2.2 Loss factor

It is also useful to define the loss factor as the normalized energy lost by the source charge $q_0$:

$$k = -\frac{U(z = 0)}{q_0^2}.$$  \hspace{1cm} (12)

Although in general the loss factor is given by the longitudinal wake at $z = 0$, for charges travelling with the velocity of light, the longitudinal wake potential is discontinuous at $z = 0$, see Fig. 3.

2.3 Beam loading theorem

When the source charge travels with the velocity of light $v = c$, the electromagnetic fields are left behind and called ‘wake fields’. Any electromagnetic perturbation produced by the charge cannot overtake the charge itself. This means that the longitudinal wake-potential vanishes in the region $z < 0$. This property is a consequence of the ‘causality principle’. Causality requires that the longitudinal wake-potential of a charge travelling with the velocity of light is discontinuous at the origin.

The beam loading theorem states that:

$$k = \frac{w_0(z \to 0)}{2}.$$  \hspace{1cm} (13)

For example, the beam loading theorem is fulfilled by the wake potential of the resonant mode. In fact the energy lost by the charge $q_0$ loading the capacitor is:

$$U = \frac{CV_0^2}{2} = \frac{q_0^2}{2C}$$
2.4 Relationship between transverse and longitudinal forces

Another important feature worth mentioning is the differential relationship existing between longitudinal and transverse forces:

\[ \nabla \cdot \mathbf{F} = \frac{\partial}{\partial z} F_\perp \]
\[ \nabla \times \mathbf{w} = \frac{\partial}{\partial z} w_\perp \]  

(14)

The above relations are known as the ‘Panofsky–Wenzel theorem’ [15].

2.5 Coupling impedance

The wake potentials are used to study the beam dynamics in the time domain \((s = vt)\). If we take the equation of motion in the frequency domain, we need the Fourier transform of the wake potentials. Since these quantities are in ohms they are called coupling impedances:

**Longitudinal impedance** [\(\Omega\)]: \(Z_{\|}(\omega) = \frac{1}{\nu} \int_{-\infty}^{\infty} w_\| (z) e^{-i\omega \tau} \, dz \) .  

(15)

**Transverse impedance** [\(\Omega/m\)]: \(Z_{\perp}(\omega) = \frac{i}{\nu} \int_{-\infty}^{\infty} w_\perp (z) e^{-i\omega \tau} \, dz \) .  

(16)

The coupling impedances (\(\Omega/m\)) of the ‘space charge’ wake is:

\[ \frac{\partial Z_{\|}(\omega)}{\partial s} = \frac{1}{\nu} \int_{-\infty}^{\infty} w_\| (z) e^{-i\omega \tau} \, dz = \frac{1 + 2 \ln (b/a)}{v4\pi \epsilon_0 c^2} \int_{-\infty}^{\infty} \delta(z)e^{-\frac{z}{c^2}} \, dz \]

(17)

where: \(\int_{-\infty}^{\infty} \delta(z)f(z)dz = f'(0)\), so that:

\[ \frac{\partial Z_{\|}(\omega)}{\partial s} = -\frac{i\omega Z_0}{4\pi c^2 \beta^2} \left(1 + 2 \ln \frac{b}{a}\right) \]  

(18)

The longitudinal coupling impedance of a resonant HOM is given by:

\[ Z_{\|}(\omega) = \frac{R_s}{1 - iQ_r \left(\frac{\omega_r - \omega}{\omega - \omega_r}\right)} \]

(19)

where \(R_s = w_\| / 2\Gamma\) is the shunt impedance and \(Q_r = \omega_r / 2\Gamma\) is the quality factor, quantities that we can obtain with the computer codes. Note that the loss factor is also determined: \(k = \omega R_s / 2Q_r\).

The transverse impedance is given by:
\[ Z_\perp(\omega) = \frac{c}{\omega^2 - iQ_r \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)} \], \quad (20) 

with \( R_{r\perp} = R_i / b^2 \) being the transverse shunt impedance.

### 2.6 Wake potential and energy loss of a bunched distribution

When we have a bunch with density \( \lambda(z) \), we may wonder what is the amount of energy lost or gained by a single charge \( e \) in the beam. To this end we calculate the effect on the charge from the whole bunch by means of the convolution integral, see Fig. 4:

\[ U(z) = -e \int_{-\infty}^{\infty} w_{||}(z' - z) \lambda(z') dz' . \quad (21) \]

\[ \text{Fig. 4: Convolution integral} \]

This permits the definition of the wake potential of a distribution:

\[ W_{||}(z) = -\frac{U(z)}{qq_0} . \quad (22) \]

The total energy lost by the bunch is computed summing up the loss of all particles:

\[ U_{\text{bunch}} = -\frac{1}{e} \int_{-\infty}^{\infty} U(z') \lambda(z') dz' . \quad (23) \]

### 3 Wake-field Effects in Linear Accelerators

#### 3.1 Energy spread

The longitudinal wake forces change the energy of individual particles depending on their position in the beam. As a consequence, the wake can induce an energy spread in the beam.

For example, the energy spread induced by the space charge force in a Gaussian bunch is given by:

\[ \frac{dU(z)}{ds} = -e \int_{-\infty}^{\infty} \frac{dw_{||}(z' - z)}{ds} \lambda(z') dz' = -\frac{eq}{4\pi\epsilon_0 \gamma^2 \sqrt{2}\pi \sigma_z} \left( 1 + 2 \ln \frac{b}{a} \right) z e^{-z^2 / 2\sigma_z^2} . \quad (24) \]

The bunch head gains energy, while the tail loses energy.
In a similar way one can show that the energy loss induced by a resonant HOM on the charges inside a rectangular, uniform bunch is given by:

\[
U(z) = -eqw_z \sin \left( \frac{\omega_l}{c} \left( \frac{l_0}{2} + z \right) \right).
\]

(25)

### 3.2 Beam break-up

A beam injected off-centre in a linac, because, for example, of focusing quadrupole misalignment, executes betatron oscillations. The displacement produces a transverse wake field in all the devices crossed during the flight, which deflects the trailing charges, see Fig. 5.

![Fig. 5: Single-bunch beam break up originated by an injection error](image)

In order to understand the effect, consider a simple model with only two charges \(q_1=Ne/2\) (leading = half bunch) and \(q_2=e\) (trailing = single charge). The leading charge executes free betatron oscillations:

\[
y_1(s) = \hat{y}_1 \cos \left( \frac{\omega_l}{c} s \right).
\]

(26)

The trailing charge, at a distance \(z\) behind, over a length \(L_w\) experiences a deflecting force proportional to the displacement \(y_1\), and dependent on the distance \(z\):

\[
\langle F_{y_{\text{self}}}(z,y_1) \rangle = \frac{Ne^2 w_z(z)}{2L_w} y_1(s).
\]

(27)

Notice that \(L_w\) is the length of the device for which the transverse wake has been computed. For example, in the case of a cavity cell \(L_w\) is the length of the cell. This force drives the motion of the trailing charge:

\[
y_2 + \left( \frac{\omega_s}{c} \right)^2 y_2 = \frac{Ne^2 w_z(z)}{2E_0 L_w} \hat{y}_1 \cos \left( \frac{\omega_s}{c} s \right).
\]

(28)

This is the typical equation of a resonator driven at the resonant frequency.

The solution is given by the superposition of the ‘free’ oscillation and a ‘forced’ oscillation which, being driven at the resonant frequency, grows linearly with \(s\), as shown in Fig. 6.

\[
y_2(s) = \hat{y}_2 \cos \left( \frac{\omega_s}{c} s \right) + y_2^{\text{forced}}
\]

(29)
\[ y_2^\text{forced} = \frac{cNe_w(z)w(z)}{4\omega_y E_0 L_w} \hat{y}_2 \sin \left( \frac{\omega_y}{c} s \right). \tag{30} \]

Fig. 6: HOMDYN [16] simulation of a typical BBU instability, 50 µm initial offset, no energy spread

At the end of the linac of length \( L \), the oscillation amplitude is grown by \( \hat{y}_1 = \hat{y}_2 \):

\[
\begin{pmatrix}
\Delta \hat{y}_2 \\
\hat{y}_2
\end{pmatrix}
= \frac{cNe_w(z) L}{4\omega_y (E_0/e) L_w}.
\tag{31}
\]

If the transverse wake is given per cell, the relative displacement of the tail with respect to the head of the bunch depends on the number of cells. It depends, of course, also on the focusing strength through the frequency \( \omega \).

To extend the analysis to a particle distribution, we write the transverse equation of motion of a single particle with the inclusion of the transverse wake field effects as [7]:

\[
\frac{d}{ds} \left[ \gamma(s) \frac{dy(z,s)}{ds} \right] + k_y^2(s) \gamma(s) y(z,s) = -\frac{e^2 N_p}{m_0 c^2 L_w} \int_{s}^{\infty} y(z',s') \omega(z' - z) d z'.
\tag{32}
\]

where \( \gamma(s) \) is the relativistic parameter, \( k_y(s) \) the beta function, \( N_p \) the number of particles of the bunch, and \( \lambda(z) \) the longitudinal bunch distribution.

The solution of the equation in the general case is unknown. We can, however, apply a perturbation method to obtain the solution at any order in the wake-field intensity. Indeed we write:

\[ y(z,s) = \sum_n y^{(n)}(z,s). \tag{33} \]

The first-order solution is found from the equation

\[
\frac{d}{ds} \left[ \gamma(s) \frac{dy^{(0)}(z,s)}{ds} \right] + k_y^2(s) \gamma(s) y^{(0)}(z,s) = 0.
\tag{34} \]
It is important to note that the above equation does not depend on \( z \) any more. This means that the bunch distribution remains constant along the structure.

If the \( s \)-dependence of \( \gamma(s) \) and \( k^2_y(s)\gamma(s) \) is moderate, we can use the WKB approximation, and the solution of the above equation with the starting conditions \( y(0) = y_m, y'(0) = 0 \) is [2]

\[
y^{(0)}(s) = \left( \frac{\gamma_0 k_y}{\gamma(s) k_y(s)} \right)^{1/2} y_m \cos[\psi(s)] \quad (35)
\]

where

\[
\psi(s) = \int_0^s k_y(s') ds'
\quad (36)
\]

represents the unperturbed transverse motion of the bunch.

The second-order differential equation is obtained by substituting the first-order solution in the right-hand side of Eq. (31) giving

\[
\frac{\partial}{\partial s} \left[ \gamma(s) \frac{d y^{(1)}(z,s)}{ds} \right] + k^2_y(s)\gamma(s)y^{(1)}(z,s) = -\frac{e^2 N_p}{m_e c^2 L_w} y^{(0)}(s) \int_z^\infty w_\perp(z'-z)\lambda(z')dz'. \quad (37)
\]

The forced solution of the above equation can be written in the form

\[
y^{(1)}(z,s) = -y_m \frac{e^2 N_p}{m_e c^2 L_w} \left( \frac{\gamma_0 k_y}{\gamma(s) k_y(s)} \right)^{1/2} G(s) \int_z^\infty w_\perp(z'-z)\lambda(z')dz' \quad (38)
\]

where

\[
G(s) = \int_0^s \frac{1}{\gamma(s) k_y(s')} \sin[\psi(s')-\psi(s')] \cos[\psi(s')] ds' = \frac{1}{2} \int_0^s \frac{\sin[\psi(s')-2\psi(s')]}{\gamma(s) k_y(s')} ds' + \frac{1}{2} \int_0^s \frac{1}{\gamma(s) k_y(s')} ds'. \quad (39)
\]

The first integral undergoes several oscillations with \( s \) and, if \( \gamma(s) \) and \( k_y(s) \) do not vary much, we can write

\[
y^{(1)}(z,s) = -y_m \frac{e^2 N_p}{2m_e c^2 L_w} \left( \frac{\gamma_0 k_y}{\gamma(s) k_y(s)} \right)^{1/2} \sin[\psi(s)] \int_0^s \frac{ds'}{\gamma(s) k_y(s')} \int_z^\infty w_\perp(z'-z)\lambda(z')dz'. \quad (40)
\]

The last integral represents the transverse wake-potential produced by the whole bunch. This solution can then be substituted again in the right-hand side of Eq. (32) to obtain a second-order equation and so on. If we consider constant \( \gamma(s) \) and \( k_y(s) \), we obtain the same result of the two-particle model when we substitute \( \lambda(z) \) with 1/2 in the particle positions.
For example, by using the wake of LEP superconducting cavities, it is possible to find that for a Gaussian bunch, the wake potential is as given in Fig.7.

![Fig.7: Transverse wake potential for LEP superconducting cavities](image)

For a conservative estimation of the BBU effect, one should use the maximum value of the curve in this case, thus eliminating the $z$ dependence.

If the BBU effect is strong, it is necessary to include higher order terms in the perturbation expansion. Assuming:

- rectangular bunch distribution, $\lambda(z)=1/l_0$, $-l_0/2 < z < l_0/2$, $l_0$ bunch length,
- monoenergetic beam,
- constant acceleration gradient $dE_0/ds = \text{const}$,
- constant beta function,
- linear wake function inside the bunch $w_\perp(z)=w_{\perp,0}z/l_0$,

the sum of Eq. (33) can be written in terms of powers of the dimensionless parameter $\eta$ also called BBU strength

$$
\eta = \frac{e^2 N_p}{k_y (dE_0/ds)} \frac{w_{\perp,0}}{L_w} \ln \left( \frac{\gamma_f}{\gamma_i} \right)
$$

(41)

with $\gamma_i$ and $\gamma_f$ respectively the initial and final relativistic parameter.

By using the method of stepping descent [5], it is possible to obtain the asymptotic expression of $\gamma(z, s)$ finding at the end of the linac,

$$
\gamma(L_m) = \gamma_m \sqrt{\frac{\gamma_f}{6\pi\gamma_i}} \eta^{-1/6} \exp \left[ \frac{3\sqrt{3}}{4} \eta^{1/3} \right] \cos \left[ k_y L_m - \frac{3}{4} \eta^{1/3} + \frac{\pi}{12} \right],
$$

(42)

the two-particle model is different from the first-order solution, and gives a tail displacement growing exponentially with $\eta$, resulting in better agreement with the simulation in Fig. 6.

### 3.3 BNS damping

The BBU instability is quite harmful and hard to get under control even at high energy with a strong focusing, and after careful injection and steering. A simple method to cure it has been proposed on the basis that the strong oscillation amplitude of the bunch tail is mainly due to ‘resonant’ driving. If the
tail and the head move with a different frequency, this effect can be significantly reduced [13], compare Fig. 8 with Fig. 6.

![Graph](image)

**Fig. 8:** HOMDYN simulation of a typical BNS damping, 50 μm initial offset, 2% energy spread

Let us assume that the tail oscillates with a frequency $\omega_y + \Delta \omega_y$, the two particle model equation of motion reads:

$$y_2'' + \left( \frac{\omega_y + \Delta \omega_y}{c} \right)^2 y_2 = \frac{N e^2 w_\perp(z)}{2 \beta^2 E_0 L_w} \hat{y}_1 \cos \left( \frac{\omega_y}{c} s \right)$$  \hspace{1cm} (43)

the solution of which is

$$y_2(s) = \frac{c^2 N e^2 w_\perp(z)}{4 \omega_y \Delta \omega_y E_0 L_w} \hat{y}_1 \left[ \cos \left( \frac{\omega_y + \Delta \omega_y}{c} s \right) - \cos \left( \frac{\omega_y}{c} s \right) \right]$$  \hspace{1cm} (44)

where the amplitude of the oscillation is limited.

Furthermore, by a suitable choice of $\Delta \omega_y$, it is possible to fully depress the oscillations of the tail. Setting

$$\hat{y}_2 \cos \left( \frac{\omega_y + \Delta \omega_y}{c} s \right) + y_2(s) = \hat{y}_1 \cos \left( \frac{\omega_y}{c} s \right)$$  \hspace{1cm} (45)

we get

$$\Delta \omega_y = \frac{c^2 N e^2 w_\perp(z)}{4 \omega_y E_0 L_w}.$$  \hspace{1cm} (46)

The extra focusing at the tail can be obtained by using a RFQ, where head and tail see a different focusing strength; exploiting the energy spread across the bunch which, because of the chromaticity, induces a spread in the betatron frequency. An energy spread correlated with the position is attainable with the external accelerating voltage, or with the wake fields.
4 LANDAU DAMPING

There is a fortunate stabilizing effect against collective instabilities called ‘Landau Damping’. The basic mechanism relies on the fact that if the particles in the beam have a spread in their natural frequencies (synchrotron or betatron), their motion cannot be coherent for a long time.

4.1 Driven oscillators

In order to understand the physical nature of this effect, we consider a simple harmonic oscillator, at rest for \( t < 0 \), driven by an oscillatory force for \( t > 0 \).

\[
\frac{d^2 x}{dt^2} + \omega^2 x = A \cos(\Omega t) .
\] (47)

The general solution is given by the superposition of the free and forced solutions, see Appendix 1:

\[
x(t) = \frac{A}{\omega^2 - \Omega^2} [\cos(\Omega t) - \cos(\omega t)] .
\] (48)

Let us assume that the external force is driving a particle population characterized by a spread of natural frequency of oscillation around a mean value \( \omega_x \). Furthermore, leave the forcing frequency \( \Omega \) inside the spectrum so that \( \delta \equiv \Omega - \omega \ll \omega_x \).

The motion of a given particle in the bunch can be approximated by:

\[
x(t) = \frac{At}{2\omega_x} \sin\left(\frac{\delta t}{2}\right) .
\] (49)

Let us observe two particles in the bunch, one with \( \delta = 0 \), and the other with \( \delta \neq 0 \). Both are at rest, and at \( t = 0 \) they start to oscillate with the same amplitude and phase (coherency). However, while the amplitude of the former charge grows indefinitely (driven at resonance), the latter reaches a maximum amplitude (beating of two close frequencies). The system of the particle has lost coherency at the time when the beating amplitude is maximum, i.e. for \( t = \pi/\delta \).

At any time \( t' \), only those oscillators inside the bandwidth \( |\delta| < \pi/\delta \), oscillate coherently. The longer we wait, the narrower the coherent bandwidth and therefore the smaller the number of ‘coherent’ particles.

4.2 Amplitude of oscillations

At any instant we can divide the bunch population into two groups: coherent particles, oscillating all together with an amplitude growing linearly with time; and ‘incoherent’ particles, with different phases, and a saturated amplitude of oscillation.

It is interesting that, although the amplitude of the coherent oscillators grows linearly with time, the average amplitude of the whole system remains bounded, as the number of coherent particles decreases inversely with time

\[
\langle x(t) \rangle_{\text{max}} = \frac{1}{N} \left[ \sum_{\text{coh}} x(t) + \sum_{\text{incoh}} x(t) \right]_{\text{max}} .
\] (50)
Consider the time when the coherent particles have the maximum amplitude. The amplitudes of the incoherent particles, being uncorrelated, have a zero average. For the coherent particles we have

\[ \langle x(t) \rangle_{\text{max}} = \frac{N_{\text{coh}}}{N} x(t)_{\text{max}} = \frac{N_{\text{coh}}}{N} A \frac{t}{2 \omega_i} . \quad (51) \]

On the other hand, the number of oscillators keeping coherency decreases with time

\[ N_{\text{coh}} = \frac{N}{\Delta \omega} \pi \frac{t}{t} \Rightarrow \langle x(t) \rangle_{\text{max}} = \frac{\pi}{\Delta \omega} \frac{A}{2 \omega_i} . \quad (52) \]

4.3 Energy of the system

What happens to the energy of the system? In this case we distinguish the coherent and incoherent particles. The energy of the coherent particles has quadratic growth with time, while the energy of the incoherent particles is bounded. In this case, although the number of coherent oscillators decreases with time, the total energy still grows linearly

\[ E(t) = E_{\text{coh}}(t) + E_{\text{incoh}}(t) \]

\[ E_{\text{coh}}(t) = N_{\text{coh}} \left[ \frac{1}{2} k x_{\text{coh}}^2(t) \right] = \frac{\pi}{2} \frac{NA^2}{\omega_i^2} \left( \frac{1}{4\Delta \omega} \right)^2 t . \quad (54) \]

To conclude, when a force drives this kind of system, initially the whole system follows the external force. Thereafter, fewer and fewer particles are driven at the resonance. The result is that although the system absorbs energy, the average amplitude remains bounded.

This mechanism also works when the driving force is produced by the bunch itself. To make the coherent instability start, the rise time of the instability has to be shorter than the ‘de-coherency’ time of the bunch: \( \tau_{\text{inst}} < \tau_{\text{decoh}} = 2\pi / \Delta \omega \).
APPENDIX A – DRIVEN OSCILLATORS

Consider an harmonic oscillator with natural frequency \( \omega \), with an external excitation at frequency \( \Omega \)

\[
\ddot{x} + \omega^2 x = A \cos(\Omega t)
\]

The general solution is

\[
x(t) = x_{\text{free}}(t) + x_{\text{driven}}(t)
\]

\[
\cos(\Omega t) \Rightarrow e^{i\Omega t}
\]

\[
x_{\text{free}}(t) = \tilde{x}_m e^{i\omega t}
\]

\[
x_{\text{driven}}(t) = \tilde{x}_m^d e^{i\Omega t}
\]

The driven solution (steady state) is found by direct substitution in the differential equation

\[
(\omega^2 - \Omega^2) \tilde{x}_m^d e^{i\Omega t} = Ae^{i\Omega t} \Rightarrow \tilde{x}_{\text{driven}}(t) = \frac{A}{(\omega^2 - \Omega^2)} e^{i\Omega t}.
\]

The general solution has to satisfy the initial condition at \( t = 0 \). In our case we assume that the oscillator is at rest for \( t = 0 \)

\[
x_{\text{free}}(t = 0) = -x_{\text{driven}}(t = 0)
\]

\[
\tilde{x}_m = -\frac{A}{\omega^2 - \Omega^2}
\]

thus we get

\[
x(t) = \frac{A}{\omega^2 - \Omega^2} \left[ e^{i\Omega t} - e^{i\omega t} \right].
\]

taking only the real part

\[
x(t) = \frac{A}{\omega^2 - \Omega^2} \left[ \cos(\Omega t) - \cos(\omega t) \right].
\]

This expression is suitable for deriving the response of the oscillator driven at resonance or at very close frequency: \( \omega = \Omega + \delta \), with \( \delta \to 0 \). Defining: \( \bar{\omega} = (\omega + \Omega)/2 \), equivalent to \( \omega = \bar{\omega} + \delta/2 \) or \( \Omega = \bar{\omega} - \delta/2 \) the solution is given by

\[
x(t) = \frac{A}{2\bar{\omega}\delta} \left\{ \cos(\bar{\omega} t)\cos(\delta t/2) + \sin(\bar{\omega} t)\sin(\delta t/2) \right\} +
\]

\[
+ \left[ \cos(\bar{\omega} t)\cos(\delta t/2) + \sin(\bar{\omega} t)\sin(\delta t/2) \right]
\]

that is

\[
x(t) = \frac{A}{\bar{\omega}} \sin(\bar{\omega} t)\sin(\delta t/2) \equiv \frac{A t}{\bar{\omega}} \sin(\bar{\omega} t) \frac{\sin(\delta t/2)}{\delta t/2}
\]

with the limit

\[
\lim_{\delta \to 0} x(t) = \frac{A t}{2\bar{\omega}} \sin(\bar{\omega} t).
\]
APPENDIX B – POWER RADIATED BY A BUNCH PASSING THROUGH A TAPER

In the case of a charge distribution, and \(\gamma \to \infty\), the electric field lines are perpendicular to the direction of motion and travel with the charge [6], as shown in Fig. B.1. In other words, the field-map does not change during the charge-flight, as long as the trajectory is parallel to the pipe axis. Under this condition the transverse field intensity can be computed in the static case, applying the Gauss and Ampere laws:

\[
\int_S E \cdot n dS = \int_V \rho dV, \quad \oint_S B \cdot dl = \mu_0 \int_S J \cdot n dS \tag{B.1}
\]

Let us consider a cylindrical beam of radius \(a\) of current \(I\), with uniform charge density \(\rho = I / \pi a^2 v\) and current density \(J = I / \pi a^2\), propagating with relativistic speed \(v = \beta c\) along the \(z\) axis of a cylindrical, perfectly conducting pipe of radius \(b\), as shown in Fig. B.1.

**Fig. B.1:** Cylindrical bunch of radius \(a\) propagating inside a cylindrical, perfectly conducting pipe of radius \(b\)

By applying the relations (B.1) one can obtain for the radial component of the electric field

\[
E_r = \frac{I}{2\pi \varepsilon_0 a^2 v} r \quad \text{for} \quad r \leq a
\]

\[
E_r = \frac{I}{2\pi \varepsilon_0 v} \quad \text{for} \quad r > a
\]

and the relation \(B_\phi = \frac{\beta}{c} E_r\) holds.

The electrostatic potential satisfying the boundary condition \(\varphi(b) = 0\) is given by:

\[
\varphi(r,z) = \int_r^b E_r(r',z) dr' = \begin{cases} 
\frac{I}{4\pi \varepsilon_0 v} \left(1 + 2 \ln \frac{b}{a} - \frac{r^2}{a^2}\right) & \text{for} \quad r \leq a \\
\frac{I}{2\pi \varepsilon_0 v} \ln \frac{b}{r} & \text{for} \quad a \leq r \leq b
\end{cases}
\]

How can a perturbation of the boundary conditions affect the beam dynamics? Consider the following example: a smooth transition of length \(L\) (taper) from a beam pipe of radius \(b\) to a larger beam pipe of radius \(d\) is experienced by the beam [6]. To satisfy the boundary condition of a perfectly conducting pipe in the tapered region, the field lines are bent as shown in Fig. B.2. Therefore there must be a longitudinal \(E_z(r,z)\) field component in the transition region.

A test particle running outside the beam charge distribution along the transition of length \(L\) will experience a voltage given by [12]:

\[\text{Voltage} = \int L E_z(r,z) dr\]
\[ V = -\int_z^{z+L} E_z(r,z')dz' = -[\varphi(r,z+L) - \varphi(r,z)] = -\frac{I}{2\pi\varepsilon_0 v} \ln \frac{d}{b} \]

that is decelerating if \( d > b \). The power lost by the beam in order to sustain the induced voltage is given by

\[ P_{\text{lost}} = VI = \frac{I^2}{2\pi\varepsilon_0 v} \ln \frac{d}{b}. \quad (B.2) \]

**Fig. B.2:** Smooth transition of length \( L \) (taper) from a beam pipe of radius \( b \) to a larger beam pipe of radius \( d \)

It means that for \( d > b \) the power is deposited to the energy of the fields: moving from left to right of the transition, the beam induces the fields in the additional room around the bunch bunch (i.e. in the region \( b < r < d \), \( 0 < z < l_0 \)) at the expense of the only available energy source, that is the kinetic energy of the beam itself.

**Fig. B.3:** During the beam propagation in the taper additional electromagnetic power flow is required to fill up the new available room.

To verify this interpretation, compute the electromagnetic power radiated by the beam to fill up the additional room available, see Fig. B.3. Integrating the Poynting vector through the surface \( \Delta S = \pi \left( d^2 - b^2 \right) \) representing the additional power passing through the right part of the beam pipe [12], one obtains

\[ P_{\text{em}} = \int_{\Delta S} \left( \frac{1}{\mu} \vec{E} \times \vec{B} \right) \cdot \hat{n}dS = \int_b^d E_z B_\theta \frac{2\pi r dr}{\mu} = \frac{I^2}{2\pi\varepsilon_0 v} \ln \frac{d}{b} \]

exactly the same expression of Eq. (B.2). Notice that if \( d < b \) the beam gains energy. If \( d \to \infty \) the power goes to infinity, an unphysical result like this is nevertheless consistent with the original assumption of an infinite energy beam (\( \gamma \to \infty \)).
References


